MATHEMATICS OF COMPUTATION
VOLUME 43, NUMBER 167
JULY 1984, PAGES 219-242

# Rates of Convergence of Gaussian Quadrature for Singular Integrands

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**Abstract.** The authors obtain the rates of convergence (or divergence) of Gaussian quadrature on functions with an algebraic or logarithmic singularity inside, or at an endpoint of, the interval of integration. A typical result is the following: For a bounded smooth weight function on $[-1, 1]$, the error in $n$-point Gaussian quadrature of $f(x) = |x - y|^{-\delta}$ is $O(n^{-2+2\delta})$ if $y = \pm 1$ and $O(n^{-1+\delta})$ if $y \in (-1, 1)$, provided we avoid the singularity. If we ignore the singularity $y$, the error is $O(n^{-1+2\delta}(\log n)^{\delta}(\log\log n)^{\delta(1+\epsilon)})$ for almost all choices of $y$. These assertions are sharp with respect to order.

**1. Introduction.** Much has been written about convergence of rules of numerical integration for integrands with integrable singularities inside or at the endpoints of the interval of integration. The first papers on the subject in recent years, by Davis and Rabinowitz [3] and Rabinowitz [11], established convergence of composite rules and Gauss rules for functions monotonic around certain singularities. Gautschi [7] verified Rabinowitz's conditions for the Fejér weights. Miller [9] introduced the idea of dominated integrability and proved that the latter condition was still sufficient for convergence of quadrature procedures. Feldstein and Miller [5] and El-Tom [4] obtained rates of convergence of compound rules on singular integrands. Chawla and Jain [1] and Rabinowitz [14] found the asymptotic form of the error of Gauss quadrature on certain functions with an algebraic singularity in their derivative.

Osgood and Shisha [10] and others took up the subject of dominated integrability. Rabinowitz [13] showed that Gaussian quadrature would converge even on functions with singularities interior to the interval of integration, provided the nearest abscissa(s) to the singularity was omitted and provided a certain relationship held between weights and abscissas. Lubinsky and Sidi [8] used a generalized Markov-Stieltjes inequality to prove that omitting the closest abscissas from left and right to the singularity guaranteed convergence of Gauss quadrature, without requiring the above relationship between weights and abscissas.

In this paper the authors use the same generalized Markov-Stieltjes inequality to investigate convergence rates of Gaussian quadrature for functions with a singularity at an endpoint of, or interior to, the interval of integration. This tool yields upper and lower bounds for the error when the integrand is absolutely monotone to the left of the singularity and completely monotone to the right. Furthermore, it yields asymptotic rates for functions which are the product of such a function and a

Received December 7, 1982; revised August 1, 1983.
1980 *Mathematics Subject Classification*. Primary 65D30.

smooth function. We shall see that, under very mild assumptions on the weight function, the error in Gaussian integration of a function with an interior singularity which is algebraic of order $\delta$ (respectively logarithmic) is $O(n^{-1+\delta})$ (respectively $O(n^{-1}\log n)$), provided only that we "avoid the singularity" by omitting the closest abscissa to the interior singularity. When we do not omit the closest abscissa, the error turns out to be $O(n^{-1+2\delta}(\log n)^{\delta}(\log\log n)^{\varepsilon\delta})$ any $\varepsilon > 1$ (respectively $o(n^{-1}\log n)$) for almost all choices of the singularity. All these results are sharp with respect to order.

For endpoint singularities, we shall prove the following: If the interval is $(-1,1)$ and the weight function is "comparable" to the Jacobi weight $(1-x)^{\nu}(1+x)^{\beta}$, then the error is $O(n^{-2\nu-2+2\delta})$ (respectively $O(n^{-2\nu-2}\log n)$) for an algebraic singularity of order $\delta$ at $x = 1$ (respectively a logarithmic singularity).

We note finally that avoiding or ignoring a singularity using some standard rule is not necessarily the best method for numerical integration of a singular integrand. Thus many of the results in this paper are of theoretical, rather than practical, interest.

**2. Notation.** Let $(a, b)$ be a finite or infinite interval. Throughout let there be given a monotone increasing and right continuous function $\alpha$: $(a,b) \to \mathbf{R}$. We assume all the moments $\int_a^b x^j\,d\alpha(x)$, $j = 0,1,2,\ldots$, exist. Then there exist orthonormal polynomials $p_n(x) = \gamma_n\prod_{j=1}^n(x - x_{nj})$, where $\gamma_n > 0$, $n = 1,2,\ldots$, that satisfy

$$\int_a^b p_n(x)p_m(x)\,d\alpha(x) = \begin{cases} 1, & m = n, \\ 0, & m \neq n. \end{cases}$$

We assume that the zeros of $p_n$ are ordered so that $a < x_{n1} < x_{n2} < \cdots < x_{nn} < b$, $n = 1,2,\ldots$. Further, we define the Christoffel numbers

$$\lambda_{nj} = \left(\sum_{k=0}^{n-1} p_k^2(x_{nj})\right)^{-1}, \qquad j = 1,2,\ldots,n;\ n = 1,2,\ldots,$$

so that

$$\int_a^b p(x)\,d\alpha(x) = \sum_{j=1}^{n}\lambda_{nj}p(x_{nj}), \tag{2.1}$$

whenever $p(x)$ is a polynomial of degree at most $2n-1$. For any function $f$: $(a,b) \to \mathbf{R}$, let

$$I[f] = \int_a^b f(x)\,d\alpha(x),$$

$$I_n[f] = \sum_{j=1}^{n}\lambda_{nj}f(x_{nj}), \qquad n = 1,2,\ldots,$$

$$E_n[f] = I[f] - I_n[f], \qquad n = 1,2,\ldots,$$

provided these numbers are defined, the integral being a proper or improper Riemann-Stieltjes integral. Thus $E_n[f]$ is the error in Gauss quadrature of order $n$ for the integrand $f$.

We shall frequently need to consider some fixed point $y \in (a,b)$ at which $f(x)$ may, or may not, have a singularity. Throughout $x_{c(n)}$, $x_{l(n)}$, $x_{r(n)}$ denote the abscissas from $\{x_{n1}, x_{n2},\ldots,x_{nn}\}$ which are, respectively, the closest to $y$, the closest from the left to $y$, and the closest from the right to $y$. More precisely

$$\begin{aligned} |x_{c(n)} - y| &= \min\{|x_{nj} - y| : j = 1, 2, \ldots, n\}, \\ y - x_{l(n)} &= \min\{y - x_{nj} : x_{nj} \leqslant y\}, \\ x_{r(n)} - y &= \min\{x_{nj} - y : x_{nj} > y\}. \end{aligned}$$

If $y < x_{n1}$, we take $x_{l(n)} = a$, and if $y \geqslant x_{nn}$, we take $x_{r(n)} = b$. When $x_{c(n)}$ is not uniquely defined by the above, which is the case only when $y$ is midway between $x_{l(n)}$ and $x_{r(n)}$, we take $x_{c(n)} = x_{l(n)}$. We let

$$I_n^*[f] = \sum_{\substack{j=1 \\ j \neq c(n)}}^{n} \lambda_{nj} f(x_{nj}),$$

so that $I_n^*$ avoids the singularity by omitting the closest abscissa to it. Further, we let

$$E_n^*[f] = I[f] - I_n^*[f].$$

Similarly, we define

$$I_n^{**}[f] = \sum_{\substack{j=1 \\ j \neq l(n), r(n)}}^{n} \lambda_{nj} f(x_{nj}),$$

so that $I_n^{**}$ avoids the singularity by omitting the closest abscissas from the left and right to $y$. Further,

$$E_n^{**}[f] = I[f] - I_n^{**}[f].$$

We let $\lambda_{c(n)}, \lambda_{l(n)}, \lambda_{r(n)}$ denote the Christoffel numbers corresponding to $x_{c(n)}, x_{l(n)}$, $x_{r(n)}$, respectively. Similarly $x_{c(n)\pm 1}, \lambda_{c(n)\pm 1}$ denote $x_{n,c(n)\pm 1}$ and $\lambda_{n,c(n)\pm 1}$ and so on. Note that $x_{r(n)} = x_{l(n)+1}$.

It is worth comparing the definition of $I_n^*$, $I_n^{**}$ above to the ideas of avoiding the singularity used in Rabinowitz [13] and Lubinsky and Sidi [8]. The rule $I_n^{**}$ above coincides with $\hat{R}_n$ used in Theorem 1 in [13]. Further, $I_n^{**}$ is similar to $K_n^*$, used in [8], except that the latter rule also includes the closest abscissas from the left and right to the singularity $y$, provided those abscissas are not too close to $y$, in the sense of (2.4B) in [8].

*Definition* 2.1. We shall say $d\alpha(x)$ is bounded above and below near $y$ if there exist positive constants $m$ and $M$ such that

$$m \leqslant \frac{\alpha(x_2) - \alpha(x_1)}{x_2 - x_1} \leqslant M \tag{2.2}$$

for all $x_1, x_2$ in a neighborhood of $y$.

The usual symbols $O, o, \sim, \cong$ will be used to compare sequences and functions. For example, if $(c_n), (d_n)$ are sequences of real numbers,

$$\begin{aligned} c_n = O(d_n) &\Leftrightarrow \limsup_{n\to\infty} |c_n/d_n| < \infty, \\ c_n = o(d_n) &\Leftrightarrow \lim_{n\to\infty} c_n/d_n = 0, \\ c_n \cong d_n &\Leftrightarrow \lim_{n\to\infty} c_n/d_n = 1. \end{aligned}$$

$c_n \sim d_n \Leftrightarrow K_1 \leqslant c_n/d_n \leqslant K_2$ for all large enough $n$, where $K_1$ and $K_2$ are positive constants.

*Definition* 2.2. Let $\mathscr{I}$ be a real interval.

(i) $R(\mathscr{I})$ denotes the class of functions $f(x)$ such that both $f$ and $|f|$ are (possibly improperly) Riemann-Stieltjes integrable with respect to $d\alpha(x)$ over $\mathscr{I}$.

(ii) If $\mathscr{I}$ is bounded and closed and if $l$ is a nonnegative integer, $C^l[\mathscr{I}]$ denotes the class of functions whose $l$th derivative is continuous in $\mathscr{I}$ with norm $\|f\| = \max\{|f(x)|: x \in \mathscr{I}\}$.

(iii) If $\mathscr{I}$ is bounded and closed and $f \in C[\mathscr{I}] \equiv C^0[\mathscr{I}]$, the modulus of continuity of $f$ in $\mathscr{I}$ is

$$\omega_f(\mathscr{I}; \varepsilon) = \max\{|f(x_1) - f(x_2)|: |x_1 - x_2| \leqslant \varepsilon, x_1, x_2 \in \mathscr{I}\} \quad \text{for any } \varepsilon > 0.$$

We say $f \in \mathrm{Lip}(\theta)$ in $\mathscr{I}$ where $0 < \theta \leqslant 1$ if $\omega_f(\mathscr{I}; \varepsilon) = O(\varepsilon^\theta)$, and we say $f \in \mathrm{Lip}(\theta; \eta)$ in $\mathscr{I}$ where $\theta \geqslant 0$ and $\eta$ is real if $\omega_f(\mathscr{I}; \varepsilon) = O(\varepsilon^\theta |\log \varepsilon|^{-\eta})$.

*Definition* 2.3. Let $\mathscr{I}$ be a real interval. Let $k$ be a positive integer. We shall say $f$: $\mathscr{I} \to \mathbf{R}$ is $k$-absolutely monotone in $\mathscr{I}$ ($k$-completely monotone in $\mathscr{I}$) if $f \in R(\mathscr{I})$ and if

$$f^{(j)}(x) \geqslant 0, \qquad x \in \mathscr{I}, j = 0, 1, 2, \ldots, k \tag{2.3}$$
$$\left((-1)^j f^{(j)}(x) \geqslant 0, x \in \mathscr{I}, j = 0, 1, 2, \ldots, k\right).$$

If $f$ is $k$-absolutely monotone in $\mathscr{I}$ ($k$-completely monotone in $\mathscr{I}$) for all positive integers $k$, we shall say $f$ is absolutely monotone in $\mathscr{I}$ (completely monotone in $\mathscr{I}$).

**3. Basic Lemmas.** The Markov-Stieltjes inequality that we need depends on the following fundamental lemma:

LEMMA 3.1. *Let $f$ be $(m + 1)$-absolutely monotone in $(a, \xi]$ with strict inequality holding in* (2.3). *Let $P(x)$ be a polynomial of degree at most $m$. Let*

$$m_1 = \textit{total multiplicity of zeros of } f - P \textit{ in } (a, \xi],$$
$$m_2 = \textit{total multiplicity of zeros of } P \textit{ in } [\xi, \infty).$$

*Then $m_1 + m_2 \leqslant m + 1$.*

*Proof*. Freud [6, Lemma I.5.3] gives a proof for $a = -\infty$. By substituting $a$ for $-\infty$ throughout his proof, we see the more general form above is true. □

Both the statement and proof of the generalized Markov-Stieltjes inequality below are essentially contained in Freud [6, pp. 32-33], but we restate and reprove it, because it is difficult to recognize from [6] the form of the inequality below.

LEMMA 3.2. *Let $f(x)$ be $(2n - 1)$-absolutely monotone in $(a, x_{nk})$ some $n \geqslant 1$, $1 \leqslant k \leqslant n$. Then*

(i)

$$\sum_{j=1}^{k-1} \lambda_{nj} f(x_{nj}) \leqslant \int_a^{x_{nk}} f(x)\, d\alpha(x).$$

(ii) *If in addition $f(x)$ is $(2n - 1)$-absolutely monotone in $(a, x_{nk}]$, then*

$$\sum_{j=1}^{k} \lambda_{nj} f(x_{nj}) \geqslant \int_a^{x_{nk}} f(x)\, d\alpha(x).$$

*Proof*. (i) Define a polynomial $p(x)$ of degree $\leqslant 2n-2$ by the $2n-1$ interpolation conditions

(3.1A) $$p(x_{nj}) = \begin{cases} f(x_{nj}), & j = 1,2,\dots,k-1, \\ 0, & j = k, k+1,\dots,n. \end{cases}$$

(3.1B) $$p'(x_{nj}) = \begin{cases} f'(x_{nj}), & j = 1,2,\dots,k-1, \\ 0, & j = k+1, k+2,\dots,n. \end{cases}$$

We shall assume initially that strict inequality holds in (2.3). Let $\xi \in (x_{n,k-1}, x_{nk})$. Then, by (3.1A,B), $f-p$ has $m_1 \geqslant 2k-2$ zeros in $(a, \xi]$ and $p$ has $m_2 \geqslant 2n-2k+1$ zeros in $[\xi, \infty)$. Thus $m_1 + m_2 \geqslant 2n-1 = \deg(p)+1$. By Lemma 3.1, we have $m_1 + m_2 \leqslant 2n-1$. Thus $m_1 = 2k-2$ and $m_2 = 2n-2k+1$, and the only zeros of $f-p$ and $p$ in $(a, \xi]$ and $[\xi, \infty)$, respectively, are already listed in (3.1A,B). As all zeros of $f-p$ in $(a, \xi]$ are double zeros, it follows that $f-p$ does not change sign in $(a, \xi]$ for any $\xi < x_{nk}$ and hence $f-p$ does not change sign in $(a, x_{nk})$. As $p(x_{nk}) = 0$, we deduce

(3.2) $$f(x) \geqslant p(x), \qquad x \in (a, x_{nk}).$$

Next, as $\xi > x_{n,k-1}$ was arbitrary, it follows that $p(x)$ has $2n-2k+1$ zeros in $(x_{n,k-1}, \infty)$, these being listed in (3.1A,B). Since $p(x_{n,k-1}) = f(x_{n,k-1}) > 0$ and as $p(x)$ has a simple zero at $x_{nk}$ and double zeros at $x_{nj}$, $j = k+1, k+2,\dots,n$, it follows that $p(x)$ changes sign at $x_{nk}$ and

(3.3) $$0 \geqslant p(x), \qquad x \in [x_{nk}, \infty).$$

Then by (2.1), (3.2) and (3.3), and by (3.1A),

$$\int_{-\infty}^{x_{nk}} f(x)\, d\alpha(x) \geqslant \int_{-\infty}^{\infty} p(x)\, d\alpha(x) = \sum_{j=1}^{n} \lambda_{nj} p(x_{nj}) = \sum_{j=1}^{k-1} \lambda_{nj} f(x_{nj}).$$

Finally, if strict inequality does not hold in (2.3), $f_\varepsilon(x) = f(x) + \varepsilon e^x$ satisfies (2.3) with strict inequality for any $\varepsilon > 0$. Applying the above inequality to $f_\varepsilon$ and letting $\varepsilon \to 0+$, we obtain the more general inequality.

(ii) is similar: One defines a polynomial $P(x)$ of degree $\leqslant 2n-2$ by

$$P(x_{nj}) = \begin{cases} f(x_{nj}), & j = 1,2,\dots,k, \\ 0, & j = k+1, k+2,\dots,n, \end{cases}$$

$$P'(x_{nj}) = \begin{cases} f'(x_{nj}), & j = 1,2,\dots,k-1, \\ 0, & j = k+1, k+2,\dots,n, \end{cases}$$

and uses Lemma 3.1 to deduce

$$f(x) \leqslant P(x), \qquad x \in (a, x_{nk}],$$
$$0 \leqslant P(x), \qquad x \in [x_{nk}, \infty). \quad \square$$

For $(2n-1)$-completely monotone functions, there is the following corollary:

LEMMA 3.3. *Let $f(x)$ be $(2n-1)$-completely monotone in $(x_{nk}, b)$ some $n \geqslant 1$, $1 \leqslant k \leqslant n$. Then*

(i)

$$\sum_{j=k+1}^{n} \lambda_{nj} f(x_{nj}) \leqslant \int_{x_{nk}}^{b} f(x)\, d\alpha(x).$$

(ii) *If, in addition*, $f(x)$ *is* $(2n-1)$*-completely monotone in* $[x_{nk}, b)$, *then*

$$\sum_{j=k}^{n} \lambda_{nj} f(x_{nj}) \geq \int_{x_{nk}}^{b} f(x)\, d\alpha(x).$$

*Proof*. (i) Make the change of variable $x \to -x$ and let $d\beta(x) = -d\alpha(-x)$, so that $\beta(x) = \alpha(b) - \alpha(-x)$, $x \in (-b, -a)$. We denote the orthonormal polynomials, zeros and Christoffel numbers for $d\beta(x)$ respectively by $\hat{p}_n$, $\hat{x}_{nj}$ and $\hat{\lambda}_{nj}$. It is easy to see $\hat{p}_n(x) = (-1)^n p_n(-x)$, $n = 1, 2, \ldots$, and so $\hat{x}_{nj} = -x_{n,n-j+1}$; $\hat{\lambda}_{nj} = \lambda_{n,n-j+1}$, $j = 1, 2, \ldots, n$; $n = 1, 2, \ldots$. Let $g(x) = f(-x)$. We see $g(x)$ is $(2n-1)$-absolutely monotone in $(-b, -x_{nk}) = (-b, \hat{x}_{n,n-k+1})$. Then Lemma 3.2(i) yields

$$\sum_{j=1}^{n-k} \hat{\lambda}_{nj} g(\hat{x}_{nj}) \leq \int_{-b}^{\hat{x}_{n,n-k+1}} g(x)\, d\beta(x)$$

$$\Rightarrow \sum_{j=k+1}^{n} \lambda_{nj} f(x_{nj}) \leq \int_{x_{nk}}^{b} f(x)\, d\alpha(x).$$

(ii) follows similarly from Lemma 3.2(ii). □

The following lemma on the asymptotic behavior of weights and abscissas will be useful in the sequel.

LEMMA 3.4. *Let* $(a, b)$ *be bounded and assume* $d\alpha(x)$ *is bounded above and below near* $y \in (a, b)$. *Then there exist positive constants* $c_1, c_2, c_3, c_4$ *and a neighborhood* $\mathcal{I}$ *of* $y$ *such that for all* $n$ *and* $j$,

$$\text{(3.4)} \qquad \begin{aligned} &\text{(i)} \quad x_{nj} \in \mathcal{I} \Rightarrow c_1/n \leq x_{n,j+1} - x_{nj} \leq c_2/n,\\ &\text{(ii)} \quad x_{nj} \in \mathcal{I} \Rightarrow c_3/n \leq \lambda_{nj} \leq c_4/n,\\ &\text{(iii)} \quad c_1/(2n) \leq \max\{y - x_{l(n)}, x_{r(n)} - y\} \leq c_2/n. \end{aligned}$$

*Proof*. (i) This is Theorem III.5.1 in Freud [6] with a linear transformation of $(a, b)$ onto $(-1, 1)$.

(ii) By the classical Markov-Stieltjes inequality

$$\lambda_{nj} \leq \int_{x_{n,j-1}}^{x_{n,j+1}} d\alpha(x)$$

(Szegö [18, p. 50] or Freud [6, p. 29]). Further as $d\alpha(x)$ is bounded below near $y$, Theorem II.2.4 in Freud [6] shows that, for large $n$, there are as many $x_{nj}$ near $y$ as we like. We deduce from (2.2) and (3.4) that, for all $x_{nj}$ in a neighborhood $\mathcal{I}$ of $y$,

$$\lambda_{nj} \leq M(x_{n,j+1} - x_{n,j-1}) \leq 2Mc_2/n = c_4/n.$$

Next by Theorem I.4.1 in Freud [6], and by (2.2),

$$\lambda_{nj} = \inf\left\{\int_{-\infty}^{\infty} P^2(x)\, d\alpha(x) \colon \deg(P) \leq n-1 \text{ and } P(x_{nj}) = 1\right\}$$

$$\geq m \inf\left\{\int_{y-\delta}^{y+\delta} P^2(x)\, dx \colon \deg(P) \leq n-1 \text{ and } P(x_{nj}) = 1\right\},$$

where $(y-\delta, y+\delta)$ is a suitable neighborhood of $y$. Now consider the transformation $u = -1 + (x - (y-\delta))/\delta$ which maps $x \in [y-\delta, y+\delta]$ onto $u \in [-1, 1]$. Each polynomial $P(x)$ of degree $\leq n-1$ satisfying $P(x_{nj}) = 1$ corresponds to a polynomial $P^*(u)$ of degree $\leq n-1$ in $u$ satisfying $P^*(u_{nj}) = 1$ where $u_{nj} = u(x_{nj})$.

Then

$$\lambda_{nj} \geqslant (m\delta)\inf\left\{\int_{-1}^{1}(P^*(u))^2\,du\colon \deg(P^*) \leqslant n-1 \text{ and } P^*(u_{nj}) = 1\right\}$$
$$= (m\delta)\lambda_n(du;\, u_{nj}),$$

using Freud's notation for the Christoffel function of the weight $du$ over $[-1,1]$. By Theorem V.6.8 in Freud [6], for the weight $\alpha'(x) \equiv 1$ in $[-1,1]$,

$$\lambda_n(du;\, u_{nj}) = \pi\left(1 - u_{nj}^2\right)^{1/2}/n + o(1/n),$$

where if $x_{nj}$ is restricted to some closed subinterval of $(y-\delta,\, y+\delta)$, then $u_{nj}$ lies in some closed subinterval of $(-1,1)$ and so the $o(1/n)$ term is uniform in such $x_{nj}$ by the theorem. This yields $\lambda_{nj} \geqslant c_3/n$ for all $n$, $j$ such that $x_{nj}$ lies in some neighborhood of $\mathscr{I}$.

(iii) Now $\max\{y - x_{l(n)}, x_{r(n)} - y\} \geqslant (x_{r(n)} - x_{l(n)})/2$ and for large $n$, $x_{r(n)} = x_{l(n)+1}$ and $x_{l(n)}$ both lie in the neighborhood $\mathscr{I}$ of $y$. Hence $(x_{r(n)} - x_{l(n)}) \geqslant c_1/n$. Similarly

$$\max\{y - x_{l(n)}, x_{r(n)} - y\} \leqslant (x_{r(n)} - x_{l(n)}) \leqslant c_2/n. \quad \square$$

**4. Interior Singularities, Part 1.** In this section, we investigate the asymptotic behavior of $E_n[f]$ where $f(x) = |x-y|^{-\delta}$ or $-\log|x-y|$. First, however, we establish our basic error estimate which may be applied to functions with a singularity on either one, or both sides of $y$.

LEMMA 4.1. *Let $f(x)$ be $(2n-1)$-absolutely monotone in $(a, y)$ and $(2n-1)$-completely monotone in $(y, b)$. Then*

(i)

$$\int_{x_{l(n)}}^{x_{r(n)}} f(x)\,d\alpha(x) \leqslant E_n^{**}[f] \leqslant \int_{x_{l(n)-1}}^{x_{r(n)+1}} f(x)\,d\alpha(x). \tag{4.1}$$

(ii) *If $y \neq x_{c(n)}$,*

$$\int_{x_{l(n)}}^{x_{r(n)}} f(x)\,d\alpha(x) - \sum_{j=l(n)}^{r(n)} \lambda_{nj} f(x_{nj}) \leqslant E_n[f] \leqslant \int_{x_{l(n)}}^{x_{r(n)}} f(x)\,d\alpha(x). \tag{4.2}$$

(iii) *If $j$ is the integer such that $j \in \{l(n), r(n)\} \setminus \{c(n)\}$, then*

$$E_n^*[f] = E_n^{**}[f] - \lambda_{nj} f(x_{nj}). \tag{4.3}$$

(iv) *If $y = x_{c(n)}$,*

$$0 \leqslant E_n^*[f] \leqslant \int_{x_{l(n)-1}}^{x_{r(n)}} f(x)\,d\alpha(x). \tag{4.4}$$

*Proof.* (i) By Lemma 3.2(i) and 3.3(i), respectively, we have

$$\sum_{j=1}^{l(n)-1} \lambda_{nj} f(x_{nj}) \leqslant \int_a^{x_{l(n)}} f(x)\,d\alpha(x),$$

$$\sum_{j=r(n)+1}^{n} \lambda_{nj} f(x_{nj}) \leqslant \int_{x_{r(n)}}^{b} f(x)\,d\alpha(x).$$

Adding, we obtain

$$I_n^{**}[f] \leqslant I[f] - \int_{x_{l(n)}}^{x_{r(n)}} f(x)\, d\alpha(x),$$

from which the lower bound in (4.1) follows. Similarly, Lemma 3.2(ii) and 3.3(ii) yield

$$\sum_{j=1}^{l(n)-1} \lambda_{nj} f(x_{nj}) \geqslant \int_a^{x_{l(n)-1}} f(x)\, d\alpha(x),$$

$$\sum_{j=r(n)+1}^{n} \lambda_{nj} f(x_{nj}) \geqslant \int_{x_{r(n)+1}}^{b} f(x)\, d\alpha(x).$$

Adding, we obtain

$$I_n^{**}[f] \geqslant I[f] - \int_{x_{l(n)-1}}^{x_{r(n)+1}} f(x)\, d\alpha(x),$$

and the upper bound in (4.1) follows.

(ii) Since $y \neq x_{c(n)}$, we have $x_{l(n)} < y < x_{r(n)}$ and Lemmas 3.2(ii) and 3.3(ii) yield

$$\sum_{j=1}^{l(n)} \lambda_{nj} f(x_{nj}) \geqslant \int_a^{x_{l(n)}} f(x)\, d\alpha(x),$$

$$\sum_{j=r(n)}^{n} \lambda_{nj} f(x_{nj}) \geqslant \int_{x_{r(n)}}^{b} f(x)\, d\alpha(x)$$

$$\Rightarrow I_n[f] \geqslant I[f] - \int_{x_{l(n)}}^{x_{r(n)}} f(x)\, d\alpha(x),$$

which yields the upper bound in (4.2). The lower bound follows from the identity

$$E_n[f] = E_n^{**}[f] - \sum_{j=l(n)}^{r(n)} \lambda_{nj} f(x_{nj})$$

and the lower bound for $E_n^{**}[f]$ in (4.1).

(iii) follows immediately from the definition of $E_n^*$ and $E_n^{**}$.

(iv) Since $y = x_{c(n)}$, we have $y = x_{l(n)}$, and by Lemmas 3.2(i), (ii), 3.3(i), (ii)

$$\int_a^{x_{l(n)-1}} f(x)\, d\alpha(x) \leqslant \sum_{j=1}^{l(n)-1} \lambda_{nj} f(x_{nj}) \leqslant \int_a^{x_{l(n)}} f(x)\, d\alpha(x),$$

$$\int_{x_{r(n)}}^{b} f(x)\, d\alpha(x) \leqslant \sum_{j=r(n)}^{n} \lambda_{nj} f(x_{nj}) \leqslant \int_{x_{l(n)}}^{b} f(x)\, d\alpha(x).$$

Adding, we obtain (4.4). □

LEMMA 4.2. *Let* $y, c, d \in (a, b), y \neq d$ *and* $z = (y - c)/(y - d)$.

(i) *Let* $f(x)$ *be monotone increasing and positive in* $(a, y)$ *and let* $c, d \in (a, y)$. *Then*

$$(4.5) \qquad (1/z + 1)^{-1} \leqslant \int_c^y f(u)\, du \Big/ \int_d^y f(u)\, du \leqslant (z + 1).$$

(ii) *Let $f(x)$ be monotone decreasing and positive in $(y, b)$ and let $c, d \in (y, b)$. Then*

$$(1/z + 1)^{-1} \leqslant \int_y^c f(u)\,du \Big/ \int_y^d f(u)\,du \leqslant (z + 1).$$

*Proof.* (i) We first prove the second inequality in (4.5). If $z \leqslant 1$, that is, if $d$ is not closer to $y$ than $c$, this inequality is trivial. So assume $z > 1$, and let $k$ be the largest integer $\leqslant z$. We can then partition the interval $[c, y]$ into $k + 1$ intervals $[c_j, c_{j+1}]$, $j = 0, 1, 2, \ldots, k$, where $c_0 = c$, $c_{k+1} = y$ and $c_{j+1} - c_j = y - d$, $j = 1, 2, \ldots, k$. Then each interval $[c_j, c_{j+1}]$ has length at most $y - d$. Further, as $f(x)$ is increasing in $(a, y)$, we see

$$\int_c^y f(u)\,du = \sum_{j=0}^{k} \int_{c_j}^{c_{j+1}} f(u)\,du \leqslant (k + 1)\int_d^y f(u)\,du \leqslant (z + 1)\int_d^y f(u)\,du.$$

By symmetry of $c$, $d$, we obtain also

$$\int_d^y f(u)\,du \leqslant (1/z + 1)\int_c^y f(u)\,du$$

and (4.5) follows.

(ii) is similar. □

We can now prove a general theorem for "2-sided" singularities:

THEOREM 4.3. *Let $(a, b)$ be a finite interval and $y \in (a, b)$. Let $d\alpha(x)$ be bounded above and below near $y$. Let $f(x)$ be absolutely monotone in $(a, y)$, completely monotone in $(y, b)$ and let $f(y) = 0$. Further assume $f(x)$ grows at roughly the same rate on both sides of $y$ as $x \to y$, that is*

$$f(y - u) \sim f(y + u) \quad \text{as } u \to 0+. \tag{4.6}$$

*Let $\mu_n = \int_{y-1/n}^{y} f(x)\,dx$, $n = 1, 2, 3, \ldots$. Then*

(i) $E_n^{**}[f] \sim \mu_n$,

(ii) $E_n^{*}[f] = O(\mu_n)$,

(iii) $E_n[f] = O(\mu_n) - \lambda_{c(n)} f(x_{c(n)})$

*and* $\lambda_{c(n)} \sim n^{-1}$.

*Proof.* The condition (4.6) entails that for some positive constants $c_5$, $c_6$, $\varepsilon$,

$$c_5 \leqslant f(y - u)/f(y + u) \leqslant c_6 \quad \text{all } u \in (0, \varepsilon). \tag{4.7}$$

(i) By (2.2) and (4.1), for large $n$,

$$E_n^{**}[f] \leqslant M\left\{\int_{x_{l(n)-1}}^{y} f(x)\,dx + \int_y^{x_{r(n)+1}} f(x)\,dx\right\}$$

$$\leqslant M\left\{\left[(y - x_{l(n)-1})n + 1\right]\int_{y-1/n}^{y} f(x)\,dx + \left[(x_{r(n)+1} - y)n + 1\right]\int_y^{y+1/n} f(x)\,dx\right\}$$

(by Lemma 4.2(i), (ii))

$$\leqslant M\{[2c_2 + 1] + [2c_2 + 1]/c_5\}\int_{y-1/n}^{y} f(x)\,dx,$$

by Lemma 3.4(i), (iii) and by (4.7).

Further, by (2.2) and (4.1), for large $n$,

$$E_n^{**}[f] \geqslant m\left\{\int_{x_{l(n)}}^{y} f(x)\,dx + \int_{y}^{x_{r(n)}} f(x)\,dx\right\}$$

$$\geqslant m\left\{\left[\left((y - x_{l(n)})n\right)^{-1} + 1\right]^{-1}\int_{y-1/n}^{y} f(x)\,dx\right.$$

$$\left. + \left[\left((x_{r(n)} - y)n\right)^{-1} + 1\right]^{-1}\int_{y}^{y+1/n} f(x)\,dx\right\},$$

by Lemma 4.2(i), (ii). Here if $y = x_{l(n)}$, the first term in the $\{\ \}$ may be interpreted as 0. From Lemma 3.4(iii) and from (4.7) we deduce

$$E_n^{**}[f] \geqslant m[2/c_1 + 1]^{-1}\min\{1, 1/c_6\}\int_{y-1/n}^{y} f(x)\,dx.$$

Thus we have shown

$$K_1\min\{1, 1/c_6\} \leqslant E_n^{**}[f]/\mu_n \leqslant K_2(1 + 1/c_5), \tag{4.8}$$

where $K_1$, $K_2$ are independent of $n$ and $f$ as $M$, $m$, $c_1$, $c_2$ are and where $c_5$, $c_6$ depend on $f$ (as in (4.7)), but are independent of $n$. This establishes (i).

(ii) By (4.3) and (i) above, we deduce

$$E_n^*[f] = O(\mu_n) - \lambda_{nj}f(x_{nj}),$$

where $j = j(n) \in \{l(n), r(n)\} \setminus \{c(n)\}$. By Lemma 3.4(iii), $|x_{nj} - y| \geqslant c_1/(2n)$. By monotonicity of $f$, if $x_{nj} < y$, we see

$$f(x_{nj}) \leqslant f(y - c_1/(2n)) \leqslant (2n/c_1)\int_{y-c_1/(2n)}^{y} f(x)\,dx$$

$$\leqslant (2n/c_1)(c_1/2 + 1)\int_{y-1/n}^{y} f(x)\,dx$$

by Lemma 4.2(i). Then by Lemma 3.4(ii), for large $n$,

$$\lambda_{nj}f(x_{nj}) \leqslant (2c_4/c_1)(c_1/2 + 1)\mu_n,$$

and so $E_n^*[f] = O(\mu_n)$. Similarly if $x_{nj} > y$.

(iii) follows from the identity

$$E_n[f] = E_n^*[f] - \lambda_{c(n)}f(x_{c(n)})$$

and from Lemma 3.4(ii) which shows $\lambda_{c(n)} \sim n^{-1}$. □

Thus the rate of convergence to 0 of the error in Gaussian quadrature, where the singularity is avoided using $I_n^*$ or $I_n^{**}$, is determined by the asymptotic behavior of $\mu_n$. As a first corollary, we have:

COROLLARY 4.4. *Let $(a, b)$ be a finite interval and $y \in (a, b)$. Let*

$$f(x) = \begin{cases} |x - y|^{-\delta}, & x \in (a, b)\setminus\{y\}, \\ 0, & x = y, \end{cases}$$

*where $0 < \delta < 1$. Assume $d\alpha(x)$ is bounded above and below near $y$. Then*

(i) $E_n^{**}[f] \sim n^{-1+\delta}$.

(ii) $E_n^*[f] = O(n^{-1+\delta})$,

*and there exists $\delta_0 \in (0, 1)$ such that, whenever $\delta \in (\delta_0, 1)$, we have*

$$E_n^*[f] \sim n^{-1+\delta}. \tag{4.9}$$

(iii) *For those positive integers n for which* $y \neq x_{c(n)}$,

$$(4.10)\qquad \begin{aligned} E_n[f] &= -\lambda_{c(n)}|x_{c(n)} - y|^{-\delta} + O(n^{-1+\delta}) \\ &= O\left(n^{-1}|x_{c(n)} - y|^{-\delta}\right), \end{aligned}$$

*where* $\lambda_{c(n)} \sim n^{-1}$.

*Proof.* First note that $f(y-u) = f(y+u) = |u|^{-\delta}$, and so (4.7) holds with $c_5 = c_6 = 1$. Further $f$ is absolutely monotone in $[a, y)$ and completely monotone in $(y, b]$, while

$$\mu_n = \int_{y-1/n}^{y} f(x)\,dx = n^{-1+\delta}/(1-\delta).$$

(i) By (4.8), as $c_5 = c_6 = 1$,

$$(4.11)\qquad K_1/(1-\delta) \leqslant E_n^{**}[f]/n^{-1+\delta} \leqslant 2K_2/(1-\delta),$$

where $K_1$ and $K_2$ are positive constants independent of $f$ and $n$.

(ii) The first part follows from Theorem 4.3(ii). To prove (4.9), we use (4.3). If $j \in \{l(n), r(n)\} \setminus \{c(n)\}$, Lemma 3.4(ii), (iii) yield

$$\lambda_{nj}|x_{nj} - y|^{-\delta} \leqslant (c_4/n)\left(c_1/(2n)\right)^{-\delta} \leqslant K_3 n^{-1+\delta},$$

where $K_3 = c_4 \max\{1, 2/c_1\}$ is independent of $n$ and $\delta$. Then by (4.3) and (4.11),

$$\{K_1/(1-\delta) - K_3\} \leqslant E_n^*[f]/n^{-1+\delta} \leqslant 2K_2/(1-\delta),$$

and for $\delta$ close enough to 1, say for $\delta \in (\delta_0, 1)$, the term in { } is positive as $K_1$ and $K_3$ are independent of $\delta$.

(iii) The first part follows from Theorem 4.3(iii). To show (4.10), it suffices to show $n^\delta = O(|x_{c(n)} - y|^{-\delta})$, but this follows from Lemma 3.4(iii) which shows $|x_{c(n)} - y| \leqslant c_2/n$. □

Next, we have a corollary for logarithmic singularities.

COROLLARY 4.5. *Let* $(a, b)$ *be a finite interval and* $y \in (a, b)$. *Let*

$$f(x) = \begin{cases} -\log|x-y|, & x \in (a,b)\setminus\{y\}, \\ 0, & x = y. \end{cases}$$

*Assume* $d\alpha(x)$ *is bounded above and below near* $y$. *Then*

(i) $E_n^{**}[f] \sim n^{-1}\log n$.

(ii) $E_n^*[f] = O(n^{-1}\log n)$.

(iii) *For those positive integers n for which* $y \neq x_{c(n)}$,

$$\begin{aligned} E_n[f] &= -\lambda_{c(n)}\log|x_{c(n)} - y| + O(n^{-1}\log n) \\ &= O\left(n^{-1}\log|x_{c(n)} - y|\right), \end{aligned}$$

*where* $\lambda_{c(n)} \sim n^{-1}$.

*Proof.* Let $d$ be a positive constant chosen so that $g(x) = f(x) + d$, $x \in (a, b)$, is nonnegative in $(a, b)$. We see $g$ is absolutely monotone in $(a, y)$ and completely monotone in $(y, b)$. Further $E_n[d] = 0$ and, using Lemma 3.4(ii), we see $E_n^{**}[d]$ and $E_n^*[d]$ are $O(n^{-1})$. By applying Theorem 4.3 to $g$ and using the linearity of $E_n$, $E_n^*$, $E_n^{**}$, we obtain the result as before. □

As a final corollary, we have the following analogue of Theorem 2 in Rabinowitz [13], for the case where $y = \cos(\pi p/q)$ with $p/q$ a rational number.

COROLLARY 4.6. *Let* $(a, b) = (-1, 1)$ *and* $d\alpha(x)$ *be a Jacobi weight given by* $\alpha'(x) = (1-x)^{\nu}(1+x)^{\beta}$, $x \in (-1, 1)$, *where* $\beta$, $\nu = \pm 1/2$. *Let* $y = \cos(\pi p/q)$, *where* $p/q$ *is a rational number in* $(0, 1)$.

(i) *If*

$$f(x) = \begin{cases} |x-y|^{-\delta}, & x \in (-1,1)\setminus\{y\}, \\ 0, & x = y, \end{cases}$$

*where* $0 < \delta < 1$, *then* $E_n[f] = O(n^{-1+\delta})$.

(ii) *If*

$$f(x) = \begin{cases} -\log|x-y|, & x \in (-1,1)\setminus\{y\}, \\ 0, & x = y, \end{cases}$$

*then* $E_n[f] = O(n^{-1}\log n)$.

*Proof.* When $y = x_{c(n)}$, we have $f(x_{c(n)}) = 0$ and so $E_n[f] = E_n^*[f]$. When $y \neq x_{c(n)}$, we have

$$E_n[f] = E_n^*[f] - \lambda_{c(n)} f(x_{c(n)}),$$

where $\lambda_{c(n)} \sim n^{-1}$. It is then evident that both (i) and (ii) follow from Corollaries 4.4 and 4.5 provided we can show that there is a positive constant $c_7$ independent of $n$ such that $|y - x_{c(n)}| \geqslant c_7/n$ if $y \neq x_{c(n)}$. Now for Jacobi weights of the above form, the abscissas $x_{nj}$ are known explicitly (Szegö [18, p. 124, (6.3.5)]). From those explicit formulae, we may write $x_{c(n)} = \cos(k\pi/(2n+i))$, where $k$ is an integer depending only on $n$ and where $i = 0$ or $i = 1$. We have of course $k/(2n+i) \to p/q$ as $n \to \infty$. Then, for large $n$ such that $y \neq x_{c(n)}$,

$$\begin{aligned} |y - x_{c(n)}| &= \left| 2\sin\left\{\pi\left(\frac{k}{2n+i} + \frac{p}{q}\right)\Big/2\right\}\sin\left\{\pi\left(\frac{k}{2n+i} - \frac{p}{q}\right)\Big/2\right\}\right| \\ &\geqslant \sin(\pi p/q)\left|\frac{k}{2n+i} - \frac{p}{q}\right| \\ &\geqslant \sin(\pi p/q)/((2n+1)q) \geqslant c_7/n, \end{aligned}$$

where $c_7 = \sin(\pi p/q)/(4q)$ and we have used the fact that $|kq - p(2n+i)| \geqslant 1$, being a nonzero integer. □

Lemma 4.1 was stated and proved for finite or infinite intervals. Much as above, one can show that for the Laguerre weights, $\alpha'(x) = x^{\nu}e^{-x}$, $E_n^{**}[|x-y|^{-\delta}] \sim n^{-1+\delta}$ and for the Hermite weight, $\alpha'(x) = e^{-x^2}$, $E_n^{**}[|x-y|^{-\delta}] \sim n^{-(1-\delta)/2}$. Similar results are possible for weights on the infinite interval studied by Freud in the 1970's. The method of Lemma 4.1 may also be applied to functions which are "piecewise" completely monotone or absolutely monotone in $(a, b)$ and to functions with more than one singularity or with one-sided singularities.

**5. Interior Singularities, Part 2.** We now prove results of a different character to those of Section 4. For example, we show that, for almost all choices of $y$,

$$E_n\left[|x-y|^{-\delta}\right] = O\left(n^{-1+2\delta}(\log n)^{\delta}(\log\log n)^{\varepsilon\delta}\right),$$

where $\varepsilon > 1$ and that this result is substantially the best possible. This is the analogue of Theorem 3 in Rabinowitz [13].

THEOREM 5.1. (i) *Assume $d\alpha(x)$ is bounded above and below near each $y$ interior to the finite interval $(a, b)$. Then, given $\varepsilon > 1$, there is a set $\mathscr{E}_\varepsilon$ in $(a, b)$ of linear Lebesgue measure zero with the following property*:

$$E_n\left[|x-y|^{-\delta}\right] = O\left(n^{-1+2\delta}(\log n)^{\delta}(\log\log n)^{\varepsilon\delta}\right) \tag{5.1}$$

*for all* $0 < \delta < 1$, *whenever* $y \notin \mathscr{E}_\varepsilon$.

*Hence if* $\delta < 1/2$, $E_n[|x-y|^{-\delta}] \to 0$ *as* $n \to \infty$ *for almost all* $y \in (a, b)$.

(ii) *Assume* $(a, b) = (-1, 1)$ *and* $d\alpha(x)$ *is a Jacobi weight given by* $\alpha'(x) = (1-x)^{\nu}(1+x)^{\beta}$, $x \in (-1, 1)$ *where* $\beta, \nu = \pm 1/2$. *Then there is a set* $\mathscr{E}$ *in* $(-1, 1)$ *of linear Lebesgue measure zero with the following property*:

$$-E_n\left[|x-y|^{-\delta}\right] \geqslant cn^{-1+2\delta}(\log n)^{\delta}(\log\log n)^{\delta}$$

*for infinitely many integers $n$ and for all* $0 < \delta < 1$, *whenever* $y \notin \mathscr{E}$. *Here $c$ is a positive constant independent of $n$, $y$ and $\delta$.*

*Hence if* $\delta \geqslant 1/2$, $E_n[|x-y|^{-\delta}] \nrightarrow 0$ *as* $n \to \infty$ *for almost all* $y \in (-1, 1)$.

*Proof.* (i) Fix $\varepsilon > 1$. Let $\rho_n = n^{-2}(\log n)^{-1}(\log\log n)^{-\varepsilon}$ for all large enough integers $n$, and let

$$\mathscr{I}_n = \bigcup_{k=1}^{n} (x_{nk} - \rho_n, x_{nk} + \rho_n)$$

for all such integers $n$. Further let

$$\mathscr{E}_\varepsilon = \left\{x \in (a, b) \colon x \in \mathscr{I}_n \text{ for infinitely many } n\right\}.$$

Note that $\mathscr{I}_n$ has linear measure at most $2n\rho_n$. Since $\sum_n 2n\rho_n < \infty$, Lemma 1 in Sprindzuk [17, p. 2] ensures that $\mathscr{E}_\varepsilon$ has linear measure zero. Further, if $y \notin \mathscr{E}_\varepsilon$, we see $|x_{c(n)} - y| \geqslant \rho_n$ for all large $n$, and by (4.10), $E_n[|x-y|^{-\delta}] = O(n^{-1}\rho_n^{-\delta})$ from which (5.1) follows.

(ii) The proof is based on the fact that, for the given Jacobi weights, the zeros $x_{nj}$ are known explicitly (Szegö [18, p. 124, (6.3.5)]). Suppose, for example, $\nu = \beta = -1/2$. Then taking account of Szegö's different ordering of the zeros,

$$x_{n,n-j+1} = \cos((j-1/2)\pi/n), \qquad j = 1, 2, \ldots, n.$$

Now writing $y = \cos(\theta\pi)$ where $\theta \in (0, 1)$, we see

$$|y - x_{n,n-j+1}| = |\cos(\theta\pi) - \cos((j-1/2)\pi/n)| \leqslant \pi|\theta - (2j-1)/(2n)|.$$

By Theorem 4 in Sprindzuk [17, p. 11] with

$$P(k) = 2k \quad \text{and} \quad \lambda_m = m^{-1}(\log m)^{-1}(\log\log m)^{-1},$$

all large enough $m$, we see that, for almost all $\theta \in (0, 1)$,

$$|\theta - (2j-1)/(2n)| < \lambda_n/(2n), \qquad j = j(n),$$

for infinitely many $n$. It follows that, for almost all $y \in [-1, 1]$,

$$|y - x_{c(n)}| \leqslant \pi n^{-2}(\log n)^{-1}(\log\log n)^{-1}$$

for infinitely many $n$. Applying Corollary 4.4(iii) and Lemma 3.4(ii),

$$\begin{aligned} -E_n\left[|x-y|^{-\delta}\right] &= \lambda_{c(n)}|x_{c(n)} - y|^{-\delta} + O(n^{-1+\delta}) \\ &\geqslant (c_3/2)n^{-1+2\delta}(\log n)^{\delta}(\log\log n)^{\delta} \end{aligned}$$

for infinitely many $n$ and for almost all $y \in [-1, 1]$. $\square$

Note that any Jacobi weight $d\alpha(x)$ is bounded above and below near each $y \in (-1,1)$. Further note that $(\log\log n)^{\varepsilon\delta}$ in (5.1) may be replaced by $(\log\log n)^{\delta} \cdot (\log\log\log n)^{\varepsilon\delta}$ and so on. Similar remarks apply to part (ii) of the above theorem. The proof of the following result is similar to that of Theorem 5.1.

THEOREM 5.2. *Assume $d\alpha(x)$ is bounded above and below near each $y$ interior to the finite interval $(a, b)$. Then there is a set $\mathscr{E}$ of linear Lebesgue measure zero (even further of Hausdorff dimension zero) such that $E_n[-\log|x-y|] = O(n^{-1}\log n)$ whenever $y \notin \mathscr{E}$.*

**6. Endpoint Singularities.** For endpoint singularities, there is no need to omit abscissas in Gaussian quadrature for singular integrands. Thus we restrict ourselves to the study of $E_n[f]$, and in this section $f(x)$ is usually $(1-x)^{-\delta}$ or $-\log(1-x)$.

LEMMA 6.1. (a) *Let $f(x)$ be $(2n)$-absolutely monotone in $(a, b)$. Then*

$$\max\left\{\int_{x_{nn}}^{b} f(x)\,d\alpha(x) - \lambda_{nn} f(x_{nn}), 0\right\} \leqslant E_n[f] \leqslant \int_{x_{nn}}^{b} f(x)\,d\alpha(x). \tag{6.1}$$

(b) *Let $f(x)$ be $(2n)$-completely monotone in $(a, b)$. Then*

$$\max\left\{\int_{a}^{x_{n1}} f(x)\,d\alpha(x) - \lambda_{n1} f(x_{n1}), 0\right\} \leqslant E_n[f] \leqslant \int_{a}^{x_{n1}} f(x)\,d\alpha(x). \tag{6.2}$$

*Proof.* (a) By Lemmas 3.2(i) and (ii),

$$\sum_{j=1}^{n-1} \lambda_{nj} f(x_{nj}) \leqslant \int_a^{x_{nn}} f(x)\,d\alpha(x) \leqslant \sum_{j=1}^{n} \lambda_{nj} f(x_{nj})$$

$$\Rightarrow I_n[f] - \lambda_{nn} f(x_{nn}) \leqslant I[f] - \int_{x_{nn}}^{b} f(x)\,d\alpha(x) \leqslant I_n[f],$$

and (6.1) follows if we can show also $I[f] \geqslant I_n[f]$. This follows either from Lemma III.1.5 in Freud [6] or Problem 9 in Szegö [18, p. 375].

(b) is similar. □

Unfortunately, the behavior of $\lambda_{nn}$, $b - x_{nn}$, $x_{nn} - x_{n,n-1}$, and so on, have not been thoroughly investigated for general weights and there seems to be no analogue of Lemma 3.4. Thus we are not able to prove results as general as those in Sections 4 and 5, but can prove results for weights comparable to a Jacobi weight.

LEMMA 6.2. *Let $(a, b)$ be a finite interval. Let $\alpha^*$: $(a, b) \to \mathbf{R}$ be a monotone increasing, right continuous function. Assume there exist positive constants $m$ and $M$ such that*

$$m \leqslant \frac{\alpha(x_2) - \alpha(x_1)}{\alpha^*(x_2) - \alpha^*(x_1)} \leqslant M \tag{6.3}$$

*for all $x_1$, $x_2$ in $(a, b)$. Let $x_{nn}^*$ denote the largest zero of the nth orthogonal polynomial for $d\alpha^*$. Then*

$$\frac{m}{M} \leqslant \frac{b - x_{nn}}{b - x_{nn}^*} \leqslant \frac{M}{m}.$$

*Proof.* Now $x_{nn}^* = \max\{\int_a^b xP(x)\,d\alpha^*(x)/\int_a^b P(x)\,d\alpha^*(x)\}$, the maximum being taken over all polynomials $P(x)$ of degree $\leqslant 2n-2$ that are nonnegative and not identically zero in $(a, b)$. See Theorem 7.72.1 in Szegö [18] for one case of this well-known result. The analogous formula holds for $x_{nn}$ with $d\alpha$ replacing $d\alpha^*$. Then

$$\begin{aligned} b - x_{nn}^* &= \min\left\{\int_a^b (b-x)P(x)\,d\alpha^*(x) \Big/ \int_a^b P(x)\,d\alpha^*(x)\right\} \\ &\geqslant \min\left\{\int_a^b (b-x)P(x)m\,d\alpha(x) \Big/ \int_a^b P(x)M\,d\alpha(x)\right\} \\ &= (m/M)(b - x_{nn}), \end{aligned}$$

in each case the minimum being taken over all polynomials $P(x)$ satisfying the previously mentioned conditions. Further we have used (6.3). Similarly we obtain $b - x_{nn}^* \leqslant (M/m)(b - x_{nn})$. □

We can now prove

THEOREM 6.3. *Let $(a, b) = (-1,1)$. Assume $\alpha(x)$ is absolutely continuous in $(-1,1)$ and that, for some positive $m$, $M$ and some $\nu, \beta > -1$, we have*

$$m \leqslant \alpha'(x)/(\alpha^*)'(x) \leqslant M, \qquad x \in (-1,1), \tag{6.4}$$

*where $(\alpha^*)'(x) = (1-x)^\nu(1+x)^\beta$ is a Jacobi weight. Then*

(a) $E_n[(1-x)^{-\delta}] = O(n^{-2\nu-2+2\delta})$ *if $\nu - \delta > -1$ and $\delta > 0$. Further if $\nu \leqslant -1/2$, there exists positive $\eta$ such that*

$$E_n\left[(1-x)^{-\delta}\right] \sim n^{-2\nu-2+2\delta} \quad \textit{whenever}\ \delta \in (1+\nu-\eta, 1+\nu).$$

(b) $E_n[-\log(1-x)] = O(n^{-2\nu-2}\log n)$.

*Proof*. Note first that if $x_{n,n-l+1}^*$ is the $(n-l+1)$th zero of the orthogonal polynomial of degree $n$ for $d\alpha^*$, Theorem 8.1.2 in Szegö [18] shows

$$\lim_{n\to\infty} n \arccos(x_{n,n-l+1}^*) = j_{l\nu},$$

where $j_{l\nu}$ is the $l$th positive zero of $J_\nu(x)$, the Bessel function of the first kind of order $\nu$. As usual, we have taken account of Szegö's different ordering of the zeros. We deduce from the Maclaurin series for $\cos x$ that

$$\lim_{n\to\infty} n^2(1 - x_{n,n-l+1}^*) = j_{l\nu}^2/2, \quad \text{if } l \text{ is fixed.} \tag{6.5}$$

Then, by (6.4), (6.5) and Lemma 6.2,

$$\left(\frac{m}{2M}\right)j_{1\nu}^2 + o(1) \leqslant n^2(1 - x_{nn}) \leqslant \left(\frac{M}{2m}\right)j_{1\nu}^2 + o(1). \tag{6.6}$$

Further by (6.4), by Theorem I.4.2 in Freud [6] and by Problem 10 in Freud [6, p. 132], we see

$$\lambda_{nn} \leqslant c_8 n^{-2\nu-2} \quad \text{provided } \nu \leqslant -1/2, \tag{6.7}$$

where $c_8$ is independent of $n$.

(a) By (6.1), (6.4) and (6.6),

$$0 \leqslant E_n\left[(1-x)^{-\delta}\right] \leqslant 2^{|\beta|}M(1+\nu-\delta)^{-1}(1-x_{nn})^{1+\nu-\delta} = O(n^{-2\nu-2+2\delta}).$$

If $\nu \leqslant -1/2$, then (6.1), (6.4), (6.6) and (6.7) yield

$$\begin{aligned} E_n\left[(1-x)^{-\delta}\right] &\geqslant (m/2)(1+\nu-\delta)^{-1}(1-x_{nn})^{1+\nu-\delta} - c_8 n^{-2\nu-2}(1-x_{nn})^{-\delta} \\ &\geqslant n^{-2\nu-2+2\delta}\left[\frac{m}{2(1+\nu-\delta)}\min\left\{1, \frac{mj_{1\nu}^2}{2M}\right\} - c_8\max\left\{1, \frac{2M}{mj_{1\nu}^2}\right\} + o(1)\right], \end{aligned}$$

and the constant in [ ] is positive for $\delta$ close to $1 + \nu$, since $c_8$, $m$ and $M$ are independent of $\delta$. □

(b) is similar to the first part of (a).

For Jacobi weights, we obtain the following more precise result.

THEOREM 6.4. *Let* $(a, b) = (-1, 1)$ *and* $\alpha'(x) = (1-x)^\nu(1+x)^\beta$, $x \in (-1, 1)$, *where* $\nu, \beta > -1$. *Let* $J_\nu(z)$ *be the Bessel function of the first kind of order* $\nu$ *and* $j_{1\nu}$ *be its first positive zero.*

(a) *Let* $0 < \delta < 1 + \nu$. *Let*

$$s_n = 2^{-\beta}\left(j_{1\nu}^2/2\right)^{-1-\nu+\delta} n^{2\nu+2-2\delta}(1+\nu-\delta)E_n\left[(1-x)^{-\delta}\right].$$

*Then*

$$\max\left\{0, 1 - c_0^2(\nu)(1+\nu-\delta)\right\} \leqslant \liminf_{n\to\infty} s_n \leqslant \limsup_{n\to\infty} s_n \leqslant 1, \tag{6.8}$$

*where*

$$c_0(\nu) = 2/\left(j_{1\nu}J_\nu'(j_{1\nu})\right). \tag{6.9}$$

(b) *Let*

$$t_n = 2^{-\beta-1}\left(j_{1\nu}^2/2\right)^{-1-\nu} n^{2\nu+2}(\log n)^{-1}(1+\nu)E_n[-\log(1-x)].$$

*Then*

$$\max\left\{0, 1 - c_0^2(\nu)(1+\nu)\right\} \leqslant \liminf_{n\to\infty} t_n \leqslant \limsup_{n\to\infty} t_n \leqslant 1. \tag{6.10}$$

*Proof.* (a) Now

$$\int_{x_{nn}}^1 (1-x)^{-\delta}\, d\alpha(x) \cong 2^\beta(1-x_{nn})^{1+\nu-\delta}/(1+\nu-\delta) \tag{6.11}$$
$$\cong 2^\beta\left(j_{1\nu}^2/(2n^2)\right)^{1+\nu-\delta}/(1+\nu-\delta),$$

by (6.5). Further (15.3.11) in Szegö [18, p. 350] shows that

$$\lambda_{nn}(1-x_{nn})^{-\delta} \cong 2^{\nu+\beta+1}(j_{1\nu}/2)^{2\nu}\left\{J_\nu'(j_{1\nu})\right\}^{-2} n^{-2\nu-2}\left(j_{1\nu}^2/(2n^2)\right)^{-\delta}. \tag{6.12}$$

Then (6.8) follows easily from (6.1), (6.11), (6.12) and (6.9).

(b) is similar. □

By computing $c_0(\nu)$ from tables, one observes that the lower bound in (6.8) is positive only for $\delta$ close to $1 + \nu$. Further, the lower bound in (6.10) seems to be zero for all nonnegative $\nu$, but it is not clear what happens as $\nu \to -1$.

In exactly the same way as above one can investigate singularities at the left endpoint of the interval of integration. Further, as Lemma 6.1 was valid for infinite, as well as finite intervals, one can use it to investigate $E_n[x^{-\delta}]$, for example, for the Laguerre weights on $(0, \infty)$.

**7. Interior Singularities for More General Functions.** We now extend the results of Sections 4 and 5 to the function $f(x) = \phi(x)g(x)$, where $g(x)$ is smooth and $\phi(x) = |x-y|^{-\delta}$ or $\phi(x) = -\log|x-y|$. Throughout, without further mention, we assume $(a, b)$ is a finite interval and $y \in (a, b)$.

LEMMA 7.1. *Let $\phi(x) \in R(a, b)$ be continuous in $(a, b)/\{y\}$ and $(y - x)\phi(x) \in C[a, b]$. Let $g \in C[a, b]$, and let $k$ be a nonnegative integer such that $g^{(k)}(y)$ exists. For $j = 1, 2, \ldots, k + 1$, let*

$$h_j(x) = \phi(x)\left[g(x) - \sum_{l=0}^{j-1} \frac{g^{(l)}(y)}{l!}(x - y)^l\right], \qquad x \in [a, b]. \tag{7.1}$$

*Then*

(a)

$$E_n[\phi g] = E_n[\phi]g(y) + \sum_{l=1}^{k} \frac{g^{(l)}(y)}{l!} E_n\left[(x - y)^l \phi\right] + E_n[h_{k+1}]$$

*provided* $y \neq x_{c(n)}$.

(b)

$$E_n^*[\phi g] = E_n^*[\phi]g(y) + \sum_{l=1}^{k} \frac{g^{(l)}(y)}{l!} E_n\left[(x - y)^l \phi\right] + E_n[h_{k+1}] + \lambda_{c(n)} h_1(x_{c(n)}). \tag{7.2}$$

(c)

$$E_n^{**}[\phi g] = E_n^{**}[\phi]g(y) + \sum_{l=1}^{k} \frac{g^{(l)}(y)}{l!} E_n\left[(x - y)^l \phi\right] + E_n[h_{k+1}] + \sum_{j=l(n)}^{r(n)} \lambda_{nj} h_1(x_{nj}).$$

*Proof.* (a) follows immediately from the definition of $h_{k+1}$.

(b) From the definition of $E_n^*$, $E_n$ and $h_{k+1}$, we see

$$E_n^*[\phi g] = E_n^*[\phi]g(y) + \sum_{l=1}^{k} \frac{g^{(l)}(y)}{l!} E_n\left[(x - y)^l \phi\right] + E_n[h_{k+1}] + \lambda_{c(n)}\left[\sum_{l=1}^{k} \frac{g^{(l)}(y)}{l!}(x_{c(n)} - y)^l \phi(x_{c(n)}) + h_{k+1}(x_{c(n)})\right],$$

which reduces to (7.2) since

$$h_1(x) = \sum_{l=1}^{k} \frac{g^{(l)}(y)}{l!}(x - y)^l \phi(x) + h_{k+1}(x).$$

(c) is similar to (b). □

Next we need an error estimate for Gaussian quadrature of functions whose derivatives (except at $y$) eventually obey the sign patterns of derivatives of absolutely monotone or completely monotone functions.

LEMMA 7.2. *Assume $d\alpha(x)$ is bounded above and below near $y$. Assume $\psi \in C[a, b]$ is infinitely differentiable in $(a, b)\setminus\{y\}$ and that there exist positive integers $p$, $q$ and $N$ such that $\psi \in C^{N-1}[a, b]$ and such that, for $j \geq N$,*

$$(-1)^p \psi^{(j)}(x) \geq 0 \quad \textit{for all } x \in (a, y),$$

$$(-1)^{q+j} \psi^{(j)}(x) \geq 0 \quad \textit{for all } x \in (y, b).$$

*Then $E_n[\psi] = O(n^{-\mu})$ where $\mu = \max\{1, N - 1\}$.*

*In particular, we may choose* $\psi(x) = (x-y)^N|x-y|^{-\delta}$ $(0 < \delta < 1)$ *or* $\psi(x) = -(x-y)^N \log|x-y|$ *for all positive integers* $N$.

*Proof.* Let

$$\chi(x) = \begin{cases} 1, & a \leqslant x \leqslant y, \\ 0, & y \leqslant x \leqslant b. \end{cases}$$

Let $P(x) = \sum_{j=0}^{N-1} b_j(x-a)^j/j!$ with $b_0, b_1, \ldots, b_{N-1}$ chosen so large that

$$\{(-1)^p \psi(x) + P(x)\}^{(j)} \geqslant 0, \qquad x \in (a, y), j = 0, 1, 2, \ldots, N-1.$$

Let $f_1(x) = \chi(x)\{(-1)^p\psi(x) + P(x)\}$, $x \in (a, b)$. We see both $\chi P(x)$ and $f_1(x)$ are absolutely monotone in $(a, y)$ and (trivially) completely monotone in $(y, b)$. Then by Lemma 4.1(i),

$$\begin{aligned} |E_n^{**}[\chi\psi]| &= |E_n^{**}[(f_1 - \chi P)(-1)^p]| \leqslant E_n^{**}[f_1] + E_n^{**}[\chi P] \\ &\leqslant \int_{x_{l(n)-1}}^{y} f_1(x)\, d\alpha(x) + \int_{x_{l(n)-1}}^{y} P(x)\, d\alpha(x) \\ &\leqslant (\|\psi\| + 2\|P\|) M(y - x_{l(n)-1}) \end{aligned}$$

(by (2.2) and where the norms are over $[a, b]$)

$$\leqslant 2Mc_2(\|\psi\| + 2\|P\|)/n = O(n^{-1}),$$

by Lemma 3.4(i), (iii). Similarly $|E_n^{**}[(1-\chi)\psi]| = O(n^{-1})$ and hence $E_n^{**}[\psi] = O(n^{-1})$. Finally

$$E_n[\psi] = E_n^{**}[\psi] - \sum_{j=l(n)}^{r(n)} \lambda_{nj} f(x_{nj}) = O(n^{-1}) - O(n^{-1})\|\psi\| = O(n^{-1}),$$

by Lemma 3.4(ii).

Next, standard estimation [2, p. 257] yields

$$|E_n[\psi]| \leqslant \left\{2\int_a^b d\alpha(x)\right\} \min_{\deg(P) \leqslant n} \|\psi - P\| = o(n^{-N+1}),$$

by Jackson's Theorem (Rivlin [16, Theorem 1.5]) since $\psi \in C^{N-1}[a, b]$.

If, for example, $\psi(x) = (x-y)^N|x-y|^{-\delta}$, we see $\psi^{(j)}(x) > 0$, $x \in (a, y)$, $j = N, N+1, N+2, \ldots, (-1)^{N+j}\psi^{(j)}(x) > 0$, $x \in (y, b)$, $j = N, N+1, N+2, \ldots$ □

Next we need a lemma on the Lipschitz class of the functions $h_1$ and $h_2$ given by (7.1).

LEMMA 7.3. *Let* $g \in C[a, b]$ *and* $\phi(x) = |x-y|^{-\delta}$, $x \in [a, b] \setminus \{y\}$, *where* $0 < \delta < 1$.

(i) *Let* $g \in \mathrm{Lip}(1-\delta)$ *in* $[a, b]$ *and* $g \in \mathrm{Lip}(1)$ *near* $y$. *Let* $h_1$ *be given by* (7.1). *Then* $h_1 \in \mathrm{Lip}(1-\delta)$ *in* $[a, b]$.

(ii) *Let* $0 < \varepsilon < \delta$ *and let* $g \in \mathrm{Lip}(1-\varepsilon)$ *in* $[a, b]$. *Further let* $g'$ *exist near* $y$ *and* $g' \in \mathrm{Lip}(\delta-\varepsilon)$ *near* $y$. *Let* $h_2$ *be given by* (7.1). *Then* $h_2 \in \mathrm{Lip}(1-\varepsilon)$ *in* $[a, b]$.

*Proof.* We first prove (ii). By hypothesis, there exist positive $N$ and $\eta$ such that

$$|g(u) - g(v)| \leqslant N|u-v|^{1-\varepsilon}, \qquad a \leqslant u, v \leqslant b, \tag{7.3}$$

$$|g'(u) - g'(v)| \leqslant N|u-v|^{\delta-\varepsilon}, \qquad y - \eta \leqslant u, v \leqslant y + \eta. \tag{7.4}$$

Recall $h_2(x) = \phi(x)[g(x) - g(y) - g'(y)(x - y)]$. We shall assume $a \leqslant u \leqslant v \leqslant y$ and consider three cases:

*Case* I: $a \leqslant u < v \leqslant y - \eta$. Now $\phi(u) = \phi(v) + \phi'(\omega)(u - v)$, where $\omega$ lies between $u$ and $v$, so

$$\begin{aligned}
(7.5)\quad & |h_2(u) - h_2(v)| \\
&= |\phi(v)[g(u) - g(v) - g'(y)(u - v)] \\
&\qquad + \phi'(\omega)(u - v)[g(u) - g(y) - g'(y)(u - y)]| \\
&\leqslant |v - y|^{-\delta}[N|u - v|^{1-\varepsilon} + |g'(y)|\,|u - v|] \\
&\quad + \delta|v - y|^{-\delta-1}|v - u|[2\|g\| + |g'(y)|(b - a)]
\end{aligned}$$

(by (7.3))

$$\begin{aligned}
&\leqslant |u - v|^{1-\varepsilon}\{\eta^{-\delta}[N + |g'(y)|(b - a)^{\varepsilon}] \\
&\qquad + \delta\eta^{-\delta-1}(b - a)^{\varepsilon}[2\|g\| + |g'(y)|(b - a)]\} \\
&= K|u - v|^{1-\varepsilon}.
\end{aligned}$$

*Case* II: $y - \eta \leqslant u < v \leqslant y$ and $y - v > v - u$. By (7.5) and differentiability of $g$ in $[y - \eta, y]$,

$$\begin{aligned}
|h_2(u) - h_2(v)| &= |\phi(v)[(g'(\omega_1) - g'(y))(u - v)] \\
&\qquad + \phi'(\omega)(u - v)[(g'(\omega_2) - g'(y))(u - y)]|
\end{aligned}$$

(where $\omega_1$ lies between $u$ and $v$ and $\omega_2$ lies between $u$ and $y$)

$$\begin{aligned}
&\leqslant |v - y|^{-\delta}N|u - y|^{\delta-\varepsilon}|u - v| + \delta|v - y|^{-\delta-1}|u - v|N|u - y|^{\delta-\varepsilon}|u - y| \\
&\leqslant N|u - v|\{2^{\delta-\varepsilon}|v - y|^{-\varepsilon} + \delta 2^{1+\delta-\varepsilon}|v - y|^{-\varepsilon}\}
\end{aligned}$$

(as $|u - y| \leqslant (y - v) + (v - u) < 2(y - v)$)

$$\leqslant 6N|u - v|^{1-\varepsilon}$$

(as $|v - y| > |u - v|$).

*Case* III: $y - \eta \leqslant u < v \leqslant y$ and $y - v \leqslant v - u$. For some $\omega$ between $u$ and $y$,

$$|h_2(u)| = |\phi(u)(g'(\omega) - g'(y))(u - y)| \leqslant N|u - y|^{1-\varepsilon}$$

(by (7.4))

$$\leqslant 2N|u - v|^{1-\varepsilon}$$

(as $|y - u| \leqslant |y - v| + |v - u| \leqslant 2|v - u|$). Similarly $|h_2(v)| \leqslant N|u - v|^{1-\varepsilon}$, and so

$$|h_2(u) - h_2(v)| \leqslant 3N|u - v|^{1-\varepsilon}.$$

From Case I, it follows that $h_2 \in \operatorname{Lip}(1 - \varepsilon)$ in $[a, y - \eta]$ and from Cases II, III, it follows that $h_2 \in \operatorname{Lip}(1 - \varepsilon)$ in $[y - \eta, y]$. Thus $h_2 \in \operatorname{Lip}(1 - \varepsilon)$ in $[a, y]$, and similarly $h_2 \in \operatorname{Lip}(1 - \varepsilon)$ in $[y, b]$ and so in $[a, b]$. This completes the proof of (ii).

The proof of (i) is very similar, but easier: one again considers Cases I, II, III as above and uses

$$\begin{aligned}
|g(u) - g(v)| &\leqslant N|u - v|^{1-\delta}, \qquad a \leqslant u, v \leqslant b, \\
|g(u) - g(v)| &\leqslant N|u - v|, \qquad y - \eta \leqslant u, v \leqslant y + \eta. \quad \square
\end{aligned}$$

Roughly the above lemma states that if $g$ has "smoothness" $r$ in $[a, b]$ and "smoothness" $r + \delta$ near $y$, then $h_1$ or $h_2$ has "smoothness" $r$ in $[a, b]$. Similarly, for $\phi(x) = -\log|x - y|$, one can prove

LEMMA 7.4. *Let $g \in C[a, b]$ and $\phi(x) = -\log|x - y|$, $x \in [a, b]\setminus\{y\}$.*

(i) *Let $g \in \mathrm{Lip}(1; -1)$ in $[a, b]$ and $g \in \mathrm{Lip}(1)$ near $y$. Let $h_1$ be given by* (7.1). *Then $h_1 \in \mathrm{Lip}(1; -1)$ in $[a, b]$.*

(ii) *Let $g \in \mathrm{Lip}(1; -1 + \eta)$ in $[a, b]$ for some $0 < \eta < 1$. Further let $g'$ exist near $y$, and $g' \in \mathrm{Lip}(0; \eta)$ near $y$. Let $h_2$ be given by* (7.1). *Then $h_2 \in \mathrm{Lip}(1; -1 + \eta)$ in $[a, b]$.*

We can now prove our main result on avoiding the singularity.

THEOREM 7.5. *Assume $d\alpha(x)$ is bounded above and below near $y$. Assume $g \in C[a, b]$.*

(i) *Let $f(x) = |x - y|^{-\delta}g(x)$, $x \in [a, b]\setminus\{y\}$, where $0 < \delta < 1$.*

(a) *If $g \in \mathrm{Lip}(1 - \delta)$ in $[a, b]$ and $g \in \mathrm{Lip}(1)$ near $y$, then*

$$E_n^{**}[f] = O(n^{-1+\delta}), \qquad E_n^{*}[f] = O(n^{-1+\delta}).$$

(b) *If, further, there exists $0 < \varepsilon < \delta$ such that $g \in \mathrm{Lip}(1 - \varepsilon)$ in $[a, b]$ and $g' \in \mathrm{Lip}(\delta - \varepsilon)$ near $y$, and if $g(y) \neq 0$, then*

$$E_n^{**}[f] \sim g(y)n^{-1+\delta}.$$

*Further $E_n^{*}[f] \sim g(y)n^{-1+\delta}$ if $\delta$ is close enough to* 1.

(ii) *Let $f(x) = (-\log|x - y|)g(x)$, $x \in [a, b]\setminus\{y\}$.*

(a) *If $g \in \mathrm{Lip}(1; -1)$ in $[a, b]$ and $g \in \mathrm{Lip}(1)$ near $y$, then*

$$E_n^{**}[f] = O(n^{-1}\log n), \qquad E_n^{*}[f] = O(n^{-1}\log n).$$

(b) *If, further, there exists $0 < \eta < 1$ such that $g \in \mathrm{Lip}(1; -1 + \eta)$ in $[a, b]$ and $g' \in \mathrm{Lip}(0; \eta)$ near $y$, and if $g(y) \neq 0$, then*

$$E_n^{**}[f] \sim g(y)n^{-1}\log n.$$

*Proof.* (i) Let $\phi(x) = |x - y|^{-\delta}$, $x \in [a, b]\setminus\{y\}$.

(a) By Lemma 7.1(c) with $k = 0$,

$$E_n^{**}[f] = E_n^{**}[\phi]g(y) + E_n[h_1] + \sum_{j=l(n)}^{r(n)} \lambda_{nj}h_1(x_{nj}).$$

Here $E_n^{**}[\phi] \sim n^{-1+\delta}$ by Corollary 4.4(i). Further $h_1 \in C[a, b]$ and $\lambda_{l(n)}, \lambda_{r(n)} = O(n^{-1})$ by Lemma 3.4(ii). Finally, as $h_1 \in \mathrm{Lip}(1 - \delta)$ in $[a, b]$, by Lemma 7.3(i), and by Jackson's Theorem [16, Theorem 1.5, p. 23],

$$|E_n[h_1]| \leqslant \left(2\int_a^b d\alpha(x)\right) \min_{\deg(P) \leqslant 2n-2} \|h_1 - P\| = O(n^{-1+\delta}).$$

Thus $E_n^{**}[f] = O(n^{-1+\delta})$. Similarly Lemma 7.1(b) may be used to show $E_n^{*}[f] = O(n^{-1+\delta})$.

(b) By Lemma 7.1(c) with $k = 1$,

$$E_n^{**}[f] = E_n^{**}[\phi]g(y) + g'(y)E_n[(x - y)\phi] + E_n[h_2] + \sum_{j=l(n)}^{r(n)} \lambda_{nj}h_1(x_{nj}). \tag{7.6}$$

By Lemma 7.2 with $\psi(x) = (x - y)\phi(x) = (x - y)|x - y|^{-\delta}$, one sees $E_n[(x - y)\phi] = O(n^{-1})$. Further, by Lemma 7.3(ii), $h_2 \in \text{Lip}(1 - \varepsilon)$ in $[a, b]$, and as usual this implies $E_n[h_2] = O(n^{-1+\varepsilon}) = o(n^{-1+\delta})$. Finally, by Lemma 3.4(ii), we see

$$\sum_{j=l(n)}^{r(n)} \lambda_{nj} h_1(x_{nj}) = O(n^{-1}).$$

Thus all terms in the right member of (7.6), other than the first, are $o(n^{-1+\delta})$. As $E_n^{**}[\phi] \sim n^{-1+\delta}$, the result follows. Similarly for $E_n^{*}[f]$.

(ii)(a), (b) are similar to (i)(a), (b), respectively. □

If, for example, $g \in C^1[a, b]$ and $g' \in \text{Lip}(\eta)$ in $[a, b]$ for some $\eta > 0$, then all the restrictions of Theorem 7.5(i)(b) or (ii)(b) on $g$ are satisfied. Thus, under fairly weak assumptions on the distribution $d\alpha$ and on the function $g$, $E_n^{**}[f] \sim n^{-1+\delta}$. The conditions on $g$ in Theorem 7.5(i)(b) and (ii)(b) can be weakened without weakening the result, but the formulation becomes more complicated and is omitted.

The following result analyzes the error when the singularity is ignored.

THEOREM 7.6. (i) *Assume $d\alpha(x)$ is bounded above and below near each $y$ interior to $[a, b]$. Then, given $\varepsilon > 1$, there is a set $\mathscr{E}_\varepsilon$ in $(a, b)$ of linear Lebesgue measure zero with the following property: If $g \in \text{Lip}(1)$ in $[a, b]$, then*

$$E_n\left[|x - y|^{-\delta} g\right] = O\left(n^{-1+2\delta}(\log n)^{\delta}(\log\log n)^{\varepsilon\delta}\right)$$

*for all $0 < \delta < 1$ whenever $y \notin \mathscr{E}_\varepsilon$.*

*Hence if $\delta < 1/2$, $E_n[|x - y|^{-\delta} g] \to 0$ as $n \to \infty$ for almost all $y \in (a, b)$.*

(ii) *Assume $(a, b) = (-1, 1)$ and $d\alpha(x)$ is a Jacobi weight given by $\alpha'(x) = (1 - x)^{\nu}(1 + x)^{\beta}$, $x \in (-1, 1)$, where $\beta, \nu = \pm 1/2$. Then there is a set $\mathscr{E}$ in $(-1, 1)$ of linear Lebesgue measure zero with the following property: If $g \in \text{Lip}(1)$ in $[a, b]$, then*

$$|E_n\left[|x - y|^{-\delta} g\right]| \geqslant c|g(y)| n^{-1+2\delta}(\log n)^{\delta}(\log\log n)^{\delta}$$

*for infinitely many integers $n$ and all $0 < \delta < 1$ whenever $y \notin \mathscr{E}$. Here $c$ is a positive constant independent of $g$, $n$, $y$ and $\delta$.*

*Thus, provided the set of zeros of $g$ has linear Lebesgue measure zero, and if $\delta \geqslant 1/2$, $E_n[|x - y|^{-\delta} g] \nrightarrow 0$ as $n \to \infty$ for almost all $y \in [a, b]$.*

*Proof.* By Lemma 7.1(a), with $k = 0$,

$$E_n\left[|x - y|^{-\delta} g\right] = g(y) E_n\left[|x - y|^{-\delta}\right] + E_n[h_1],$$

where $h_1$ is given by (7.1) and $\phi(x) = |x - y|^{-\delta}$. Using Lemma 7.3(i), we see $h_1 \in \text{Lip}(1 - \delta)$ and hence $E_n[h_1] = O(n^{-1+\delta})$ for all $y \in (a, b)$. The statements (i), (ii) then follow from Theorem 5.1(i) (ii). □

In a similar fashion, one can use Theorem 5.2 to prove the following result for ignoring a logarithmic singularity:

THEOREM 7.7. *Assume $d\alpha(x)$ is bounded above and below near each $y$ interior to $(a, b)$. Then there is a set $\mathscr{E}$ of linear Lebesgue measure zero (even further of Hausdorff dimension zero) with the following property: If $g \in \text{Lip}(1)$ in $[a, b]$, then*

$$E_n\left[(-\log|x - y|) g\right] = O\left(n^{-1}\log n\right) \quad \text{whenever } y \notin \mathscr{E}.$$

**8. Endpoint Singularities for More General Functions.** In extending the results of Section 6 to more general functions, we shall assume throughout that $(a, b) = (-1, 1)$ and that $\alpha(x)$ is absolutely continuous there. Further, we shall assume that $d\alpha$ is comparable to a Jacobi weight, that is, there exist positive $m$, $M$ and real $\nu$, $\beta > -1$ such that

$$m \leqslant \alpha'(x)/\{(1-x)^{\nu}(1+x)^{\beta}\} \leqslant M, \qquad x \in (-1, 1). \tag{8.1}$$

LEMMA 8.1. *Let* $\psi \in C[-1,1]$ *be infinitely differentiable in* $[-1,1]$, *and assume there exist positive integers p and N such that*

$$(-1)^{p}\psi^{(j)}(x) \geqslant 0, \qquad x \in [-1,1), j = N, N+1, N+2, \ldots.$$

*Then*

$$E_n[\psi] = O(n^{-2(\nu+1)}).$$

*In particular, we can choose* $\psi(x) = (1-x)^{N-\delta}$ *or* $\psi(x) = -(1-x)^{N}\log(1-x)$.

*Proof.* By choosing a suitable polynomial $P(x)$ of degree at most $N-1$, we can ensure that $f(x) = (-1)^{p}\psi(x) + P(x)$ is absolutely monotone in $[-1,1)$. Then, by Lemma 6.1(a), by (8.1), and as $E_n[P] = 0$ for large $n$, we see

$$|E_n[\psi]| = E_n[f] \leqslant M2^{|\beta|}\|f\|\int_{x_{nn}}^{1}(1-x)^{r}\,dx$$

$$= O\left((1-x_{nn})^{\nu+1}\right) = O(n^{-2(\nu+1)}),$$

by (6.6).

Finally if, for example, $\psi(x) = (1-x)^{N-\delta}$, we see

$$(-1)^{N}\psi^{(N+j)}(x) \geqslant 0, \qquad x \in [-1,1), j = 0, 1, 2, \ldots. \quad \square$$

The above lemma is by no means best possible for integrands of low continuity. For example, for the Legendre weight, Chawla and Jain [1, Eq. (18), p. 95] proved $E_n[(1-x)^{-\delta}] = O(n^{-4+2\delta})$, whereas the above result gives only $E_n[(1-x)^{-\delta}] = O(n^{-2})$. We can now prove our main result for endpoint singularities.

THEOREM 8.2. (i) *Let* $0 < \delta < \min\{1, 1+\nu\}$, *and let l be the smallest integer* $\geqslant 2(1+\nu-\delta)$. *Let* $g \in C^{l}[-1,1]$ *and assume there exists* $\eta > 0$ *such that* $g^{(l)}(x) \in$ Lip$(\delta; \eta)$ *near* 1. *Then*

$$E_n\left[(1-x)^{-\delta}g\right] = O(n^{-2(1+\nu-\delta)}).$$

(ii) *Let k be the smallest integer* $\geqslant 2(1+\nu)$. *Let* $g \in C^{k}[-1,1]$ *and assume there exists* $\eta > 1$ *such that* $g^{(k)}(x) \in$ Lip$(0; \eta)$ *near* 1. *Then*

$$E_n[(\log(1-x))g] = O(n^{-2(1+\nu)}\log n).$$

*Proof.* (i) Let

$$G(x) = g(x) - \sum_{j=0}^{l}\frac{g^{(j)}(1)}{j!}(x-1)^{j}, \qquad x \in [-1, 1).$$

We see $G \in C^{l}[-1,1]$. Further,

$$G(x) = \left\{g(x) - \sum_{j=0}^{l-1}\frac{g^{(j)}(1)}{j!}(x-1)^{j}\right\} - \frac{g^{(l)}(1)}{l!}(x-1)^{l}$$

$$= \{g^{(l)}(u) - g^{(l)}(1)\}(x-1)^{l}/l!,$$

where $u$ lies between $x$ and 1. Then we deduce that for $x$ close to 1, and for some positive constant $K$,

$$|\phi(x)G(x)| \leqslant K|x-1|^{-\delta}|x-1|^{l+\delta}|\log|x-1||^{-\eta}/l!$$
$$= O\left(|x-1|^{l}|\log|x-1||^{-\eta}\right).$$

It follows that $(\phi G)(x)$ has a zero of order $l$ at $x = 1$ and further that

$$|(\phi G)^{(l)}(x)| = O(|\log|x-1||^{-\eta}) \to 0 \quad \text{as } x \to 1-.$$

Hence also $(\phi G)^{(l)}(1) = 0$ and $\phi G \in C^{l}[-1,1]$. As usual, Jackson's Theorem yields

$$E_n[\phi G] = o(n^{-l}) = o(n^{-2(1+\nu-\delta)}).$$

Finally, using the definition of $G$, we see

$$E_n\left[(1-x)^{-\delta}g\right] = \sum_{j=0}^{l} \frac{g^{(j)}(1)}{j!} E_n\left[(1-x)^{j-\delta}\right] + E_n[\phi G]$$
$$= O(n^{-2(\nu+1-\delta)}) + O(n^{-2(\nu+1)}) + O(n^{-2(1+\nu-\delta)}),$$

by Theorem 6.3(a) and Lemma 8.1.

(ii) is similar. □

If, for example, $d\alpha(x)$ is the Legendre weight $d\alpha(x) \equiv dx$ in $[-1,1]$ and $\delta = 1/2$, the above result shows $E_n[(1-x)^{-1/2}g] = O(n^{-1})$ provided $g \in C^1[-1,1]$ and $g' \in \mathrm{Lip}(1/2;\, \eta)$ near $x = 1$. It seems certain that the restrictions on $g$ above can be substantially weakened.

Similarly one can discuss singularities at the left endpoint of the interval of integration. The methods of Sections 6 and 8 may also be applied to integrands with a singularity at $\infty$ and for Laguerre or Hermite weights.

**9. Conclusion.** In this paper, upper and lower bounds for the error in Gaussian integration were obtained, using a generalized Markov-Stieltjes inequality. These estimates lead to asymptotic results for the error in Gaussian integration whether the singularity is ignored or avoided. They also suggest derivative-free correction terms for numerical integration of singular integrands of certain types. This idea, which is not investigated here, could improve existing methods for evaluating singular integrals.

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