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Finite Element Methods of Optimal Order for Problems with Singular Data

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Abstract. An adapted finite element method is proposed for a class of elliptic problems with singular data. The idea is to subtract the main singularity from the solution and to solve for the remainder using suitable mesh-refinements. Optimal order error estimates are proved.

1. Introduction and Results. Let Ω be a bounded domain in R^N with smooth boundary Γ and consider the following problem: Given $x_0 \in \Omega$ find $u = u(x)$ such that

$$(1.1a)\qquad Lu(x) \equiv -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial u}{\partial x_i}\right) + \sum_{i=1}^{N} a_i(x)\frac{\partial u}{\partial x_i} + a(x)u = \delta(x - x_0) \quad \text{in } \Omega,$$

$$(1.1b)\qquad lu(x) \equiv \sum_{i,j=1}^{N} a_{ij}(x)\frac{\partial u}{\partial x_i} n_j(x) = 0 \quad \text{on } \Gamma,$$

where δ is the Dirac distribution (unit impulse), $n = (n_j)$ is the outward unit normal to Γ, and a_{ij}, a_i, and a are smooth (C^∞ regular) functions on $\overline{\Omega}$, with $a_{ij} = a_{ji}$, and such that the associated bilinear form

$$A(v, w) \equiv \int_\Omega \left(\sum_{i,j} a_{ij}\frac{\partial v}{\partial x_i}\frac{\partial w}{\partial x_j} + \sum_i a_i \frac{\partial v}{\partial x_i} w + avw \right) dx$$

satisfies the ellipticity-coercivity condition

$$(1.2)\qquad A(v, v) \geqslant c\|v\|_{1,\Omega}^2 \quad \text{for all } v \in H^1(\Omega),$$

where c is a positive constant and $\|\cdot\|_{1,\Omega}$ the usual norm in $H^1(\Omega)$, the space of functions with square-integrable first-order derivatives in Ω. It is well known (cf., e.g., [8]) that problem (1.1) admits a unique (distributional) solution u, which is also determined by the corresponding variational equations

$$(1.3)\qquad A(u, \psi) = \psi(x_0) \quad \text{for all } \psi \in W_\infty^1(\Omega),$$

where $W_\infty^1(\Omega)$ is the space of functions with bounded first-order derivatives on $\overline{\Omega}$.

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In general, problem (1.1) (or (1.3)) cannot be solved exactly, so we are faced with the problem of finding an accurate approximate solution. We shall consider the following standard approach for this (cf., e.g., [1], [7], [11]): Given a finite-dimensional space $S_h \subset H^1(\Omega) \cap C(\overline{\Omega})$ find $u_h \in S_h$ such that

$$(1.4) \qquad A(u_h, \chi) = \chi(x_0) \quad \text{for all } \chi \in S_h.$$

By the coercivity of $A(\cdot, \cdot)$, there exists a unique such u_h determined by the linear system of equations

$$\sum_{i=1}^{M} U_i A(\chi_i, \chi_j) = \chi_j(x_0) \quad \text{for } j = 1, \dots, M,$$

where $u_h = \sum_{i=1}^{M} U_i \chi_i$, and $\{\chi_j\}_{j=1}^{M}$ is an arbitrary basis for S_h. For appropriate finite element spaces S_h, where h is the associated mesh-size parameter, there exist a priori estimates for the error $u_h - u$ in terms of h. One problem in deriving these estimates is that the singularity of u at x_0, which is of order $\log|x - x_0|^{-1}$ for $N = 2$, and $|x - x_0|^{-N+2}$ for $N \neq 2$, frustrates the usual type of error analysis. For $N = 1$ this is a minor problem since then the singularity is concentrated at x_0 and u is continuous. Choosing x_0 as one of the nodal points and using continuous piecewise polynomials for S_h, the usual analysis carries through, and, for example, for polynomials of degree $r - 1$ we have $\|u_h - u\|_{1,\Omega} \leqslant Ch^{r-1}$. For $N > 1$, however, the standard method of analysis fails since then the solution does not even belong to $H^1(\Omega)$. Nevertheless, Babuška [1] was able to show, for $N = 2$, $L = -\Delta + I$ (minus Laplacian plus identity), and finite element spaces S_h possessing standard approximation and inverse properties (such as piecewise linears on a quasi-uniform triangulation of Ω), that

$$\|u_h - u\|_{0,\Omega} \leqslant C_\varepsilon h^{1-\varepsilon},$$

where $\|\cdot\|_{0,\Omega}$ is the $L_2(\Omega)$-norm and $\varepsilon > 0$ is arbitrary. Later, Scott [11] improved and generalized this result by showing that for elliptic operators of order $2m$, normal covering boundary conditions (cf., e.g., [9]), and dimension $N \geqslant 2$, one has

$$\|u_h - u\|_{s,\Omega} \leqslant C(x_0) h^{2m-s-N/2} \quad \text{for } 2m - r \leqslant s < 2m - N/2,$$

where $C(x_0)$ tends to infinity as x_0 approaches Γ, $r \geqslant 2m$ is the order of approximation of $S_h \subset H^s(\Omega)$, and the Sobolev norm index s may also be negative (cf., e.g., [9] for the definition). Despite its generality, Scott's result falls short in a certain respect. For instance, if $m = 1$, $r = 2$ (piecewise linears), and $N = 2$ or 3, we obtain no information about $\nabla(u_h - u)$, and for $N \geqslant 4$, no information whatsoever.

Recently, in [7], we proposed the use of adapted finite element spaces for the given problem. Denoted by $S_h(x_0, \alpha, r)$ where $h \in (0, \frac{1}{2}]$, $\alpha \in [0, 1)$, and $r \geqslant 2$ are parameters, these spaces can be described as follows: Given positive constants c and C let Ω be divided into elements τ such that

$$(1.5) \qquad c \operatorname{diam}(\tau) \leqslant h\big(\operatorname{dist}(x_0, \tau)\big)^\alpha + h^{1/(1-\alpha)} \leqslant C \operatorname{diam}(\tau) \quad \text{for all } \tau,$$

where each τ is the restriction to Ω of the interior of a N-simplex $\hat{\tau}$ (cf. [4]), with

$$(1.6) \qquad (\operatorname{diam}(\hat{\tau}))^N \leqslant C \int_\tau dx,$$

and the intersection of any two such simplices is either a face of both, or of lower dimension. For $S_h = S_h(x_0, \alpha, r)$ we take the space of all continuous functions on $\bar{\Omega}$ which reduces to polynomials of degree at most $r-1$ on each $\tau \subset \Omega$. By (1.5) the mesh is refined (graded) around x_0 in such a way that elements at distance d from x_0 have diameters of order hd^α, but a minimum diameter of order $h^{1/(1-\alpha)}$. Hence h is a parameter for the maximal global mesh-size, and α determines the degree of refinement. Of course, such a mesh-refinement enables a better approximation in S_h of any given function which is irregular near x_0. The condition (1.6) is used to derive local inverse estimates.

The following results were obtained in [7] for $u_h \in S_h(x_0, \alpha, r)$ being the solution of (1.4). For $\alpha > (r-2)/(r-1)$,

$$\|\nabla(u_h - u)\|_{L_1(\Omega)} \leqslant Ch^{r-1}, \tag{1.7}$$

and for $\alpha > (r-2)/r$,

$$\|u_h - u\|_{L_1(\Omega)} \leqslant Ch^r, \tag{1.8}$$

where C is a constant independent of h and x_0. Further, if $\alpha > (r-1)/r$, and if $d \equiv |x - x_0| \geqslant ch^{1/(1-\alpha)}$ and $\operatorname{dist}(x, \Gamma) \geqslant d$ for a suitable $c > 0$, then

$$|u_h(x) - u(x)| \leqslant Ch^r(\ln 1/h)^{\bar{r}} d^{-N}, \tag{1.9}$$

where $\bar{r} = 1$ if $r = 2$, $\bar{r} = 0$ if $r > 2$.

In this paper, we analyze a method to approximate the solution of (1.1) to the accuracy (1.7), (1.8), and (1.9) which requires a lesser degree of mesh-refinement than the one in [7]. One reason for introducing such a method is that the computational effects of strong mesh-refinements are not yet very well known. Recall that the condition number for the stiffness matrix $(A(\chi_i, \chi_j))$ depends on the mesh-size.

In order to describe the method we first note that the solution of (1.1) can be written in the form

$$u = u_0 + v, \tag{1.10}$$

where u_0 is the fundamental singularity of u defined by (in the sequel we only consider the case $N \geqslant 2$)

$$u_0(x) = \begin{cases} \dfrac{|\det(Q)|}{2\pi} \ln\left(|\hat{x} - \hat{x}_0|^{-1}\right) & \text{if } N = 2, \\[2ex] \dfrac{|\det(Q)|}{(N-2)\sigma_N} |\hat{x} - \hat{x}_0|^{-N+2} & \text{if } N > 2, \end{cases} \tag{1.11}$$

where Q is the inverse of $A^{1/2}$, the positive square root of $A \equiv (a_{ij}(x_0))$, $\hat{x} - \hat{x}_0 = Q(x - x_0)$, and σ_N is the surface area of the unit ball $B_1(0) \subset R^N$. For example, if $L = -\Delta + I$ we can take $Q = I$ and thus obtain

$$u_0(x) = \frac{1}{(N-2)\sigma_N} |x - x_0|^{-N+2} \qquad (N > 2),$$

which we recognize as the fundamental solution of $-\Delta$. It is a matter of straightforward calculation to verify that in the general case u_0 satisfies

$$-\sum_{i,j=1}^{N} a_{ij}(x_0) \frac{\partial^2 u_0}{\partial x_j \partial x_i} = \delta(x - x_0) \quad \text{in } \Omega. \tag{1.12}$$

In view of (1.10) we are led to seek an approximate solution of (1.1) in the form

$$\tilde{u}_h = u_0 + v_h, \quad \text{with } v_h \in S_h = S_h(x_0, \alpha, r), \tag{1.13}$$

and such that

$$A(\tilde{u}_h, \chi) = \chi(x_0) \quad \text{for all } \chi \in S_h. \tag{1.14}$$

Again, by the coercivity of $A(\cdot, \cdot)$, there is a unique such $\tilde{u}_h$. In fact, to seek $\tilde{u}_h$ is to seek $v_h = \sum_{i=1}^{M} V_i\chi_i \in S_h$ such that

$$\sum_{i=1}^{M} V_i A(\chi_i, \chi_j) = \chi_j(x_0) - A(u_0, \chi_j) \quad \text{for } j = 1, \ldots, M, \tag{1.15}$$

where $\{\chi_j\}_{j=1}^{M}$ is a basis for S_h.

We shall prove the following error estimates for this method:

THEOREM 1. *Let u be the solution of* (1.1) *and $\tilde{u}_h = u_0 + v_h$ that of* (1.14), *with $v_h \in S_h(x_0, \alpha, r)$. Then, for $\alpha > (r-3)/(r-1)$,*

$$\|\nabla \tilde{u}_h - \nabla u\|_{L_1(\Omega)} \leqslant Ch^{r-1}, \tag{1.16}$$

and for $\alpha > (r-3)/r$,

$$\|\tilde{u}_h - u\|_{L_1(\Omega)} \leqslant Ch^{r}, \tag{1.17}$$

where C may depend on the given problem as well as on α, r, and the constants in (1.5) *and* (1.6), *but **not** on h.*

Further, we have the following pointwise error estimate:

THEOREM 2. *Let u and $\tilde{u}_h$ be as in Theorem* 1 *with $\alpha > (r-3)/r$. Then*

$$|\tilde{u}_h(x) - u(x)| \leqslant Ch^r(\ln 1/h)^{\bar{r}}|x - x_0|^{-N} \quad \text{for } x \in \overline{\Omega},\ x \neq x_0,$$

where C is independent of x and h, and $\bar{r} = 1$ if $r = 2$, $\bar{r} = 0$ if $r > 2$.

Remark. The constants C in Theorems 1 and 2 become infinite as x_0 approaches Γ. In order to have a method which is effective also when x_0 is close to (or even on) the boundary Γ one can modify the definition of u_0 according to

$$u_0(x) = \begin{cases} \dfrac{|\det(Q)|}{2\pi}\left(\ln\left(|\hat{x} - \hat{x}_0|^{-1}\right) + \ln\left(|\hat{x} - \hat{x}_0^*|^{-1}\right)\right) & \text{if } N = 2, \\ \dfrac{|\det(Q)|}{(N-2)\sigma_N}\left(|\hat{x} - \hat{x}_0|^{-N+2} + |\hat{x} - \hat{x}_0^*|^{-N+2}\right) & \text{if } N > 2, \end{cases}$$

where $\hat{x}_0^*$ is the "Q-reflexion" of $\hat{x}_0$ in Γ defined by $\hat{x}_0^* = Qx_0^*$, $x_0^* = 2z - x_0$, and z minimizes $|Q(y - x_0)|$, $y \in \Gamma$. For this modification of the method the estimates of Theorems 1 and 2 hold with constants C independent of x_0.

The proofs of Theorems 1 and 2 are given in Sections 4 and 5, respectively. Sections 2 and 3 are devoted to preparatory work.

2. Preliminaries. Throughout this paper we shall denote by c and C various positive constants which are independent of h (but which may depend on the data of the given problem (1.1), the constants in (1.5) and (1.6), and on the parameters α and r). Similarly, C_1 and C_* will denote two specific such constants.

Besides the usual L_p-norms

$$\|v\|_{L_p(\Omega')} = \left(\int_{\Omega'} |v(x)|^p \, dx \right)^{1/p} \quad \text{for } p = 1 \text{ and } p = 2,$$

and

$$\|v\|_{L_\infty(\Omega')} = \operatorname*{ess\,sup}_{x \in \Omega'} |v(x)|,$$

we shall use the Sobolev norms

$$\|v\|_{k,\Omega'} = \left(\sum_{|\beta| \leqslant k} \|D^\beta v\|^2_{L_2(\Omega')} \right)^{1/2},$$

where $|\beta| = \beta_1 + \cdots + \beta_N$ is the length of the multi-index $\beta = (\beta_1, \ldots, \beta_N)$, and

$$D^\beta = \left(\frac{\partial}{\partial x_1} \right)^{\beta_1} \cdots \left(\frac{\partial}{\partial x_N} \right)^{\beta_N}.$$

In particular, $\|\cdot\|_{0,\Omega'}$ denotes the usual $L_2(\Omega')$-norm. The Sobolev space $H^k(\Omega')$ is the space of all functions w such that $\|w\|_{k,\Omega'}$ is finite.

In the proofs we shall consider subdomains of Ω defined by

$$D_j \equiv \left\{ x \in \Omega \colon 2^{-(j+1)} < |x - x_0| < 2^{-j} \right\},$$

and

$$\Omega_j \equiv \left\{ x \in \Omega \colon |x - x_0| < 2^{-j} \right\} \quad \text{for } j \text{ integer},$$

and set $d_j \equiv 2^{-j}$ and $h_j \equiv h d_j^\alpha$. Accordingly, d_j is proportional to the diameter of D_j and Ω_j, and as long as j is not too large so that h_j is smaller than the minimal mesh-size $h^{1/(1-\alpha)}$, h_j is proportional to the maximal mesh-size on D_j and Ω_j. We shall frequently use the obvious facts that $d_j \leqslant C d_{j+1}$ and $h_j \leqslant C h_{j+1}$, and we also note that since Ω is bounded, there is an integer j_1 such that D_j is empty for $j < j_1$.

Due to the variable mesh-size a typical interpolant in $S_h(x_0, \alpha, r)$ approximates a given function with variable degree of accuracy over Ω. The following three results are quoted from [7].

LEMMA 1. *Given w there is an interpolant $w_I \in S_h(x_0, \alpha, r)$ of w such that for $j \leqslant J_1$,*

$$\|w - w_I\|_{1,\Omega_j} \leqslant C h_j \|w\|_{2,\Omega_{j-1}},$$

and

$$\|w - w_I\|_{1,D_j} \leqslant C h_j^{m-1} \|w\|_{m,D_j^1} \quad \text{for } 2 \leqslant m \leqslant r,$$

where J_1 is determined by $2^{-J_1} = C_1 h^{1/(1-\alpha)}$ for a suitable sufficiently large constant C_1, and $D_j^1 \equiv \Omega_{j-1} \setminus \overline{\Omega}_{j+2}$.

The next lemma shows a similar property for the Galerkin approximation.

LEMMA 2. *If $j \leqslant J_1$, and if $v_h \in S_h(x_0, \alpha, r)$ and v satisfy*

$$A(v_h - v, \chi) = 0 \quad \text{for all } \chi \in S_h(x_0, \alpha, r) \text{ with support in } \overline{D_j^1},$$

then

$$\|v_h - v\|_{1,D_j} \leqslant C h_j^{r-1} \|v\|_{r,D_j^1} + C d_j^{-1} \|v_h - v\|_{0,D_j^1}.$$

From (1.6) we have the following inverse property:

LEMMA 3. *For any τ as in* (1.6) *and any polynomial p of degree at most $r-1$, we have*

$$\|p\|_{L_\infty(\tau)} \leqslant C(\operatorname{diam}(\tau))^{-N/q}\|p\|_{L_q(\tau)} \quad \textit{for } q \in [1,\infty).$$

For the proofs of these results we refer to [7].

3. A Step of Reduction. Recall that by definition $u = u_0 + v$ and $\tilde{u}_h = u_0 + v_h$. Hence, we can estimate $\tilde{u}_h - u$ by estimating $v_h - v$. We shall do this by estimating, each individually, $v_h - \tilde{v}$ and $\tilde{v} - v$, where $\tilde{v}$ is an appropriate approximation of v. In this section we introduce such a $\tilde{v}$ and derive estimates for $\tilde{v} - v$. For the analysis of $v_h - \tilde{v}$ we also investigate the regularity of $\tilde{v}$.

Setting

$$\Phi(x) = \sum_{i,j=1}^{N} \frac{\partial}{\partial x_j}\left((a_{ij}(x) - a_{ij}(x_0))\frac{\partial u_0}{\partial x_i}\right) - \sum_{i=1}^{N} a_i(x)\frac{\partial u_0}{\partial x_i} - a(x)u_0 \quad \text{in } \Omega,$$

and

$$\phi(x) = -\sum_{i,j=1}^{N} a_{ij}(x)\frac{\partial u_0}{\partial x_i}n_j(x) \quad \text{on } \Gamma,$$

we see from (1.1) and (1.12) that

$$Lv = \Phi \quad \text{in } \Omega, \tag{3.1a}$$

$$lv = \phi \quad \text{on } \Gamma. \tag{3.1b}$$

We shall use the following facts:

LEMMA 4. *Let Φ, ϕ, and v be defined as above. Then ϕ is a smooth function* (*with degree of smoothness depending on* $\operatorname{dist}(x_0, \Gamma)$), *and the following estimates hold for Φ and v* (*for $|\beta| \leqslant r$, say*):

$$|D^\beta\Phi(x)| \leqslant C|x - x_0|^{-N+1-|\beta|}, \tag{3.2}$$

$$|D^\beta v(x)| \leqslant \begin{cases} C\left(\ln\left(|x - x_0|^{-1}\right) + 1\right) & \textit{if } -N+3-|\beta| = 0, \\ C\left(|x - x_0|^{-N+3-|\beta|} + 1\right) & \textit{if } -N+3-|\beta| \neq 0. \end{cases} \tag{3.3}$$

Proof. The smoothness of ϕ and the estimate (3.2) follows at once from the definitions. The estimate (3.3) can be obtained, implicitly, from [8]. For completeness, we show in an appendix that the result follows easily from the properties of the Green's function.

We now introduce $\tilde{v}$, requiring that $\tilde{v}$ be close to v, that $\tilde{v} \in H^2(\Omega)$, that $\tilde{v}$ possess at least r derivatives away from x_0, and that v_h (which is the Galerkin approximation of v) be also the Galerkin approximation of $\tilde{v}$. Therefore, let $\tilde{v}$ be the solution of

$$L\tilde{v} = \tilde{\Phi} \quad \text{in } \Omega, \tag{3.4a}$$

$$l\tilde{v} = \phi \quad \text{on } \Gamma, \tag{3.4b}$$

where $\tilde{\Phi}$ is defined as follows: Set $\varepsilon = h^{1/(1-\alpha)}$ and let Ω_h be the smallest mesh-domain covering $B_\varepsilon(x_0) \cap \Omega$, $B_\varepsilon(x_0) \equiv \{x: |x - x_0| < \varepsilon\}$. By a mesh-domain we mean (the interior of the closure of) a union of elements. Set

$$\tilde{\Phi}(x) = \begin{cases} \pi_\tau \Phi & \text{on } \Omega_h, \\ \Phi & \text{outside } \Omega_h, \end{cases} \tag{3.5}$$

where π_τ is the local L_2-projection onto $P_{r-1}(\tau)$, the space of polynomials of degree at most $r-1$ restricted to τ.

It is well known that problem (3.4) admits a (unique) solution, and the following estimates hold for $\tilde{\Phi}$ and $\tilde{v}$:

LEMMA 5. *Let $\tilde{\Phi}$ be defined by* (3.5) *and let $\tilde{v}$ be the solution of* (3.4). *Then* (*with* $\varepsilon = h^{1/(1-\alpha)}$)

$$\|\tilde{\Phi}\|_{L_2(\Omega)} \leqslant C\varepsilon^{-N/2+1}(\ln 1/\varepsilon)^{\bar{N}/2}, \tag{3.6}$$

$$\|\tilde{\Phi}\|_{L_1(\Omega_h)} \leqslant C\varepsilon, \tag{3.7}$$

$$\|\tilde{v}\|_{2,\Omega} \leqslant C\varepsilon^{-N/2+1}(\ln 1/\varepsilon)^{\bar{N}/2}, \tag{3.8}$$

where $\bar{N}=1$ if $N=2$, $\bar{N}=0$ if $N>2$, and for $j \leqslant J_1$ and $r \geqslant 2$ (*cf. Lemma* 1),

$$\|\tilde{v}\|_{r,D_j} \leqslant Cd_j^{-N/2+3-r}. \tag{3.9}$$

Proof. We have first

$$\|\tilde{\Phi}\|_{L_2(\Omega)} \leqslant \|\Phi\|_{L_2(\Omega\setminus\Omega_h)} + \sum_{\tau\subset\Omega_h} \|\tilde{\Phi}\|_{L_2(\tau)},$$

and by (3.2), with $R = |x-x_0|$,

$$\|\Phi\|^2_{L_2(\Omega\setminus\Omega_h)} \leqslant C\int_\varepsilon^{\operatorname{diam}(\Omega)} (R^{-N+1})^2 R^{N-1}\,dR \leqslant C\varepsilon^{-N+2}\left(\ln\frac{1}{\varepsilon}\right)^{\bar{N}}.$$

Using Lemma 3, we obtain

$$\begin{aligned}\|\tilde{\Phi}\|^2_{L_2(\tau)} &= \int_\tau \tilde{\Phi}\tilde{\Phi}\,dx = \int_\tau \Phi\tilde{\Phi}\,dx \leqslant \|\Phi\|_{L_1(\tau)}\|\tilde{\Phi}\|_{L_\infty(\tau)} \\ &\leqslant C\|\Phi\|_{L_1(\tau)}\varepsilon^{-N/2}\|\tilde{\Phi}\|_{L_2(\tau)},\end{aligned}$$

and hence, using (3.2),

$$\sum_{\tau\subset\Omega_h} \|\tilde{\Phi}\|_{L_2(\tau)} \leqslant C\varepsilon^{-N/2}\|\Phi\|_{L_1(\Omega_h)} \leqslant C\varepsilon^{-N/2+1}.$$

Together our estimates now prove (3.6).

The estimate (3.7) follows at once from the proof of (3.6), and (3.8) follows from (3.6) by the standard H^2-regularity estimate, since ϕ is smooth (cf., e.g., [9]).

For the proof of (3.9) we note that such an estimate holds for v, because of (3.3). Hence,

$$\|\tilde{v}\|_{r,D_j} \leqslant \|\tilde{v}-v\|_{r,D_j} + Cd_j^{-N/2+3-r}.$$

Further, we have

$$(\tilde{v}-v)(x) = \int_{\Omega_h} g(x,y)(\tilde{\Phi}(y)-\Phi(y))\,dy, \tag{3.10}$$

where g is the associated Green's function; i.e., $g(x, y)$ is the solution of

$$A(\psi, g(x, \cdot)) = \psi(x) \quad \text{for all } \psi \in W^1_\infty(\Omega).$$

It is known (cf., e.g., [8]) that such a g exists and that

$$|D^\beta_x D^\gamma_y g(x, y)| \leqslant \begin{cases} C \ln\left(|x - y|^{-1} + 1\right) & \text{if } -N + 2 - |\beta| - |\gamma| = 0, \\ C|x - y|^{-N+2-|\beta|-|\gamma|} & \text{if } -N + 2 - |\beta| - |\gamma| < 0. \end{cases} \tag{3.11}$$

Hence, for $x \in D_j$ and $|\beta| \leqslant r$,

$$|D^\beta(\tilde{v} - v)(x)| \leqslant \sup_{y \in \Omega_h} |D^\beta_x g(x, y)| \left(\|\tilde{\Phi}\|_{L_1(\Omega_h)} + \|\Phi\|_{L_1(\Omega_h)} \right) \leqslant C d_j^{-N+2-r} \varepsilon,$$

where we have also used (3.2) and (3.7) in the last step. Together our estimates show (3.9) which completes the proof of Lemma 5.

We shall now see that $\tilde{v}$ is appropriately close to v.

Lemma 6. *Let $\tilde{v}$ be the solution of* (3.4) *and v that of* (3.1), *or, equivalently, set $v = u - u_0$. Then*

$$\|\nabla(\tilde{v} - v)\|_{L_1(\Omega)} \leqslant C\varepsilon^2, \tag{3.12}$$

and

$$\|\tilde{v} - v\|_{L_1(\Omega)} \leqslant C\varepsilon^3 \ln 1/\varepsilon. \tag{3.13}$$

Proof. Let $B_{C\varepsilon}(x_0)$ be the ball of smallest radius such that $\Omega_h \subset B_{C\varepsilon}(x_0)$ and set $B = B_{2C\varepsilon}(x_0) \cap \Omega$. We have at once that

$$\|\nabla^i(\tilde{v} - v)\|_{L_1(\Omega)} \leqslant \|\nabla^i(\tilde{v} - v)\|_{L_1(B)} + \|\nabla^i(\tilde{v} - v)\|_{L_1(\Omega \setminus B)} \quad \text{for } i = 0, 1,$$

and, by (3.11) and a change of order of integration,

$$\begin{aligned} \|\nabla^i(\tilde{v} - v)\|_{L_1(B)} &\leqslant \sup_{y \in \Omega_h} \|\nabla^i_x g(\cdot, y)\|_{L_1(B)} \|\tilde{\Phi} - \Phi\|_{L_1(\Omega_h)} \\ &\leqslant C\varepsilon^{2-i} (\ln 1/\varepsilon)^{\bar{N}(1-i)} \left(\|\tilde{\Phi}\|_{L_1(\Omega_h)} + \|\Phi\|_{L_1(\Omega_h)} \right) \\ &\leqslant C\varepsilon^{3-i} (\ln 1/\varepsilon)^{\bar{N}(1-i)} \quad \text{for } i = 0, 1. \end{aligned}$$

Replacing $g(x, \cdot)$ by its expansion

$$g(x, y) = g(x, x_0) + (y - x_0)\nabla_y g(x, x_0)^t + \tfrac{1}{2}(y - x_0)\nabla^2_y g(x, \eta)(y - x_0)^t,$$

where $\eta = \theta x_0 + (1 - \theta)y$, $0 \leqslant \theta \leqslant 1$, and t denotes transpose, and using the fact that $\tilde{\Phi} - \Phi$ is orthogonal on Ω_h to the linear part of the expansion, we see from (3.10) that

$$(\tilde{v} - v)(x) = \int_{\Omega_h} (y - x_0)\nabla^2_y g(x, \eta)(y - x_0)^t (\tilde{\Phi}(y) - \Phi(y))\, dy.$$

Hence, again by (3.11),

$$\begin{aligned} |\nabla^i(\tilde{v} - v)(x)| &\leqslant C\varepsilon^2 \|\nabla^i_x \nabla^2_y g(x, \cdot)\|_{L_\infty(\Omega_h)} \left(\|\tilde{\Phi}\|_{L_1(\Omega_h)} + \|\Phi\|_{L_1(\Omega_h)} \right) \\ &\leqslant C\varepsilon^3 |x - x_0|^{-N-i} \quad \text{for } x \in \Omega \setminus B,\ i = 0, 1. \end{aligned}$$

Integration over $\Omega \setminus B$ shows that

$$\|\nabla^i(\tilde{v} - v)\|_{L_1(\Omega\setminus B)} \leqslant C\varepsilon^{3-i}(\ln 1/\varepsilon)^{1-i} \quad \text{for } i = 0, 1.$$

This completes the proof of the lemma.

We close this section by noting that also the final claim on $\tilde{v}$ is satisfied, namely that its Galerkin approximation is v_h. For by (3.1), (3.4), and the definition of $\tilde{\Phi}$, we have

$$A(\tilde{v} - v, \chi) = (L(\tilde{v} - v), \chi) = (\tilde{\Phi} - \Phi, \chi) = 0 \quad \text{for all } \chi \in S_h,$$

and by (1.3) and (1.4),

$$A(v_h - v, \chi) = 0 \quad \text{for all } \chi \in S_h,$$

so that

$$A(v_h - \tilde{v}, \chi) = 0 \quad \text{for all } \chi \in S_h. \tag{3.14}$$

4. Proof of Theorem 1. In view of Lemma 6 and the fact that $\tilde{u}_h - u = v_h - v$, it is sufficient to show that for the appropriate α's

$$\|\nabla(v_h - \tilde{v})\|_{L_1(\Omega)} \leqslant Ch^{r-1}, \tag{4.1}$$

and

$$\|v_h - \tilde{v}\|_{L_1(\Omega)} \leqslant Ch^r, \tag{4.2}$$

respectively.

Given a positive constant C_* let J be determined by

$$C_* h^{1/(1-\alpha)} \leqslant d_J < 2C_* h^{1/(1-\alpha)}.$$

Thus, h_J, d_J, and $h^{1/(1-\alpha)}$ are of the same order; but, by choosing C_* sufficiently large, $h_J d_J^{-1}$ is suitably small, since

$$h_J d_J^{-1} = h d_J^{\alpha-1} \leqslant 1/C_*^{1-\alpha}. \tag{4.3}$$

The constant C_* will be determined later. For the moment we only require that the results of Lemma 1, Lemma 2, and (3.9) apply for $j \leqslant J$, i.e., that $C_* \geqslant C_1$.

In order to prove (4.1) we first use Schwarz's inequality to change from the L_1-norm to a weighted L_2-norm. Setting $e = v_h - \tilde{v}$ and

$$S = \sum_{j \leqslant J} d_j^{N/2}\|e\|_{1,D_j},$$

we have

$$\|\nabla e\|_{L_1(\Omega)} = \sum_{j\leqslant J} \|\nabla e\|_{L_1(D_j)} + \|\nabla e\|_{L_1(\Omega_{J+1})} \leqslant CS + d_J^{N/2}\|e\|_{1,\Omega_{J+1}}.$$

We shall show that for $\alpha > (r-3)/(r-1)$ and a suitable choice of C_*,

$$S \leqslant \tfrac{1}{2}S + Cd_J^{N/2}\|e\|_{1,\Omega_J} + Ch^{r-1}, \tag{4.4}$$

and

$$d_J^{N/2}\|e\|_{1,\Omega} \leqslant Ch^{r-1}. \tag{4.5}$$

Obviously, the desired result then follows.

By Lemma 2, we have

$$(4.6)\qquad \begin{aligned} S &\leqslant C\sum_{j<J} d_j^{N/2}\Big(h_j^{r-1}\|\tilde{v}\|_{r,D_j^1} + d_j^{-1}\|e\|_{0,D_j^1}\Big) + d_J^{N/2}\|e\|_{1,D_J} \\ &\leqslant C\sum_{j\leqslant J} d_j^{N/2}h_j^{r-1}\|\tilde{v}\|_{r,D_j} + C\sum_{j\leqslant J} d_j^{N/2-1}\|e\|_{0,D_j} + d_J^{N/2}\|e\|_{1,D_J}, \end{aligned}$$

and by Lemma 5 and our assumption $\alpha > (r-3)/(r-1)$,

$$(4.7)\qquad \begin{aligned} \sum_{j\leqslant J} d_j^{N/2}h_j^{r-1}\|\tilde{v}\|_{r,D_j} &\leqslant C\sum_{j_1\leqslant j\leqslant J} h_j^{r-1}d_j^{3-r} \\ &\leqslant Ch^{r-1}\sum_{j_1\leqslant j} d_j^{\alpha(r-1)+3-r} \leqslant Ch^{r-1}. \end{aligned}$$

In order to estimate $\|e\|_{0,D_j}$ we use duality. Let e_j equal $e/\|e\|_{0,D_j}$ on D_j and vanish outside D_j, and let w solve

$$(4.8)\qquad A(\psi, w) = (\psi, e_j) \quad \text{for all } \psi \in H^1(\Omega).$$

Hence, $\|e\|_{0,D_j} = (e, e_j) = A(e, w)$, and by (3.14) and Lemma 1,

$$\|e\|_{0,D_j} = A(e, w - w_I) \leqslant C\sum_{i\leqslant J} \|e\|_{1,D_i}h_i\|w\|_{2,D_i^1} + C\|e\|_{1,\Omega_{J+1}}h_J\|w\|_{2,\Omega_J}.$$

It is well known that problem (4.8) admits a unique solution w such that

$$(4.9)\qquad \|w\|_{2,\Omega} \leqslant C\|e_j\|_{0,\Omega},$$

and with the representation

$$(4.10)\qquad w(x) = \int_\Omega g^*(x, y)e_j(y)\,dy,$$

with a g^* (the Green's function for the adjoint problem) such that

$$(4.11)\qquad \big|D_x^\beta g^*(x, y)\big| \leqslant C|x - y|^{-N} \quad \text{for } |\beta| \leqslant 2.$$

Hence, for w we have the estimates

$$\begin{aligned} \|w\|_{2,D_i} &\leqslant Cd_i^{-N/2}d_j^{N/2} \quad \text{for } i \leqslant j, \\ \|w\|_{2,\Omega_i} &\leqslant Cd_i^{N/2}d_j^{-N/2} \quad \text{for } i \geqslant j. \end{aligned}$$

For $i = j-1, j$, and $j+1$ this follows from (4.9), since e_j has L_2-norm equal to one, and for the other i's from the representation (4.10) and the estimate (4.11). We have thus the following estimate:

$$(4.12)\qquad \begin{aligned} \|e\|_{0,D_j} \leqslant\ & Cd_j^{N/2}\sum_{i\leqslant j} \|e\|_{1,D_i}h_id_i^{-N/2} + Cd_j^{-N/2}\sum_{j<i\leqslant J} \|e\|_{1,D_i}h_id_i^{N/2} \\ &+ C\|e\|_{1,\Omega_{J+1}}h_Jd_J^{N/2}d_j^{-N/2}. \end{aligned}$$

Using obvious arguments, we obtain

$$\begin{aligned} \|e\|_{0,D_j} \leqslant\ & Cd_j^{N/2}\max_{i\leqslant j}\big(h_id_i^{-N}\big)\sum_{i\leqslant j} \|e\|_{1,D_i}d_i^{N/2} \\ &+ Ch_jd_j^{-N/2}\sum_{j<i\leqslant J} \|e\|_{1,D_i}d_i^{N/2} + C\|e\|_{1,\Omega_{J+1}}h_Jd_J^{N/2}d_j^{-N/2} \\ \leqslant\ & Ch_jd_j^{-N/2}S + C\|e\|_{1,\Omega_{J+1}}h_Jd_J^{N/2}d_j^{-N/2}, \end{aligned}$$

and hence, using (4.3),

$$C \sum_{j \leqslant J} d_j^{N/2-1} \|e\|_{0,D_j} \leqslant CS \sum_{j \leqslant J} h_j d_j^{-1} + C\|e\|_{1,\Omega_{J+1}} h_J d_J^{N/2} \sum_{j \leqslant J} d_j^{-1}$$

$$\leqslant CSh_J d_J^{-1} + C\|e\|_{1,\Omega_{J+1}} h_J d_J^{N/2} d_J^{-1} \leqslant (C/C_*^{1-\alpha})S + C d_J^{N/2} \|e\|_{1,\Omega_{J+1}}.$$

For a suitable choice of C_* and together with (4.6) and (4.7) this shows (4.4).

We now prove (4.5). By the coercivity and the continuity of $A(\cdot,\cdot)$, and by (3.14), we have

$$\|e\|_{1,\Omega}^2 \leqslant CA(e,e) = CA(e, \tilde{v} - \chi) \leqslant C\|\tilde{v} - \chi\|_{1,\Omega}^2 \quad \text{for all } \chi \in S_h.$$

Lemma 1 then shows

$$\|e\|_{1,\Omega}^2 \leqslant C \sum_{j<J} h_j^{2(r-1)} \|\tilde{v}\|_{r,D_j^1}^2 + Ch_J^2 \|\tilde{v}\|_{2,\Omega_{J-1}}^2,$$

and by Lemma 5,

$$(4.13) \qquad \|e\|_{1,\Omega}^2 \leqslant C \sum_{j_1 \leqslant j \leqslant J} h_j^{2(r-1)} d_j^{2(-N/2+3-r)} + Ch_J^2 h^{(-N+2)/(1-\alpha)} \left(\ln \frac{1}{h}\right)^{\bar{N}},$$

since $\varepsilon = h^{1/(1-\alpha)}$. But for $\alpha > (r-3)/(r-1)$,

$$\sum_{j_1 \leqslant j \leqslant J} h_j^{2(r-1)} d_j^{2(-N/2+3-r)} \leqslant d_J^{-N} h^{2(r-1)} \sum_{j_1 \leqslant j} d_j^{2\alpha(r-1)+2(3-r)} \leqslant C d_J^{-N} h^{2(r-1)},$$

and

$$h_J^2 h^{(-N+2)/(1-\alpha)} (\ln 1/h)^{\bar{N}} \leqslant C(C_*) d_J^{-N} h^{4/(1-\alpha)} (\ln 1/h)^{\bar{N}} \leqslant C(C_*) d_J^{-N} h^{2(r-1)}.$$

Together these estimates show (4.5) which completes the proof of (4.1).

We now turn to the proof of (4.2). We have at once

$$(4.14) \qquad \|e\|_{L_1(\Omega)} \leqslant C \sum_{j \leqslant J} d_j^{N/2} \|e\|_{0,D_j} + d_J^{N/2} \|e\|_{0,\Omega_{J+1}}.$$

Applying (4.12) and changing order of summation we obtain

$$\sum_{j \leqslant J} d_j^{N/2} \|e\|_{0,D_j} \leqslant C \sum_{i \leqslant J} \|e\|_{1,D_i} h_i d_i^{-N/2} \sum_{i \leqslant j \leqslant J} d_j^N$$

$$+ C \sum_{i \leqslant J} \|e\|_{1,D_i} h_i d_i^{N/2} \sum_{j_1 \leqslant j < i} 1 + C\|e\|_{1,\Omega_{J+1}} h_J d_J^{N/2} \sum_{j_1 \leqslant j \leqslant J} 1,$$

and hence (for convenience we now assume that $j_1 \geqslant 1$),

$$\sum_{j \leqslant J} d_j^{N/2} \|e\|_{0,D_j} \leqslant C \sum_{i \leqslant J} i h_i d_i^{N/2} \|e\|_{1,D_i} + CJh_J d_J^{N/2} \|e\|_{1,\Omega_{J+1}}.$$

We shall show that the single term in (4.14) can be estimated in the same way. Repeating the arguments used to derive (4.12), we obtain

$$\|e\|_{0,\Omega_{J+1}} \leqslant C \sum_{i \leqslant J} \|e\|_{1,D_i} h_i \|w\|_{2,D_i^1} + C\|e\|_{1,\Omega_{J+1}} h_J \|w\|_{2,\Omega_J},$$

where w now is the solution of the problem

$$A(\psi, w) = (\psi, e_{J+1}) \quad \text{for all } \psi \in H^1(\Omega),$$

for an appropriate e_{J+1} with L_2-norm equal to one. Hence by (4.9),

$$d_J^{N/2} \|e\|_{0,\Omega_{J+1}} \leqslant C d_J^{N/2} \left(\sum_{i \leqslant J} h_i \|e\|_{1,D_i} + h_J \|e\|_{1,\Omega_{J+1}} \right),$$

which is an even better estimate than we required. We have thus shown that

$$\|e\|_{L_1(\Omega)} \leqslant C \sum_{j \leqslant J} jh_j d_j^{N/2} \|e\|_{1,D_j} + CJh_J d_J^{N/2} \|e\|_{1,\Omega_{J+1}}.$$

Now, set

$$S' = \sum_{j \leqslant J} jh_j d_j^{N/2} \|e\|_{1,D_j}.$$

We shall show that for $\alpha > (r-3)/r$ and a sufficiently large C_*,

$$S' \leqslant \tfrac{1}{2} S' + CJh_J d_J^{N/2} \|e\|_{1,\Omega_J} + Ch^r, \tag{4.15}$$

and

$$Jh_J d_J^{N/2} \|e\|_{1,\Omega} \leqslant Ch^r. \tag{4.16}$$

Clearly the desired result then follows.

By Lemma 2,

$$S' \leqslant C \sum_{j \leqslant J} jh_j^r d_j^{N/2} \|\tilde{v}\|_{r,D_j} + C \sum_{j \leqslant J} jh_j d_j^{N/2-1} \|e\|_{0,D_j} + Jh_J d_J^{N/2} \|e\|_{1,\Omega_J},$$

and by Lemma 5 and our assumption $\alpha > (r-3)/r$,

$$\sum_{j \leqslant J} jh_j^r d_j^{N/2} \|\tilde{v}\|_{r,D_j} \leqslant Ch^r \sum_{j_1 \leqslant j \leqslant J} jd_j^{r\alpha+3-r} \leqslant Ch^r.$$

Using (4.12), we have

$$\begin{aligned} \sum_{j \leqslant J} jh_j d_j^{N/2-1} \|e\|_{0,D_j} \leqslant\ & C \sum_{i \leqslant J} \|e\|_{1,D_i} h_i d_i^{-N/2} \sum_{i \leqslant j \leqslant J} jh_j d_j^{N-1} \\ & + C \sum_{i \leqslant J} \|e\|_{1,D_i} h_i d_i^{N/2} \sum_{j_1 \leqslant j < i} jh_j d_j^{-1} \\ & + C \|e\|_{1,\Omega_{J+1}} h_J d_J^{N/2} \sum_{j_1 \leqslant j \leqslant J} jh_j d_j^{-1}, \end{aligned}$$

which shows that

$$\sum_{j \leqslant J} jh_j d_j^{N/2-1} \|e\|_{0,D_j} \leqslant C \sum_{i \leqslant J} \|e\|_{1,D_i} ih_i^2 d_i^{N/2-1} + C \|e\|_{1,\Omega_{J+1}} Jh_J^2 d_J^{N/2-1},$$

and, hence, that

$$C \sum_{j \leqslant J} jh_j d_j^{N/2-1} \|e\|_{0,D_j} \leqslant (C/C_*^{1-\alpha}) S' + CJh_J d_J^{N/2} \|e\|_{1,\Omega_{J+1}}.$$

For C_* sufficiently large our estimates now together show (4.15).

It remains to prove (4.16). Since for $j \leqslant J$

$$Jh_J d_J^{N/2} \leqslant jh_j d_j^{N/2},$$

we obtain from (4.13) that

$$J^2 h_J^2 d_J^N \|e\|_{1,\Omega}^2 \leqslant C \sum_{j_1 \leqslant j \leqslant J} j^2 h_j^{2r} d_j^{2(3-r)} + CJ^2 h_J^4 d_J^N h^{(-N+2)/(1-\alpha)} \left(\ln \frac{1}{h}\right)^{\bar{N}},$$

so for $\alpha > (r-3)/r$,

$$J^2 h_J^2 d_J^N \|e\|_{1,\Omega}^2 \leqslant Ch^{2r} \sum_{j_1 \leqslant j} j^2 d_j^{2\alpha r + 2(3-r)} + C(C_*) J^2 h^{6/(1-\alpha)} \left(\ln \frac{1}{h}\right)^{\bar{N}} \leqslant Ch^{2r}.$$

This completes the proof of (4.2) and hence of Theorem 1.

5. Proof of Theorem 2. We shall show that for $\alpha > (r-3)/r$,

$$|\tilde{u}_h(x) - u(x)| \leqslant Ch^r (\ln 1/h)^{\bar{r}} |x - x_0|^{-N}.$$

In view of Theorem 1 and the fact that $\tilde{u}_h - u$ equals $v_h - v$, it is sufficient to show that

$$(5.1) \qquad |v_h(x) - v(x)| \leqslant Ch^r (\ln 1/h)^{\bar{r}} d^{-N} + Cd^{-N} \|v_h - v\|_{L_1(\Omega)},$$

where $d = |x - x_0|$.

We first consider the case when $d > ch^{1/(1-\alpha)}$ and c is a sufficiently large constant. Let B be the intersection of Ω and $B_{d/2}(x)$, i.e., set

$$B = \left\{ x \in \Omega \colon |x - x_0| < \tfrac{1}{2} d \right\}.$$

Following the arguments in [10] (cf. Corollary 5.1 of [10] and the remark below), we deduce that

$$(5.2) \quad |v_h(x) - v(x)| \leqslant C(hd^\alpha)^r \left(\ln \frac{1}{h}\right)^{\bar{r}} \max_{|\beta| \leqslant r} \|D^\beta v\|_{L_\infty(B)} + Cd^{-N} \|v_h - v\|_{L_1(B)}.$$

Hence (5.1) follows from (3.3) and our assumption $\alpha > (r-3)/r$.

In the case $d \leqslant ch^{1/(1-\alpha)}$ we first use Lemma 3. Thinking of $v(x)$ as a constant on $\bar{\tau} \ni x$ we obtain

$$\begin{aligned} |v_h(x) - v(x)| &\leqslant Ch^{-N/(1-\alpha)} \|v_h - v(x)\|_{L_1(\tau)} \\ &\leqslant Ch^{-N/(1-\alpha)} \left(\|v_h - v\|_{L_1(\tau)} + \|v - v(x)\|_{L_1(\tau)} \right). \end{aligned}$$

It remains to show that

$$\|v - v(x)\|_{L_1(\tau)} \leqslant Ch^{N/(1-\alpha)} h^r d^{-N}.$$

For $N > 2$ this follows at once from (3.3), since

$$\|v\|_{L_1(\tau)} + h^{N/(1-\alpha)} |v(x)| \leqslant Ch^{N/(1-\alpha)} d^{-N} h^r.$$

For $N = 2$ we first note that for $y \in \tau$ and a suitable curve S we have, using (3.3),

$$|v(y) - v(x)| = \left| \int_S v'(s)\, ds \right| \leqslant Ch^{1/(1-\alpha)} \max\left(\ln \frac{1}{d}, \ln \frac{1}{|y - x_0|} \right),$$

and hence

$$\|v - v(x)\|_{L_1(\tau)} \leqslant Ch^{3/(1-\alpha)} \ln 1/d \leqslant Ch^{2/(1-\alpha)} h^r d^{-2}.$$

This completes the proof.

Remark. For x bounded away from Γ the local estimate (5.2) follows at once from Corollary 5.1 in [10]. Following the arguments in [10] and using cut-off functions which satisfy an appropriate boundary condition (the vanishing of the conormal derivative on Γ), it is easy to see that (5.2) holds also in the general case. Such cut-off functions were also used in [6].

Appendix.

Proof of (3.3): Recall that by definition $v = u - u_0$, $Lv = \Phi$, and $lv = \phi$. We shall show that for, say, $|\beta| \leqslant r$

$$|D^\beta v(x)| \leqslant \begin{cases} C\left(|x - x_0|^{-N+3-|\beta|} + 1\right) & \text{if } |\beta| \neq 3 - N, \\ C\ln\left(|x - x_0|^{-1} + 1\right) & \text{if } |\beta| = 3 - N. \end{cases}$$

Let $x \neq x_0$ be given and set $d = |x - x_0|$. Since $v = u - u_0$, and both u and u_0 are smooth functions away from x_0, it is sufficient to consider the case when d is suitably small. Let ω be a smooth cut-off function such that

$$\omega = 1 \quad \text{in } A_1, \quad \operatorname{supp}(\omega) \subset \bar{A}_2, \quad \text{and} \quad \|D^\gamma \omega\|_{L_\infty(\Omega)} \leqslant C d^{-|\gamma|},$$

where $A_i = \{y \in \Omega: (i+1)^{-1} d < |y - x_0| < (i+1)d\}$. Let $g(y, z)$ be the Green's function for L and l; i.e., let g be the solution of $L_z^* g(y, z) = \delta(z - y)$ in Ω, $l_z^* g(y, z) = 0$ on Γ. Using Green's formula and a splitting of Φ we can write v as the sum of three terms

$$v = \int_\Omega g\omega\Phi\, dx + \int_\Omega g(1 - \omega)\Phi\, dz + \int_\Gamma g\phi\, d\Gamma(z) = v_1 + v_2 + v_3,$$

where the latter identity defines v_1, v_2, and v_3. For d sufficiently small, x is bounded away both from Γ and the support of $(1 - \omega)\Phi$, and thus, in a neighborhood of x, we can differentiate v_2 and v_3 under the integral signs. Using (3.2), (3.11), and straightforward calculations, we obtain

$$\begin{aligned} |D^\beta v_2(x)| &\leqslant \int_\Omega |D^\beta g(x, z)|\, |(1 - \omega(z))\Phi(z)| dz \\ &\leqslant C \int_{\Omega \setminus A_1} \max\left(\ln|x - z|^{-1}, |x - z|^{-N+2-|\beta|}\right) |z - x_0|^{-N+1} dz \\ &\leqslant \begin{cases} C(d^{-N+3-|\beta|} + 1) & \text{if } |\beta| \neq 3 - N, \\ C \ln 1/d & \text{if } |\beta| = 3 - N, \end{cases} \end{aligned}$$

and

$$|D^\beta v_3(x)| \leqslant \int_\Gamma |D^\beta g(x, z)|\, |\phi(z)| d\Gamma(z) \leqslant C.$$

Similarly, we obtain for $y \in \Omega \setminus A_3$,

$$\text{(6.1)} \quad |D^\gamma v_1(y)| \leqslant \begin{cases} C d^{-N+3-|\gamma|} (\ln 1/d)^\mu & \text{if } |y - x_0| < d/4, \\ C d |y - x_0|^{-N+2-|\gamma|} \left\{\ln\left(|y - x_0|^{-1} + 1\right)\right\}^u & \text{if } |y - x_0| > 4d, \end{cases}$$

where $\mu = 1$ if $|\gamma| = 2 - N$ and $\mu = 0$ otherwise. In order to estimate $D^\beta v_1(x)$ we proceed as follows: For $|\beta| \leqslant 1$ we have at once that

$$\begin{aligned} |D^\beta v_1(x)| &\leqslant \|D^\beta g(x, \cdot)\|_{L_1(A_2)} \|\omega\Phi\|_{L_\infty(A_2)} \\ &\leqslant C d^{2-|\beta|} (\ln 1/d)^{\bar{N}(1-|\beta|)} d^{-N+1} \leqslant C(d^{-N+3-|\beta|} + 1). \end{aligned}$$

For $|\beta| > 1$, set $D^\beta = D^\sigma D^\mu$ and $w = D^\mu v_1$ for some σ and μ such that $|\sigma| = 1$. Since $Lw = D^\mu(\omega\Phi) + \eta$ for some $\eta = \sum_{|\gamma|\leqslant|\beta|} b_\gamma D^\gamma v_1$, where b_γ are certain derivatives of the coefficients of L, we have

$$w = \int_\Omega g D^\mu(\omega\Phi)\,dx + \int_\Omega g\eta\,dz + \int_\Gamma glw\,d\Gamma(z) \equiv w_1 + w_2 + w_3.$$

Here

$$|D^\sigma w_1(x)| \leqslant \|D_x^\sigma g(x,\cdot)\|_{L_1(A_2)} \|D^\mu(\omega\Phi)\|_{L_\infty(A_2)} \leqslant Cdd^{-N+1-|\mu|} = Cd^{-N+3-|\beta|},$$

and

$$|D^\sigma w_3(x)| \leqslant \int_\Gamma |D_x^\sigma g(x,z)|\,|lw(z)|\,d\Gamma(z) \leqslant C,$$

since lw is smooth. In order to estimate $D^\sigma w_2(x)$ we first use (6.1) to obtain

$$\left|\int_{\Omega\setminus A_3} D_x^\sigma g(x,z)\eta(z)\,dz\right| \leqslant Cd^{-N+4-|\beta|},$$

and hence

$$|D^\sigma w_2(x)| \leqslant Cd^{-N+4-|\beta|} + Cd \sup_{\substack{|\gamma|\leqslant|\beta| \\ y\in A_3}} |D^\gamma v_1(y)|.$$

We have thus shown

$$|D^\beta v_1(x)| = |D^\sigma w(x)| \leqslant Cd^{-N+3-|\beta|} + Cd \sup_{\substack{|\gamma|\leqslant|\beta| \\ y\in A_3}} |D^\gamma v_1(y)|.$$

Since we may as well assume that the supremum is attained for $\gamma = \beta$ and $y = x$, and since d is small, it follows that

$$|D^\beta v_1(x)| \leqslant Cd^{-N+3-|\beta|}.$$

This completes the proof.

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