High-Order Local Rate of Convergence
By Mesh-Refinement in the Finite Element Method

By Kenneth Eriksson

Abstract. We seek approximations of the solution u of the Neumann problem for the equation $Lu = f$ in Ω with special emphasis on high-order accuracy at a given point $x_0 \in \bar{\Omega}$. Here Ω is a bounded domain in R^N ($N \geqslant 2$) with smooth boundary, and L is a second-order, uniformly elliptic, differential operator with smooth coefficients. An approximate solution u_h is determined by the standard Galerkin method in a space of continuous piecewise polynomials of degree at most $r - 1$ on a partition $\Delta_h(x_0, \alpha)$ of Ω. Here h is a global mesh-size parameter, and α is the degree of a certain systematic refinement of the mesh around the given point x_0, where larger α's mean finer mesh, and $\alpha = 0$ corresponds to the quasi-uniform case with no refinement. It is proved that, for suitable (sufficiently large) α's the high-order error estimate $(u - u_h)(x_0) = O(h^{2r-2})$ holds. A corresponding estimate with the same order of convergence is obtained for the first-order derivatives of $u - u_h$. These estimates are sharp in the sense that the required degree of refinement in each case is essentially the same as is needed for the local approximation to this order near x_0. For the estimates to hold, it is sufficient that the exact solution u have derivatives to the rth order which are bounded close to x_0 and square integrable in the rest of Ω. The proof of this uses high-order negative-norm estimates of $u - u_h$. The number of elements in the considered partitions is of the same order as in the corresponding quasi-uniform ones. Applications of the results to other types of boundary value problems are indicated.

0. Introduction. Let Ω be a bounded domain in R^N, $N \geqslant 2$, with smooth boundary $\partial\Omega$ and consider the Neumann problem to find u such that

$$(0.1) \qquad Lu = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial u}{\partial x_i}\right) + \sum_{i=1}^{N} a_i \frac{\partial u}{\partial x_i} + au = f \quad \text{in } \Omega,$$

$$(0.2) \qquad \frac{\partial u}{\partial n_c} = \sum_{i,j=1}^{N} a_{ij}\frac{\partial u}{\partial x_i}n_j = 0 \quad \text{on } \partial\Omega.$$

Here L is assumed to be uniformly elliptic with smooth coefficients, $n = (n_j)$ and n_c denote the exterior normal and conormal to $\partial\Omega$, respectively. Assume also that the bilinear form

$$A(v,w) = \int_{\Omega}\left\{\sum_{i,j=1}^{N} a_{ij}\frac{\partial v}{\partial x_i}\frac{\partial w}{\partial x_j} + \sum_{i=1}^{N} a_i\frac{\partial v}{\partial x_i}w + avw\right\}dx$$

associated with L is coercive over $H^1(\Omega)$; that is, for some constant $c > 0$,

$$A(v,v) \geqslant c\|v\|^2_{H^1(\Omega)} \quad \forall v \in H^1(\Omega).$$

In order to approximate u, let $\{S_h\}$, h small and positive, be a one-parameter family of finite-dimensional subspaces of $H^1(\Omega)$ and define the Galerkin approximation $u_h \in S_h$ of u by

$$(0.3) \qquad\qquad A(u_h,v_h) = (f,v_h) \quad \forall v_h \in S_h,$$

where $(\cdot,\cdot)$ is the standard inner product in $L_2(\Omega)$. Multiplication of (0.1) by v_h and integration by parts, using (0.2), shows that (0.3) holds for the exact solution u. Hence,

$$(0.4) \qquad\qquad A(u - u_h,v_h) = 0 \quad \forall v_h \in S_h.$$

Therefore, we shall also refer to u_h as the "A-projection" of u.

In this paper, S_h will consist of the restrictions to $\overline{\Omega}$ of all continuous piecewise polynomials of degree at most $r - 1$ on a simplicial mesh, with h the maximal diameter of the simplices. From the approximation properties of S_h, the a priori estimate

$$\|u - u_h\|_{H^1(\Omega)} \leqslant Ch^{r-1}\|u\|_{H^r(\Omega)}$$

is easily obtained. Using the Aubin-Nitsche lemma (see [7, p. 137]), one can derive a sharper estimate in the $L_2(\Omega)$-norm: namely,

$$\|u - u_h\|_{L_2(\Omega)} \leqslant Ch^r\|u\|_{H^r(\Omega)}.$$

Optimal, or nearly optimal, a priori error estimates can be attained also in the maximum norm, but require a more sophisticated analysis. With an additional nondegeneracy condition on the elements, one can prove that

$$(0.5) \qquad\qquad \|u - u_h\|_{L_\infty(\Omega)} \leqslant C(\ln(1/h))^{\bar{r}}h^r\|u\|_{W^r_\infty(\Omega)},$$

where $\bar{r} = 1$ if $r = 2$, $\bar{r} = 0$ for $r > 2$. In the case of $N = 2$ and for a special form A, a proof of this was given by Scott in [19]. For a survey on related results where other boundary value problems are also considered, see Nitsche [11].

In this paper we shall consider a simplicial mesh which has been refined in a certain way close to a given fixed point $x_0 \in \overline{\Omega}$. We shall then be able to show higher-order convergence at x_0. Our main result will be

$$(0.6) \qquad\qquad |(u - u_h)(x_0)| \leqslant Ch^{2r-2}\|u\|_{W^r_\infty(\Omega)}.$$

This holds provided the degree of the prescribed refinement is such that the elements closest to x_0 have diameters of order $h^{2-2/r+\varepsilon}$ for some $\varepsilon > 0$. In the case $\varepsilon = 0$, (0.6) holds with a logarithmic modification. It is clear that only the case $r > 2$ is of interest in this context, even though, when $r = 2$ and $\varepsilon > 0$, (0.6) is slightly better than (0.5) as an estimate for $(u - u_h)(x_0)$. We shall also demonstrate that if the refinement is such that the elements closest to x_0 have diameters of order $h^{2+\varepsilon}$, then the same bound as in (0.6) can be obtained for the gradient of $u - u_h$:

$$(0.7) \qquad\qquad |\nabla(u - u_h)(x_0)| \leqslant Ch^{2r-2}\|u\|_{W^r_\infty(\Omega)}.$$

This result is of interest also for $r = 2$. Next, by localizing the arguments to a neighborhood $\Omega^0 \subset \Omega$ of x_0, we derive the estimate

$$(0.8) \qquad |(u - u_h)(x_0)| \leqslant Ch^{2r-2}\|u\|_{W^r_\infty(\Omega^0)} + C\|u - u_h\|^*_{-m,\Omega^0},$$

where $\| \cdot \|^*_{-m,\Omega^0}$ denotes a weak negative norm to be defined in Section 9. Using known high-order error estimates for $u - u_h$ in such negative norms, this will show

$$(0.9) \qquad |(u - u_h)(x_0)| \leqslant Ch^{2r-2}\{\|u\|_{W^r_\infty(\Omega^0)} + \|u\|_{H^r(\Omega)}\},$$

so that the regularity requirement on u away from x_0 can be relaxed, with a high rate of convergence retained at x_0. The corresponding results are shown for the gradient of $u - u_h$. Since (0.8) and its counterpart for the gradient require only local information on $u - u_h$, our results can also be applied in other situations where negative-norm estimates are known: for example, in connection with various procedures for treating the Dirichlet problem. The above results suggest that the Galerkin solution is a fairly local approximator of u and that pollution effects are moderate, at least for regular problems.

The assumptions on the refinement are that the mesh size decays toward x_0 approximately as h times some power α of the distance, and that local inverse estimates hold. As is seen from the above discussion, the degree α of refinement needed for our results is essentially the same as is required for local approximation to this high order near x_0. Hence, our results are sharp in this sense. Furthermore, it has been shown in Eriksson [5] that the refinement does not seriously increase the total number of elements in the partition, so that the amount of work needed to solve for u_h is essentially the same as for a quasi-uniform mesh.

One may argue that a weak point about the mesh-refinement procedure is that different grids are needed for points x_0 at different locations. In practice, however, it is often the case that one is primarily interested in the solution only in the neighborhood of one specific point. For instance, in fracture mechanics one often knows a priori the point which is the most critical one for the structure and wants detailed information about the solution near this point. On the other hand, if more than one point is of interest, one may, of course, use a mesh which has been locally refined around each one of the points under consideration. Our analysis does not quite cover this case, but it is reasonable to believe that, under the appropriate assumptions, one then obtains $O(h^{2r-2})$ convergence at all the centers of refinement simultaneously, using the same one mesh. Let us also mention that there exist commercial codes for automatic generation of meshes with local refinements like the ones we consider here.

Mesh-refinements have previously been used in connection with elliptic problems in order to enable functions in S_h to fit known singularities in the exact solution properly. See, for example, Schatz and Wahlbin [15], where the singularities of u are caused by the corners of the domain. For a singular two-point boundary value problem, see Schreiber [18]. In Eriksson [5], [6] mesh-refinements were used in order to approximate Green's function to a high-order of accuracy. The idea in these papers is very much related to the present work.

As mentioned above, $O(h^{2r-2})$ estimates are known in many cases for $u - u_h$ as measured in a suitable negative norm; see Bramble and Osborn [3]. Pointwise superconvergence at mesh points is common for one-dimensional problems. In higher dimensions, $O(h^{2r-2})$ error estimates have been obtained by various averaging procedures applied to an already existing approximation u_h of u; see Bramble and Schatz [4], Louis and Natterer [10], Thomée [20], and references therein. Here the post-processing procedure proposed by Bramble and Schatz is easy to perform, but requires a uniform mesh and can, therefore, in general, only give interior results. On the other hand, the method suggested by Louis and Natterer requires that a fundamental solution be a priori known, which, in practice, limits the possibility of application to the case of constant coefficients and no lower-order terms.

The methods of proof employed here are reminiscent of those used by Schatz and Wahlbin in a series of papers: [13], [14], [15], [16]. Let $\mathcal{G}$ be the Green function for (0.1) and (0.2), and let $\mathcal{G}_h \in S_h$ be its A^*-projection. Then the starting point for the proof of our estimate (0.6) will be

$$(u - u_h)(x_0) = A(u - \chi_h, \mathcal{G} - \mathcal{G}_h).$$

Here χ_h can be any function in S_h, and therefore, by the approximation properties of S_h due to the refinement, the essential step in the proof will be to estimate $\mathcal{G} - \mathcal{G}_h$ in a weighted W_1^1-norm. An important tool for this will be the use of local H^1-estimates similar to those obtained by Nitsche and Schatz in [12]. In fact, to avoid difficulties at the point x_0, we shall not use the exact Green function but a smooth approximation of it.

An outline of the paper is as follows. Notation and some results on elliptic regularity are given in Sections 1 and 2. The exact hypotheses on the mesh-refinements are presented in Section 3, and some desired properties of the corresponding piecewise-polynomial spaces are presented in Section 4.

In Section 5, the main result (0.6) is stated in precise terms and then proved. Section 6 is devoted to the proof of the weighted W_1^1-norm estimate mentioned above. The improved convergence (0.7) for the gradient is proved in Section 7. For completeness, a negative-norm estimate referred to in the later sections is derived in Section 8. The local estimate (0.8) is obtained in Section 9 and its counterpart for the gradient, in Section 10.

1. Notation and Preliminaries. For a domain $\Omega' \subset R^N$ and m a nonnegatve integer, $W_q^m(\Omega')$ will denote the usual Sobolev space with norm

$$\|v\|_{W_q^m(\Omega')} = \begin{cases} \left\{ \displaystyle\sum_{|\beta| \leq m} \|D^\beta v\|_{L_q(\Omega')}^q \right\}^{1/q} & \text{if } 1 \leq q < \infty, \\ \displaystyle\max_{|\beta| \leq m} \|D^\beta v\|_{L_\infty(\Omega')} & \text{if } q = \infty, \end{cases}$$

where $\beta = (\beta_1, \ldots, \beta_N)$ with β_i nonnegative integers, $|\beta| = \sum_{i=1}^N \beta_i$. $D^\beta = D_x^\beta = (\partial/\partial x_1)^{\beta_1} \cdots (\partial/\partial x_N)^{\beta_N}$, and $\| \cdot \|_{L_q(\Omega')}$ is the norm in the Banach space $L_q(\Omega')$. We shall also use the seminorm $| \cdot |_{W_q^m(\Omega')}$, defined as the corresponding norm, but with β ranging over $|\beta| = m$ only. For $q = 2$ we write $W_2^m(\Omega') = H^m(\Omega')$, and for the corresponding norm and seminorm, $\| \cdot \|_{W_2^m(\Omega')} = \| \cdot \|_{H^m(\Omega')} = \| \cdot \|_{m,\Omega'}$ and $| \cdot |_{W_2^m(\Omega')} = | \cdot |_{m,\Omega'}$, respectively.

The space of restrictions to $\overline{\Omega}'$ of infinitely differentiable functions on R^N will be denoted $C^\infty(\Omega')$, and $C_0^\infty(\Omega')$ will consist of all functions in $C^\infty(\Omega')$ with compact support in Ω'.

For $m \geqslant 0$, m an integer, we define the negative norms $\| \cdot \|_{-m,\Omega'}$ and $\|| \cdot \||_{-m,\Omega'}$ by

$$\|v\|_{-m,\Omega'} = \sup_{w \in C_0^\infty(\Omega')} \frac{(v,w)_{\Omega'}}{\|w\|_{m,\Omega'}}$$

and

$$\|v\|_{-m,\Omega'} = \sup_{w \in C^\infty(\Omega')} \frac{(v,w)_{\Omega'}}{\|w\|_{m,\Omega'}},$$

where $(v,w)_{\Omega'} = \int_{\Omega'} vw \, dx$.

Throughout this paper we shall let Ω be a fixed bounded domain in R^N, $N \geqslant 2$, with C^∞ boundary $\partial\Omega$. When $\Omega' = \Omega$ we shall shorten the notation and write $H^m(\Omega) = H^m$, $\| \cdot \|_{m,\Omega} = \| \cdot \|_m$, $(\cdot,\cdot)_\Omega = (\cdot,\cdot)$, etc.

For $\Omega' \subset \Omega$ we shall consider the bilinear form $A_{\Omega'} = A(\cdot,\cdot)_{\Omega'}$, defined by

$$(1.1) \qquad A(v,w)_{\Omega'} = \int_{\Omega'} \left\{ \sum_{i,j=1}^N a_{ij} \frac{\partial v}{\partial x_i} \frac{\partial w}{\partial x_j} + \sum_{i=1}^N a_i \frac{\partial v}{\partial x_i} w + avw \right\} dx,$$

where we require that the coefficients a_{ij}, a_i, and a be functions in $C^\infty(\Omega)$ satisfying $a_{ij} = a_{ji}$. Although we shall mainly be interested in the form $A = A_\Omega$, we make the following assumption: There is a constant $c_A > 0$ such that

$$(1.2) \qquad A(v,v)_{\Omega'} \geqslant c_A \|v\|_{1,\Omega'}^2 \quad \forall v \in H^1(\Omega'), \forall \Omega' \subset \Omega.$$

The reason for our interest in the bilinear form A is its connection with the Neumann problem for the second-order partial differential equation

$$-\sum_{i,j=1}^N \frac{\partial}{\partial x_j} \left(a_{ij} \frac{\partial u}{\partial x_i} \right) + \sum_{i=1}^N a_i \frac{\partial u}{\partial x_i} + au = f.$$

Moreover, our main result will have applications to other boundary value problems for this equation, such as the Dirichlet problem.

We shall later need the fact that the coefficients a_{ij} satisfy the following condition: There is a constant $c_A > 0$ such that

$$(1.3) \qquad \sum_{i,j=1}^N a_{ij}(x)\lambda_i\lambda_j \geqslant c_A |\lambda|^2 \quad \forall \lambda \in R^N, \forall x \in \overline{\Omega};$$

i.e., the associated differential equation is uniformly elliptic. This is, in fact, not an extra assumption on the coefficients a_{ij}, but follows easily from (1.2) with the same constant c_A. Indeed, take $\Omega' = \{x \in \Omega: |x - \overline{x}| < \varepsilon\}$ and $v = m_\varepsilon^{-1/2}\lambda \cdot (x - \overline{x})$, where $m_\varepsilon = \int_{\Omega'} dx$. Then, since a_{ij} are smooth by assumption, a simple computation shows that then for small ε,

$$A(v,v)_{\Omega'} = \sum_{i,j=1}^N a_i(\overline{x})\lambda_i\lambda_j + O(\varepsilon),$$

and

$$\|v\|_{1,\Omega'}^2 = |\lambda|^2 + O(\varepsilon^2).$$

Inserting this into (1.2) and letting ε tend to zero yields the desired inequality (1.3).

Remark 1.1. Assumption (1.2) is unnecessarily restrictive for the purposes of this paper. In fact, the following global coercivity condition suffices: There is a constant $c_A' > 0$ such that

$$(1.4) \qquad A(v, v)_\Omega \geqslant c_A' \|v\|_{1,\Omega}^2 \quad \forall v \in H^1(\Omega).$$

In Sections 9 and 10, where (1.2) is used for $\Omega' \neq \Omega$, it is possible to employ instead the modified bilinear form $A_{k,\Omega'}(v, w) = A(v, w)_{\Omega'} + k(v, w)_{\Omega'}$, which satisfies (1.2) if k is sufficiently large (cf. Appendix 1 of [13]).

We end this section with a list of some further notation and definitions. We shall write

$$m(K) = \int_K dx, \qquad \text{diam}(K) = \sup_{x, y \in K} |x - y|,$$

$$\text{dist}(K, K') = \inf_{x \in K, y \in K'} |x - y|, \quad \text{and} \quad B_d(y) = \{ x \in R^N : |x - y| < d \}.$$

Moreover, $P_r(K)$ will be the set of all polynomials of degree $\leqslant r$ restricted to K. Throughout the paper we shall use the letters c and C to denote various positive constants. For f and g two positive functions, we shall write $f \sim g$ if there are constants c and C such that $f \geqslant cg$ and $f \leqslant Cg$. We also define the unit N-simplex T by

$$T = \left\{ x \in R^N : x_i \geqslant 0, i = 1, \dots, N, \text{ and } \sum_{i=1}^{N} x_i \leqslant 1 \right\}.$$

An arbitrary N-simplex is obtained by a nonsingular affine transformation of T. A face of a simplex is any one of the $N + 1$ $(N - 1)$-simplices constituting its boundary.

2. Elliptic Regularity. In this section we shall be concerned with the variational problem

$$(2.1) \qquad A(G, v)_{\Omega'} = (g, v)_{\Omega'}, \quad \forall v \in H^1(\Omega'),$$

and with the regularity of the solution G in terms of g.

By the boundedness of the coefficients, the form $A_{\Omega'}$ is continuous on $W_p^1(\Omega') \times W_q^1(\Omega')$ for $1 \leqslant p, q \leqslant \infty$ and $p^{-1} + q^{-1} = 1$. In fact, by Hölder's inequality, there is a constant C_A, independent of Ω', such that

$$(2.2) \quad A(v, w)_{\Omega'} \leqslant C_A \|v\|_{W_p^1(\Omega')} \|w\|_{W_q^1(\Omega')} \quad \forall (v, w) \in W_p^1(\Omega') \times W_q^1(\Omega').$$

In particular, the form is continuous on $H^1(\Omega')$. We shall need (2.2) for $p = 2$ and $p = \infty$ only.

The results of the following lemma are special cases of the Lax-Milgram lemma (cf., e.g., Ciarlet [7]).

LEMMA 2.1. *Let the coefficients of the bilinear form A be such that* (1.2) *and* (2.2) *(with $p = 2$) hold. Then, for given $\Omega' \subset \Omega$ and $g \in L_2(\Omega')$, there is a unique $G \in H^1(\Omega')$ satisfying* (2.1). *Furthermore, given any $G \in H^1(\Omega')$ and a closed subspace S of $H^1(\Omega')$, the $A_{\Omega'}$-projection of G on S is well-defined; i.e., there exists a unique $G_S \in S$ for which*

$$A(G - G_S, v_S)_{\Omega'} = 0 \quad \forall v_S \in S.$$

For Ω' with a sufficiently smooth boundary $\partial\Omega'$ we have the following well-known, elliptic, regularity result (cf., e.g., Schechter [17]).

LEMMA 2.2. *Let the coefficients of the form A be as in Section 1 and assume that (1.2) holds. Furthermore, let $\Omega' \subset \Omega$ have a C^∞ boundary $\partial\Omega'$, and let $s \geqslant 0$ be an integer. Then there is a constant C such that, if $g \in H^s(\Omega')$ and G is defined by (2.1), then*

$$(2.3) \qquad \|G\|_{s+2,\Omega'} \leqslant C\|g\|_{s,\Omega'}.$$

We shall make extensive use of local estimates on G away from the support of g. We have the following lemma:

LEMMA 2.3. *Let the assumptions be as in Lemma 2.2, let $s > 0$, and let β be any multi-index of length $|\beta| \leqslant s$. Then there is a constant C such that the following holds: Let $g \in L_2(\Omega')$, let $x \in \Omega'$, and set $R = \operatorname{dist}(x, \operatorname{supp}(g))$. Then, for G defined by (2.1),*

$$(2.4) \qquad |D^\beta G(x)| \leqslant CR^{-N+2-s}\|g\|_{L_1(\Omega')}.$$

Proof. The solution G of (2.1) is given by

$$(2.5) \qquad G(x) = \int_{\Omega'} \mathscr{G}(x, y) g(y)\, dy,$$

where $\mathscr{G}(x, y)$ is the Green function. It is known (cf., e.g., Krasovskiĭ [9]) that

$$(2.6) \qquad |D_x^\beta \mathscr{G}(x, y)| \leqslant C|x - y|^{-N+2-s}.$$

Differentiation of (2.5) under the integral sign, application of Hölder's inequality, and (2.6) give

$$|D^\beta G(x)| = \left| \int_{\Omega'} D_x^\beta \mathscr{G}(x, y) g(y)\, dy \right|$$

$$\leqslant \|D_x^\beta \mathscr{G}(x, \cdot)\|_{L_\infty(\operatorname{supp}(g))} \|g\|_{L_1(\Omega')} \leqslant CR^{-N+2-s}\|g\|_{L_1(\Omega')},$$

which was to be proved. $\square$

We finally note that in all of this section we could have considered the adjoint problem as well. That is, for $g \in L_2(\Omega')$, there is a $G \in H^1(\Omega')$ (in general, different from our previous G) such that

$$A(v, G)_{\Omega'} = (v, g)_{\Omega'} \quad \forall v \in H^1(\Omega'),$$

and the regularity results above hold for this G as well. Moreover, the $A_{\Omega'}^*$-projection $G_S \in S$ of G is well-defined by

$$A(v_S, G - G_S)_{\Omega'} = 0 \quad \forall v_S \in S.$$

3. Mesh-Refinement Around a Point. In this section we define certain partitions of Ω which are refined around a given fixed point $x_0 \in \overline{\Omega}$. To these partitions there will correspond finite-dimensional function spaces, to be introduced in Section 4.

We shall divide Ω into simplices, which will be modified at the boundary Ω, since the simplices cannot fit exactly the curved parts of $\partial\Omega$. For a small positive parameter h, we shall consider a family of such partitions defined as follows: Let Ω be covered by the union $\hat{\Omega}_h$ of the simplices $\hat{\tau}_i^h$, $i = 1, \ldots, I(h)$, where we require that

the $\hat{\tau}_i^h$ have disjoint interiors and satisfy the standard assumption that, for $i = 1,\ldots,I(h)$, any face of $\hat{\tau}_i^h$ is either a part of $\partial\hat{\Omega}_h$ or corresponds exactly to a face of an adjacent simplex. As our partition we take the sets, with int$(\cdot)$ = the interior of $(\cdot)$,

$$\tau_i^h = \text{int}(\hat{\tau}_i^h \cap \Omega), \qquad i = 1,\ldots,I(h).$$

The desired features of the partition are determined by the following two requirements: For some fixed positive constants c_S, c_R, and C_R, for $x_0 \in \overline{\Omega}$, for α with $0 \leqslant \alpha < 1$, and for h small and positive, we have

(SA) $$m(\tau_i^h) \geqslant c_S \text{diam}^N(\hat{\tau}_i^h), \qquad i = 1,\ldots,I(h),$$

(RA) $$c_R \text{diam}(\tau_i^h) \leqslant h \sup_{x \in \tau_i^h} |x - x_0|^\alpha \leqslant C_R \text{diam}(\tau_i^h), \qquad i = 1,\ldots,I(h).$$

We use the notation $\Delta_h(x_0, \alpha)$ for partitions obtained as above and satisfying (SA) and (RA).

The "shape" assumption (SA) implies, on the one hand, that the simplices do not degenerate, and, on the other hand, near the boundary, that a fixed minimum portion of each simplex is inside Ω. This can be restated in the following equivalent but more useful way: There are fixed positive constants c_S' and C_S' and, for each $\tau = \tau_i^h$, a nonsingular affine mapping A_τ with $A_\tau(x) = B_\tau x + b_\tau$, $B_\tau = (b_{ij})$, $B_\tau^{-1} = (b_{ij}')$, such that

(SA)' $$c_S'T \subset A_\tau(\tau) \subset A_\tau\hat{\tau} = T, \quad |b_{ij}| \leqslant C_S'\text{diam}^{-1}(\tau), \quad |b_{ij}'| \leqslant C_S'\text{diam}(\tau).$$

The refinement assumption (RA) = (RA)$_{x_0,\alpha}$ roughly says that an element at distance r from x_0 has diameter approximately equal to hr^α. Since Ω is bounded, the global mesh size is bounded by Ch. Elements close to x_0 are forced to have diameter of order $h^{1/(1-\alpha)}$. Hence, the parameter α is a measure of the refinement.

For a discussion of the possibility of constructing a mesh satisfying (SA) and (RA), we refer to Eriksson [5]. There, it is also shown that, for a fixed $\alpha < 1$, the total number of elements in such a partition is proportional to h^{-N} as in the quasi-uniform case. Hence, the size of the finite element matrix and, thereby, the amount of work required to solve for the Galerkin approximation are not seriously increased.

For our proofs we shall need various subsets of Ω. For integers j we define

$$\Omega_j = \Omega \cap B_j \quad \text{where } B_j = B_{d_j}(x_0), d_j = 2^{-j},$$

$$D_j = \Omega \cap A_j \quad \text{where } A_j = B_j \setminus \overline{B}_{j+1},$$

and

$$D_j^k = D_{j-k} \cup \cdots \cup D_{j+k}, \qquad k = 0,1,2,\ldots.$$

In the technical work below we shall need to know that the local mesh size on D_j is small in relation to d_j, in a sense to be made precise later. This will turn out to be the case for all $j \leqslant J_1$ if we define the function $J_1 = J_1(h) = J_1(h, \alpha)$ by

$$2^{-J_1} = C_1 h^{1/(1-\alpha)},$$

where the constant C_1 is chosen sufficiently large. The significance of the quantity J_1 will be clear in Section 4, where we shall also fix C_1. Defining $h_j = hd_j^\alpha$, it follows

easily from (RA) that, for $j \leqslant J_1$, h_j is proportional to the local mesh size on D_j, and that

$$h_j d_j^{-1} = h d_j^{\alpha-1} \leqslant h\left(C_1 h^{1/(1-\alpha)}\right)^{\alpha-1} = 1/C_1^{1-\alpha},$$

which also expresses, if C_1 is large, the relative smallness of the mesh on D_j. We finally note that, since Ω is bounded, the sets D_j will be empty for j less than some integer j_1.

4. Spaces. Inverse Estimates. Approximation Properties. Let $\Delta_h = \Delta_h(x_0, \alpha)$ be a partition as described in Section 3, and let r, with $r \geqslant 2$, be an integer. We shall denote by $S_h = S_h(r, x_0, \alpha)$ the finite-dimensional space of functions in $C(\overline{\Omega})$ which reduce to polynomials in P_{r-1} in each of the elements $\tau = \tau^h \in \Delta_h$. Moreover, if $\Omega' \subset \Omega$, the space of all restrictions to $\overline{\Omega}'$ of functions in S_h will be written $S_h(\Omega')$, and $S_h^0(\Omega')$ will consist of all functions in S_h vanishing outside $\overline{\Omega}'$. We shall use the notation $\hat{\chi}_h$ for the obvious piecewise-polynomial extension to $\hat{\Omega}_h$ of a function $\chi_h \in S_h$; we use $\hat{S}_h$ for the corresponding space.

The following inverse estimate is standard for interior elements.

PROPOSITION 4.1. *Let the partition Δ_h and the corresponding piecewise-polynomial space S_h be as above. Then there is a constant C_2, depending only on N, r, c_S', and C_S', such that, for $1 \leqslant p$, $q \leqslant \infty$, integers $0 \leqslant t < s \leqslant r$, $v_h \in S_h$, and any element $\tau \in \Delta_h$, the following holds*:

$$(4.1) \qquad |v_h|_{W_p^s(\tau)} \leqslant C_2 \left[\operatorname{diam}(\tau)\right]^{t-s-N(q^{-1}-p^{-1})} |v_h|_{W_q^t(\tau)}.$$

Furthermore, if $m > 0$ is an integer, there is a constant $C_3 = C_3(N, m, c_S', C_S')$ such that, for $v_h \in S_h$ and any element τ,

$$(4.2) \qquad \|v_h\|_{0,\tau} \leqslant C_3 \operatorname{diam}^{-m}(\tau)\|v_h\|_{-m,\tau}.$$

Proof. Let the affine transformation A_τ be as in Section 3. Set $y = A_\tau(x) = B_\tau x + b_\tau$ and define the operator a_τ by $a_\tau v(y) = v(A_\tau^{-1}(y)) = v(x)$. Repeated use of the chain rule and (SA)' yields the estimates

$$\left|D_x^\alpha \hat{v}_h(x)\right|^p \leqslant N^{|\alpha|p}\left[C_S' \operatorname{diam}^{-1}(\tau)\right]^{|\alpha|p} \sum_{|\beta|=|\alpha|} \left|D_y^\beta a_\tau \hat{v}_h(y)\right|^p$$

and

$$\left|\det B_\tau^{-1}\right| \leqslant N! C_S'^N \operatorname{diam}^N(\tau).$$

A change of variables then gives

$$(4.3) \qquad |v_h|_{W_p^s(\tau)} \leqslant C \operatorname{diam}^{N/p-s}(\tau)|a_\tau \hat{v}_h|_{W_p^s(T)}.$$

Similarly, we obtain

$$\left|a_\tau \hat{v}_h\right|_{W_q^t(c_S'T)} \leqslant C \operatorname{diam}^{t-N/q}(\tau)|v_h|_{W_q^t(\tau)}.$$

The desired estimate (4.1) now follows from the fact that, on the finite-dimensional space $P_{r-1}(T)$ and for s and t as considered, the seminorm $|\cdot|_{W_p^s(T)}$ is dominated by $C|\cdot|_{W_q^t(c_S'T)}$ for some constant $C = C(N, r, c_S')$.

Considering the equivalence of norms on $P_{r-1}(T)$, we obtain from (4.3), with $s = 0$ and $p = 2$,

$$\|v_h\|_{0,\tau} \leqslant C \operatorname{diam}^{N/2}(\tau)\|a_\tau v_h\|_{-m, c_S'T}.$$

From the seminorm inequality (4.3), we easily obtain, for any function w,

$$\|w\|_{m,\tau} \leqslant C \operatorname{diam}^{N/2-m}(\tau)\|a_\tau w\|_{m,T}.$$

Hence,

$$\|v_h\|_{0,\tau} \leqslant C \operatorname{diam}^{N/2}(\tau) \sup_{a_\tau w \in C_0^\infty(c_S'T)} \frac{(a_\tau v_h, a_\tau w)}{\|a_\tau w\|_{m,T}}$$

$$\leqslant C \operatorname{diam}^{-m}(\tau) \sup_{a_\tau w \in C_0^\infty(c_S'T)} \frac{(v_h, w)}{\|w\|_{m,\tau}}$$

$$\leqslant C \operatorname{diam}^{-m}(\tau)\|v_h\|_{-m,\tau},$$

with a constant C depending only on N, m, c_S' and C_S'. This shows (4.2) and completes the proof. $\square$

We shall say that D_h is a mesh domain if $D_h = \operatorname{int}(\bigcup_{i \in M} \bar{\tau}_i^h)$, for some index set M, and we denote by $\hat{D}_h$ the corresponding union of simplices $\hat{\tau}_i^h$. We shall need the following local consequence of (4.1): Let D' and D'' be domains such that $D' \subset D_h \subset D'' \subset D_j^1$, where D_h is a mesh domain. Then there is a constant $C = C(C_2, C_R)$ such that

(4.4) $$\|v_h\|_{1,D'} \leqslant C h_j^{-1}\|v_h\|_{0,D''}.$$

In fact, since $\operatorname{diam}(\tau^h) \geqslant ch_j$ for $\tau^h \subset D_j^1$, we have, by (4.2),

$$\|v_h\|_{1,D_h}^2 = \sum_{\tau^h \subset D_h} \|v_h\|_{1,\tau^h}^2 \leqslant C^2 h_j^{-2} \sum_{\tau^h \subset D_h} \|v_h\|_{0,\tau^h}^2$$

$$= C^2 h_j^{-2}\|v_h\|_{0,D_h}^2,$$

from which (4.4) easily follows.

Together, (4.1) and (4.2) yield the following: Let $D' \subset D_h \subset D''$ and assume that, for each $\tau^h \subset D_h$, one has $\operatorname{diam}(\tau^h) \geqslant c'h$. Then there is a constant $C = C(c', C_2, C_3)$, such that

(4.5) $$\|v_h\|_{1,D'} \leqslant C h^{-m-1}\|v_h\|_{-m,D''}.$$

To see this, we recall from Lemma 1.1 of [13] that

$$\sum_{\tau^h \subset D_h} \|v_h\|_{-m,\tau}^2 \leqslant \|v_h\|_{-m,D_h}^2.$$

Hence, by (4.1) and (4.2),

$$\|v_h\|_{1,D_h}^2 = \sum_{\tau^h \subset D_h} \|v_h\|_{1,\tau^h}^2 \leqslant C^2 h^{-2(m+1)} \sum_{\tau^h \subset D_h} \|v_h\|_{-m,\tau^h}^2$$

$$\leqslant C^2 h^{-2(m+1)}\|v_h\|_{-m,D_h}^2,$$

which gives (4.5).

We now turn to approximation properties. The following result was proved in Eriksson [5]:

PROPOSITION 4.2. *Let $\Delta_h = \{\tau_i^h\}$ be a partition and S_h an associated piecewise-poly-nomial space as above. Then there are constants C_4 and C_4' depending only on $N, \Omega, r,$*

c_S', and C_S', and, for each $\tau = \tau^h \in \Delta_h$, a neighborhood O_τ of τ with $\mathrm{diam}(O_\tau) \leqslant C_4' \mathrm{diam}(\tau)$, such that the following holds: For each $v \in L_1$ there is a $\chi_h \in S_h$ such that if $0 \leqslant l \leqslant m \leqslant r$ and $1 \leqslant p \leqslant \infty$, and if $v \in W_p^m(O_\tau)$, then

$$(4.6) \qquad \|v - \chi_h\|_{W_p^l(\tau)} \leqslant C_4 \mathrm{diam}^{m-l}(\tau) \|v\|_{W_p^m(O_\tau)}.$$

We have the following consequence of estimate (4.6):

COROLLARY 4.2. *There is a constant C depending only on N, c_S, c_R, C_R, C_4, and C_4' such that the following holds: Let $v \in L_1(\Omega)$ and let χ_h be given by Proposition 4.2. Furthermore, let $D' \subset D'' \subset \Omega_{j-1}$ and assume that*

$$(4.7) \qquad \tau \cap D' \neq \varnothing \quad \text{only if } O_\tau \subset D''.$$

Then, for $0 \leqslant l \leqslant 1 \leqslant m \leqslant r$ and $v \in H^m(D'')$,

$$(4.8) \qquad \|v - \chi_h\|_{l,D'} \leqslant Ch_j^{m-l} \|v\|_{m,D''}.$$

Moreover, if $v \in W_\infty^m(D'')$, then

$$(4.9) \qquad \|v - \chi_h\|_{W_\infty^1(D')} \leqslant Ch_j^{m-1} \|v\|_{W_\infty^m(D'')}.$$

Proof. Let D_h be the smallest mesh domain for which $D' \subset D_h$, and note that then, for $\tau^h \subset D_h$, $\mathrm{diam}(\tau^h) \leqslant Ch_j$ with $C = C(c_R)$. Since it is clear that there exists an upper bound, depending only on N, c_S, c_R, C_R, and C_4' for the number of τ for which O_τ intersects any given point, we have, by (4.6) and (4.7),

$$\|v - \chi_h\|_{l,D'}^2 \leqslant \sum_{\tau^h \in D_h} \|v - \chi_h\|_{l,\tau^h}^2$$

$$\leqslant Ch_j^{2(m-l)} \sum_{\tau^h \in D_h} \|v\|_{m,O_\tau h}^2 \leqslant Ch_j^{2(m-l)} \|v\|_{m,D''}^2,$$

which proves (4.8). Estimate (4.9) is proved in the same manner. □

Recall from Section 3 the definition of J_1, $2^{-J_1} = C_1 h^{1/(1-\alpha)}$, which was introduced to separate the treatment of the elements closest to x_0 from that of the remaining elements of the refined mesh. We shall now fix the constant C_1 in a manner which will make (4.7) (and, hence, (4.8) and (4.9)) valid when needed in the sequel. More precisely, the proof of our crucial Lemma 6.2 below uses (4.8) and (4.9) for $j \leqslant J_1$ when $\mathrm{dist}(D', \Omega \setminus D'') \geqslant c_1 d_j$. In order to justify this, we may choose

$$C_1 = \left(4 \cdot c_1^{-1} \cdot C_4' \cdot c_R^{-1}\right)^{1/(1-\alpha)}.$$

It is an easy computation to show that (4.7) is then satisfied for $j \leqslant J_1$. If (4.7) holds, we also know that there exists a mesh domain D_h with $D' \subset D_h \subset D''$, and, hence, if $D' \subset D'' \subset D_j^1$, the inverse estimate (4.4) can be used.

Note also that Proposition 4.2 implies that χ_h vanishes outside a small neighborhood of $\mathrm{supp}(v)$. More precisely, if $D = \mathrm{supp}(v)$, we have

$$\mathrm{supp}(\chi_h) \subset \bigcup_{\{\tau : O_\tau \cap D \neq \varnothing\}} \bar{\tau}.$$

We end this section with a "superapproximation" result. Again, we refer to Eriksson [5] for the proof.

PROPOSITION 4.3. *Let S_h be as above, and let the constant C_η be given. Then there is a constant $C = C(N, r, \alpha, c'_S, C'_S, c_R, C_R, C_\eta)$ such that the following holds: Let D_h be a mesh domain with $D_h \subset D_j^1$, and let $\eta \in C^\infty(R^N)$ be constant outside $\hat{D}_h$ (on each component of $\mathrm{int}(\hat{\Omega}_h) \setminus \hat{D}_h$) and satisfy*

$$\|\eta\|_{W_\infty^k} \leqslant C_\eta d_j^{-k}, \qquad k = 0, \dots, r.$$

Then, given $v_h \in S_h$, there is a $\chi_h \in S_h$ such that $\mathrm{supp}(\eta v_h - \chi_h) \subset \overline{D}_h$ and

$$(4.10) \qquad \|\eta v_h - \chi_h\|_{1, D_h} \leqslant C h_j d_j^{-1} \|v_h\|_{1, D_h} + C h_j d_j^{-2} \|v_h\|_{0, D_h}.$$

5. Main Result. High-Order Local Rate of Convergence.
After the preparatory first sections, having defined the special finite-dimensional spaces $S_h = S_h(r, x_0, \alpha)$ and listed some of their properties, we are now ready to state and prove our main result, Theorem 5.1. The theorem states that if u is a smooth function and $u_h \in S_h(r, x_0, \alpha)$ is its A-projection or, equivalently, its Galerkin approximation, then

$$(u - u_h)(x_0) = O(h^{2r-2}),$$

provided α is sufficiently large—i.e., provided the underlying mesh is sufficiently refined close to x_0.

For a quasi-uniform mesh it is known that the pointwise convergence is of order $O(h^r)$ if $r > 2$ (with a logarithmic modification if $r = 2$). This was proved by Scott [19] for $N = 2$ and $A(u, v) = (\nabla u, \nabla v) + (u, v)$. Our result thus shows that refining the mesh around a point x_0 considerably improves the rate of convergence of $u - u_h$ at this point, at least for $r > 2$.

THEOREM 5.1. *Let Ω and the bilinear form A be as in Section 1. Furthermore, let x_0 be an arbitrary point in $\overline{\Omega}$, and for h small and positive let there be given a family of partitions $\Delta_h(x_0, \alpha)$ of Ω as in Section 3. Finally, for an integer $r \geqslant 2$ let $S_h = S_h(r, x_0, \alpha)$ be the corresponding finite-dimensional spaces as defined in Section 4. Then, for given α with $(r - 2)/(2r - 2) < \alpha < 1$, there is a constant C such that, for $u \in W_\infty^r$ and $u_h \in S_h$, defined by*

$$(5.1) \qquad A(u - u_h, \chi_h) = 0 \quad \forall \chi_h \in S_h,$$

the estimate

$$(5.2) \qquad |(u - u_h)(x_0)| \leqslant C h^{2r-2} \|u\|_{W_\infty^r}$$

holds. Moreover, C is independent of x_0. For $\alpha = (r - 2)/(2r - 2)$, estimate (5.2) holds with C replaced by $C \ln(h^{-1})$.

Remark. Obviously, for $u \in W_\infty^r(\Omega)$, by continuous extension $u(x_0)$ is defined also for $x_0 \in \partial\Omega$.

Note that Lemma 2.1 insures that u_h is well-defined by (5.1). Note also that the very existence of a function $u_h \in S_h(r, x_0, \alpha)$ satisfying inequality (5.2), for $\alpha \geqslant (r - 2)/(2r - 2)$, is an immediate consequence of the approximation property (4.6) and the fact that $\mathrm{diam}(\tau) \leqslant C h^{1/(1-\alpha)} \leqslant C h^{(2r-2)/r}$ for τ close to x_0 and such α's. The point of our theorem is that this high order of convergence holds for the A-projection u_h; i.e., that the high rate of convergence of x_0 is not destroyed by "pollution" from the coarser mesh away from x_0.

A brief outline of the proof of Theorem 5.1 is as follows. We first write

$$(u - u_h)(x_0) = A(u - u_h, \mathscr{G}) = A(u - \chi_h, \mathscr{G} - \mathscr{G}_h),$$

where $\mathscr{G} = \mathscr{G}_{x_0}$ is the Green function, $\mathscr{G}_h \in S_h$ the A^*-projection of $\mathscr{G}$, and χ_h an arbitrary function in S_h. By the continuity of A, we then have to estimate $\|u - \chi_h\|_{W_\infty^1}$ and $\|\mathscr{G} - \mathscr{G}_h\|_{W_1^1}$, where, by the approximation properties of S_h, the first of these norms is bounded by Ch^{r-1}. We thus wish to prove the same estimate for $\|\mathscr{G} - \mathscr{G}_h\|_{W_1^1}$. In view of the singular character of $\mathscr{G}$, this might seem too optimistic, but the idea now is that the finer mesh close to x_0 will compensate for the singularity of $\mathscr{G}$ to keep $\mathscr{G} - \mathscr{G}_h$ small in all of Ω. To follow the above sketch of a proof, however, would require a larger lower bound for α than the one given in Theorem 5.1 (cf. the Corollary in [5]). In order to obtain (5.2) also for the smaller α's, we shall take advantage of the fact that the $u - \chi_h$ part can be made smaller than h^{r-1} close to the point x_0, since the mesh is refined there. Also, to deal with the singular behavior of $\mathscr{G}$ near x_0, we shall use an inverse estimate and L_2 duality to arrive at a situation where only smooth approximations of $\mathscr{G}$ have to be considered. More precisely, we have the following lemma, with the aid of which we shall then prove Theorem 5.1. For notation, see Section 3.

LEMMA 5.2. *Let the basic assumptions be as in Theorem* 5.1. *Then for given* α, *with* $(r-2)/(2r-2) < \alpha < 1$ *and with* $J_1 = J_1(h, \alpha)$ *as in Section* 4, *there is a constant* C *and an integer* $J = J(h, \alpha) \leqslant J_1$ *such that the following holds: Let* $x_0 \in \bar{\tau}_0$, *and let* $g \in C_0^\infty(\tau_0)$ *be such that* $\|g\|_0 = h^{-N/(2-2\alpha)}$. *Define* $G \in H^1$ *and* $G_h \in S_h = S_h(r, x_0, \alpha)$ *by*

$$(5.3) \quad A(v, G) = (v, g) \quad \forall v \in H^1, \qquad A(\chi_h, G - G_h) = 0 \quad \forall \chi_h \in S_h.$$

Then there is a constant C *independent of* h *such that*

$$(5.4) \qquad \sum_{j \leqslant J} h_j^{r-1} \|G - G_h\|_{W_1^1(D_j)} + h_J^{r-1} \|G - G_h\|_{W_1^1(\Omega_{J+1})} \leqslant Ch^{2r-2}.$$

For $\alpha = (r-2)/(2r-2)$, *estimate* (5.4) *holds with* C *replaced by* $C\ln(h^{-1})$. *Moreover*, C *is independent of* x_0.

In order to understand the normalization of g, one may think of g and G as smooth approximations of the Dirac measure at x_0 and the corresponding Green function, respectively. We devote all of Section 6 to the proof of Lemma 5.2, and we proceed now to prove (5.2).

Proof of Theorem 5.1. By a sequence of standard arguments, using the triangle inequality, the inverse property (4.1) with $s = t = 0$, $p = \infty$, and $q = 2$, Hölder's inequality, and the assumption $\mathrm{diam}(\tau_0) \sim h^{1/(1-\alpha)}$, we have, for any $\chi_h \in S_h$,

$$
\begin{aligned}
|(u - u_h)(x_0)| &\leqslant |(u - \chi_h)(x_0)| + |(\chi_h - u_h)(x_0)| \\
&\leqslant |(u - \chi_h)(x_0)| + C\,\mathrm{diam}^{-N/2}(\tau_0)\|\chi_h - u_h\|_{L_2(\tau_0)} \\
(5.5) \qquad &\leqslant |(u - \chi_h)(x_0)| + C\,\mathrm{diam}^{-N/2}(\tau_0)\|u - \chi_h\|_{L_2(\tau_0)} \\
&\quad + C\,\mathrm{diam}^{-N/2}(\tau_0)\|u - u_h\|_{L_2(\tau_0)} \\
&\leqslant C\|u - \chi_h\|_{L_\infty(\tau_0)} + Ch^{-N/(2-2\alpha)}\|u - u_h\|_{L_2(\tau_0)}.
\end{aligned}
$$

By Proposition 4.2 there is a $\chi_h \in S_h$ such that

$$\|u - \chi_h\|_{L_\infty(\tau_0)} \leqslant C \operatorname{diam}^r(\tau_0)\|u\|_{W_\infty^r},$$

which, by the refinement assumption $\operatorname{diam}(\tau_0) \leqslant Ch^{1/(1-\alpha)}$ and the choice of $\alpha \geqslant (r-2)/(2r-2)$, can be continued to

$$(5.6) \qquad \|u - \chi_h\|_{L_\infty(\tau_0)} \leqslant C \operatorname{diam}^r(\tau_0)\|u\|_{W_\infty^r} \leqslant Ch^{2r-2}\|u\|_{W_\infty^r}.$$

It now remains to estimate $\|u - u_h\|_{L_2(\tau_0)}$. By L_2 duality we have

$$(5.7) \qquad \|u - u_h\|_{L_2(\tau_0)} = \sup(u - u_h, g)h^{N/(2-2\alpha)},$$

where the sup is taken over all g as in Lemma 5.2. With G, G_h, and u_h defined by (5.3) and (5.1), we can write, for any $\chi_h \in S_h$,

$$(5.8) \quad (u - u_h, g) = A(u - u_h, G) = A(u - u_h, G - G_h) = A(u - \chi_h, G - G_h).$$

Let the integer J be as in Lemma 5.2. Using Corollary 4.2, we see that there is a $\chi_h \in S_h$ such that, by the continuity of A,

$$A(u - \chi_h, G - G_h)$$

$$\leqslant C \bigg\{ \sum_{j \leqslant J} \|u - \chi_h\|_{W_\infty^1(D_j)}\|G - G_h\|_{W_1^1(D_j)}$$

$$(5.9) \qquad\qquad\qquad + \|u - \chi_h\|_{W_\infty^1(\Omega_{J+1})}\|G - G_h\|_{W_1^1(\Omega_{J+1})} \bigg\}$$

$$\leqslant C\|u\|_{W_\infty^r} \bigg\{ \sum_{j \leqslant J} h_j^{r-1}\|G - G_h\|_{W_1^1(D_j)} + h_J^{r-1}\|G - G_h\|_{W_1^1(\Omega_{J+1})} \bigg\}.$$

The theorem now follows from (5.5) through (5.9) and by Lemma 5.2. $\square$

6. Proof of Lemma 5.2. Let us first recall from Section 3 the notation $d_j = 2^{-j}$, $h_j = hd_j^\alpha$, $\Omega_j = B_{d_j}(x_0) \cap \Omega$, $D_j = \Omega_j \setminus \overline{\Omega}_{j+1}$, and $D_j^k = D_{j-k} \cup \cdots \cup D_{j+k}$. For given α with $0 \leqslant \alpha < 1$, we have also defined the function $J_1 = J_1(h)$ by $2^{-J_1} = C_1 h^{1/(1-\alpha)}$, where the constant C_1 was chosen large enough to imply, for all $j \leqslant J_1$, the relative smallness of the mesh on D_j necessary in order to derive, under the assumption $\operatorname{dist}(D', \Omega \setminus D'') > c_1 d_j$, the local inverse estimate (4.4) and the approximation properties (4.8) and (4.9). It was also pointed out that, for $j \leqslant J_1$, the mesh size on D_j is proportional to h_j and $h_j d_j^{-1} \leqslant C$. It follows immediately from our definitions that

$$(6.1) \qquad\qquad d_j = 2d_{j+1} \quad \text{and} \quad h_j = 2^\alpha h_{j+1}.$$

This will be used frequently throughout the proofs, sometimes without explicit mention.

For the proof of Lemma 5.2, we shall need the following two preliminary lemmas. Lemma 6.1 improves the standard H^1-error estimate

$$\|G - G_h\|_1 \leqslant Ch\|g\|_0$$

for the special G considered. Lemma 6.2 shows local H^1-estimates for $G - G_h$ without any specific assumptions on G. Proofs of these lemmas can be found in Eriksson [5]. Just take as the form in [5] the adjoint of A. (Cf. also the proof of Lemma 9.2 below.)

LEMMA 6.1. *For given α, with $0 \leqslant \alpha < 1$, there is a constant C such that the following holds for any integer $J \leqslant J_1$: Let $g \in L_2$ be supported in τ_0, where $\bar{\tau}_0$ contains x_0. Define G by*

$$(6.2) \qquad\qquad A(v, G) = (v, g) \quad \forall v \in H^1,$$

and let $G_h \in S_h(r, x_0, \alpha)$ be the A^*-projection of G. Then, for $E = G - G_h$,

$$(6.3) \qquad\qquad \|E\|_1 \leqslant Ch_J \|g\|_0.$$

LEMMA 6.2. *For given α with $0 \leqslant \alpha < 1$ and $r \geqslant 2$, there is a constant C such that the following holds for any integer $j \leqslant J_1$: Let $G \in H^r(D_j^1)$, and let $G_h \in S_h = S_h(r, x_0, \alpha)$ satisfy*

$$(6.4) \qquad\qquad A(\chi_h, G - G_h) = 0 \quad \forall \chi_h \in S_h^0(D_j^1).$$

Then, for $E = G - G_h$,

$$(6.5) \qquad\qquad \|E\|_{1, D_j} \leqslant Ch_j^{r-1}\|G\|_{r, D_j^1} + Cd_j^{-1}\|E\|_{0, D_j^1}.$$

Proof of Lemma 5.2. For fixed α with $(r-2)/(2r-2) < \alpha < 1$, we shall prove the existence of a constant C and an integer $J \leqslant J_1$ such that

$$(6.6) \qquad \sum_{j \leqslant J} h_j^{r-1}\|E\|_{W_1^1(D_j)} + h_J^{r-1}\|E\|_{W_1^1(\Omega_{J+1})} \leqslant Ch^{2r-2},$$

where $E = G - G_h$ is defined as in Lemma 6.1 with the additional normalization $\|g\|_0 = h^{-N/2-2\alpha}$. By the refinement assumption $\mathrm{diam}(\tau_0) \sim h^{1/(1-\alpha)}$ and the normalization of g, we have, by Schwarz' inequality,

$$(6.7) \qquad\qquad \|g\|_{L_1(\tau_0)} \leqslant \mathrm{diam}^{N/2}(\tau_0)\|g\|_0 \leqslant C.$$

The choice of the integer $J \leqslant J_1$ will be made later on in the proof by fixing a constant C_* (sufficiently large) and defining J by the inequalities

$$(6.8) \qquad\qquad 2^{-J} = d_J \leqslant C_* h^{1/(1-\alpha)} < 2d_J = 2^{-(J-1)}.$$

By the definition of h_j we then also have

$$(6.9) \qquad\qquad h_J \leqslant C_*^\alpha h^{1/(1-\alpha)} < 2^\alpha h_J.$$

We now set out to prove (6.6) by first estimating the term associated with Ω_{J+1}. By using, in turn, Schwarz' inequality, Lemma 6.1, the normalization of g, (6.8), and (6.9), we have, for $(r-2)/(2r-2) \leqslant \alpha < 1$,

$$h_J^{r-1}\|E\|_{W_1^1(\Omega_{J+1})} \leqslant h_J^{r-1}d_J^{N/2}\|E\|_1 \leqslant Ch_J^r d_J^{N/2}\|g\|_0$$

$$(6.10) \qquad\qquad \leqslant Ch_J^r d_J^{N/2}h^{-N/(2-2\alpha)} \leqslant CC_*^{N/2}h_J^r$$

$$\leqslant CC_*^{N/2+\alpha r}h^{r/(1-\alpha)} \leqslant C(C_*)h^{2r-2},$$

which is the desired estimate.

In order to get a similar estimate for the sum in (6.6), we first apply Schwarz' inequality to each term to obtain

$$(6.11) \qquad \sum_{j \leqslant J} h_j^{r-1}\|E\|_{W_1^1(D_j)} \leqslant C \sum_{j \leqslant J} h_j^{r-1}d_j^{N/2}\|E\|_{1, D_j}.$$

We shall proceed to show, with

$$S = \sum_{j \leqslant J} h_j^{r-1} d_j^{N/2} \|E\|_{1,D_j},$$

that, provided C_* is chosen sufficiently large, we have

(6.12) $$S \leqslant \tfrac{1}{2} S + Ch^{2r-2}.$$

This would imply $S \leqslant Ch^{2r-2}$ and, thus, together with (6.11) and (6.10), show the desired estimate (6.6). To complete the proof of Lemma 5.2, we shall also prove that (6.12) holds for $\alpha = (r-2)/(2r-2)$ if C is replaced by $C\ln(h^{-1})$. Moreover, as is easily seen by tracking constants, we shall prove (6.6) with C depending only on N, Ω, A, α, r, and given constants. That is, the final claim of Lemma 5.2 will be fulfilled.

In order to show (6.12) we first split the sum S and then make use of Lemma 6.2. We have

(6.13)
$$\begin{aligned}
S &= \sum_{j=J-2}^{J} + \sum_{j \leqslant J-3} = S_1 + \sum_{j \leqslant J-3} h_j^{r-1} d_j^{N/2} \|E\|_{1,D_j} \\
&\leqslant S_1 + C \sum_{j \leqslant J-3} h_j^{2r-2} d_j^{N/2} \|G\|_{r,D_j^1} \\
&\quad + C \sum_{j \leqslant J-3} h_j^{r-1} d_j^{N/2-1} \|E\|_{0,D_j^1} \\
&\leqslant S_1 + S_2 + C_5 \sum_{j \leqslant J-2} h_j^{r-1} d_j^{N/2-1} \|E\|_{0,D_j},
\end{aligned}$$

where we have also used (6.1) in the last step.

The sum S_1, only consisting of three terms, has essentially already been estimated. In fact, by (6.1) and part of (6.10), we have

(6.14) $$S_1 \leqslant 3h_{J-2}^{r-1} d_{J-2}^{N/2} \|E\|_1 \leqslant Ch_J^{r-1} d_J^{N/2} \|E\|_1 \leqslant C(C_*)h^{2r-2}.$$

For the estimation of S_2 we first note that, since D_j and $\mathrm{supp}(g)$ are separated by a distance of order d_j, we have, from Lemma 2.3,

$$\|G\|_{W_\infty^r(D_j)} \leqslant Cd_j^{-N+2-r} \|g\|_{L_1(\tau_0)}.$$

By Hölder's inequality and (6.7), this shows

(6.15) $$\|G\|_{r,D_j} \leqslant Cd_j^{N/2} \|G\|_{W_\infty^r(D_j)} \leqslant Cd_j^{-N/2+2-r}.$$

Let j_1 be the smallest integer for which $D_j \neq \varnothing$. Using (6.15) we obtain

(6.16)
$$\begin{aligned}
S_2 &\leqslant C \sum_{j \leqslant J-2} h_j^{2r-2} d_j^{N/2} \|G\|_{r,D_j} \leqslant C \sum_{j=j_1}^{J} h_j^{2r-2} d_j^{2-r} \\
&\leqslant Ch^{2r-2} \sum_{j=j_1}^{J} d_j^{\alpha(2r-2)+2-r} \leqslant Ch^{2r-2},
\end{aligned}$$

where we have used that $\alpha > (r-2)/(2r-2)$ in the last step. In the case $\alpha = (r-2)/(2r-2)$, all terms in the sum equal 1, and so, since by (6.8) the number of

terms is proportional to $\ln(h^{-1})$, we have

$$S_2 \leqslant C \ln(h^{-1})h^{2r-2}.$$

It now remains to show that

$$(6.17) \qquad C_5 \sum_{j \leqslant J-2} h_j^{r-1} d_j^{N/2-1} \|E\|_{0,D_j} \leqslant \frac{1}{2} S + C h^{2r-2}.$$

For this we first write, by duality,

$$(6.18) \qquad \|E\|_{0,D_j} = \sup(f_j, E),$$

where the supremum is taken over all functions f_j supported in D_j and of unit L_2-norm. For such f_j we define F_j by

$$(6.19) \qquad A(F_j, v) = (f_j, v) \quad \forall v \in H^1$$

and note that new applications of Hölder's inequality and Lemma 2.3, together with the normalization of f_j, give, for $i \neq j-1, j, j+1$,

$$(6.20) \qquad \begin{aligned} \|F_j\|_{r,D_i} &\leqslant C d_i^{N/2} \|F_j\|_{W_\infty^r(D_i)} \leqslant C d_i^{N/2} d_j^{-N+2-r} \|f_j\|_{L_1(D_j)} \\ &\leqslant C d_i^{N/2} d_j^{-N/2+2-r}. \end{aligned}$$

By (6.19), the definition of E, and the H^1-continuity of A, we have, for arbitrary $\chi_h \in S_h$,

$$(6.21) \qquad \begin{aligned} (f_j, E) = A(F_j, E) &= A(F_j - \chi_h, E) \\ &\leqslant C \|F_j - \chi_h\|_{1,D_j^2} \|E\|_{1,D_j^2} + C \sum_{\substack{i \leqslant J \\ i \neq j-2,\ldots,j+2}} \|F_j - \chi_h\|_{1,D_i} \|E\|_{1,D_i} \\ &\quad + C \|E\|_{1,\Omega_{J+1}} \|F_j - \chi_h\|_{1,\Omega_{J+1}} \\ &= I_1 + I_2 + I_3. \end{aligned}$$

We next intend to estimate I_1, I_2, and I_3 separately, with χ_h approximating F_j as in Proposition 4.2.

By the approximation property (4.8), elliptic H^2-regularity, and the normalization of f_j, we first have, for I_1,

$$I_1 \leqslant C \|E\|_{1,D_j^2} h_j \|F_j\|_2 \leqslant C \|E\|_{1,D_j^2} h_j \|f_j\|_0 \leqslant C h_j \|E\|_{1,D_j^2}.$$

We next use (4.8) and (6.20) to estimate I_2. We have

$$\begin{aligned} I_2 &\leqslant C \sum_{\substack{i \leqslant J \\ i \neq j-2,\ldots,j+2}} \|E\|_{1,D_i} h_i^{r-1} \|F_j\|_{r,D_i^1} \\ &\leqslant C \sum_{\substack{i \leqslant J \\ i \neq j-2,\ldots,j+2}} \|E\|_{1,D_i} h_i^{r-1} d_i^{N/2} d_j^{-N/2+2-r} \\ &\leqslant C d_j^{-N/2+2-r} \sum_{i \leqslant J} h_i^{r-1} d_i^{N/2} \|E\|_{1,D_i} = C d_j^{-N/2+2-r} S. \end{aligned}$$

It remains to estimate I_3. Since we shall only consider F_j with $j \leqslant J-2$, i.e., with $\mathrm{supp}(f_j)$ separated from Ω_J, we use Lemma 2.3 to derive

$$\|F_j\|_{r,\Omega_J} \leqslant C d_J^{N/2} d_j^{-N/2+2-r}.$$

From this, the approximation property (4.8), and again using part of (6.10), we have

$$I_3 \leqslant C\|E\|_{1,\Omega_{j+1}} h_j^{r-1} \|F_j\|_{r,\Omega_j}$$

$$\leqslant C\|E\|_{1,\Omega_{j+1}} h_j^{r-1} d_j^{N/2} d_j^{-N/2+2-r}$$

$$\leqslant C d_j^{-N/2+2-r} h_j^{r-1} d_j^{N/2} \|E\|_1 \leqslant C(C_*) h^{2r-2} d_j^{-N/2+2-r}.$$

Collecting (6.18), (6.21), and the above estimates for I_1, I_2, and I_3, we have shown

$$(6.22) \qquad \|E\|_{0,D_j} \leqslant Ch_j \|E\|_{1,D_j^2} + C d_j^{-N/2+2-r} S + C(C_*) h^{2r-2} d_j^{-N/2+2-r}.$$

Estimate (6.22) inserted into the sum in (6.17) gives

$$C_5 \sum_{j \leqslant J-2} h_j^{r-1} d_j^{N/2-1} \|E\|_{0,D_j}$$

$$\leqslant C \sum_{j \leqslant J} \left(h_j d_j^{-1} \right) h_j^{r-1} d_j^{N/2} \|E\|_{1,D_j}$$

$$(6.23) \qquad + CS \sum_{j \leqslant J} h_j^{r-1} d_j^{1-r} + C(C_*) h^{2r-2} \sum_{j \leqslant J} h_j^{r-1} d_j^{1-r}$$

$$\leqslant C_6 \left(\max_{j \leqslant J} \left(h_j d_j^{-1} \right) + \sum_{j \leqslant J} h_j^{r-1} d_j^{1-r} \right) S + C(C_*) h^{2r-2} \sum_{j \leqslant J} h_j^{r-1} d_j^{1-r},$$

where the constant C_6 is independent of the choice of C_*. As is easily seen from (6.23), all that remains in order to obtain (6.17) is to show that, for a suitable choice of C_*, we have

$$(6.24) \qquad C_6 \left(\max_{j \leqslant J} \left(h_j d_j^{-1} \right) + \sum_{j \leqslant J} h_j^{r-1} d_j^{1-r} \right) \leqslant \frac{1}{2}.$$

By the definition of h_j, the assumption $\alpha < 1$, (6.8), and for C_* sufficiently large, we have

$$(i) \qquad \max_{j \leqslant J} \left(h_j d_j^{-1} \right) = \max_{j \leqslant J} \left(h d_j^{\alpha-1} \right) = h d_J^{\alpha-1} < \frac{2^{1-\alpha}}{C_*^{1-\alpha}} \leqslant \frac{1}{4C_6}.$$

Since $\alpha < 1$ and $r \geqslant 2$, we have, again for C_* sufficiently large,

$$\sum_{j \leqslant J} h_j^{r-1} d_j^{1-r} = h^{r-1} \sum_{j \leqslant J} d_j^{(\alpha-1)(r-1)} = h^{r-1} d_J^{(\alpha-1)(r-1)} \left(1 - 2^{(\alpha-1)(r-1)} \right)^{-1}$$

$$(ii) \qquad \qquad < \frac{2^{(1-\alpha)(r-1)} \left(1 - 2^{(\alpha-1)(r-1)} \right)^{-1}}{C_*^{(1-\alpha)(r-1)}} \leqslant \frac{1}{4C_6},$$

where we have again used the right inequality of (6.8). Now fixing the constant C_* so that both (i) and (ii) hold, we have obtained the desired inequality (6.24) and, thereby, finally proved Lemma 5.2. □

7. High-Order Local Convergence for Derivatives. In this section we shall prove that, for α sufficiently large, we have $O(h^{2r-2})$ convergence at x_0 also for the first-order derivatives. More precisely, we have the following result:

THEOREM 7.1. *Let Ω and the bilinear form A be as in Section 1. Furthermore, let x_0 be an arbitrary point in $\overline{\Omega}$, let $r \geqslant 2$ be an integer, and for h small and positive let there be given a family of partitions $\Delta_h(x_0, \alpha)$ of Ω and the corresponding finite-dimensional*

spaces $S_h = S_h(r, x_0, \alpha)$ *as in Sections 3 and 4. Then, for given* α *with* $1/2 < \alpha < 1$, *there is a constant* C *such that, for* $u \in W_\infty^r$ *and* $u_h \in S_h$, *the A-projection of* u, *the estimate*

$$(7.1) \qquad |\nabla(u - u_h)(x_0)| \leqslant Ch^{2r-2}\|u\|_{W_\infty^r}$$

holds. If $\alpha = 1/2$, *estimate* (7.1) *holds with* C *replaced by* $C\ln(h^{-1})$. *Moreover, the constant* C *is independent of* x_0.

For the proof we shall need the following technical result:

LEMMA 7.2. *Let the element* τ, *the simplex* $\hat{\tau}$, *and the affine transformation* A_τ *satisfy assumption* (SA)$'$ *of Section 3, and let* $r \geqslant 2$ *be an integer. Then there is a constant* $C = C(N, r, c_S', C_S')$ *such that, for any* $p \in P_r(\hat{\tau})$, *we have*

$$\|p\|_{L_\infty(\tau)} \leqslant C \sup(p, g),$$

where the supremum is taken over all $g \in C_0^\infty(\tau)$ *with*

$$(7.2) \qquad \mathrm{diam}^{N/2}(\tau)\|g\|_0 + \mathrm{diam}^{(N+2)/2}(\tau)\|g\|_1 \leqslant 1.$$

Proof. Define, as before, the operator a_τ by $a_\tau v(y) = v(A_\tau^{-1}(y))$. Then

$$\|p\|_{L_\infty(\tau)} \leqslant \|p\|_{L_\infty(\hat{\tau})} = \|a_\tau p\|_{L_\infty(T)}.$$

With c_S' given in assumption (SA)$'$, let $B_{2c'}(y')$ be the largest ball contained in $c_S'T$, and set $B = B_{c'}(y')$. By the equivalence of norms on $P_r(T)$, there is a constant $C = C(N, r, c_S')$ such that

$$\|p'\|_{L_\infty(T)} \leqslant C\|p'\|_{L_\infty(B)} \quad \forall p' \in P_r.$$

Let M be a fixed function in $C_0^\infty(B_1(0))$ satisfying

$$(p', M) = p'(0) \quad \forall p' \in P_r$$

(cf. Hilbert [8]) and set $M_z(y) = c'^{-N}M((y - z)/c')$. Then

$$(p', M_z) = p'(z) \quad \forall p' \in P_r,$$

and, hence,

$$\|p'\|_{L_\infty(B)} = \sup_{z \in B}(p', M_z) \quad \forall p' \in P_r.$$

Noting that $a_\tau p \in P_r$, we now obtain

$$\|p\|_{L_\infty(\tau)} \leqslant C \sup_{z \in B}(a_\tau p, M_z) = C \sup_{z \in B}(p, |\det B_\tau| a_\tau^{-1} M_z),$$

where we have changed variables in the last step. To prove (7.2) it is then sufficient to show that, for some (small) constant δ, the functions $g = \delta |\det B_\tau| a_\tau^{-1} M_z$, for $z \in B$, satisfy $g \in C_0^\infty(\tau)$ and (7.2). However, since $M \in C_0^\infty(B_1(0))$, it follows from the definitions of B, M_z, and a_τ^{-1} and from (SA)$'$ that

$$g = \delta |\det B_\tau| a_\tau^{-1} M_z \in C_0^\infty(\tau).$$

Moreover, since M is bounded, so is $a_\tau^{-1}M_z$, and hence by Hölder's inequality and for suitable δ's,

$$\mathrm{diam}^{N/2}(\tau)\|g\|_{0,\tau} \leqslant \mathrm{diam}^N(\tau)\|g\|_{L_\infty} \leqslant \delta \, \mathrm{diam}^N(\tau)|\det B_\tau| \, \|a_\tau^{-1}M_z\|_{L_\infty} \leqslant 1/2.$$

Similarly,

$$\text{diam}^{(N+2)/2}(\tau)\|g\|_{1,\tau} \leqslant \delta C \, \text{diam}^{N+1}(\tau)|\det B_\tau| \, \|a_\tau^{-1} M_z\|_{W_\infty^1}$$

$$\leqslant C\delta \, \text{diam}(\tau)\|a_\tau^{-1} M_z\|_{W_\infty^1} \leqslant 1/2,$$

which completes the proof. □

Proof of Theorem 7.1. We shall prove (7.1) with ∇ replaced by any $\partial_j = \partial/\partial x_j$. Let $x_0 \in \bar\tau_0$. We shall use Lemma 7.2 with $\tau = \tau_0$ and $p = \partial_j(\chi_h - u_h) \in P_{r-1}(\hat\tau_0)$. Using the triangle inequality, Lemma 7.2, and Hölder's inequality, we then have, for any $\chi_h \in S_h$,

$$|(\partial_j u - \partial_j u_h)(x_0)| \leqslant |(\partial_j u - \partial_j \chi_h)(x_0)| + |(\partial_j \chi_h - \partial_j u_h)(x_0)|$$

$$\leqslant |(\partial_j u - \partial_j \chi_h)(x_0)| + C \sup(\partial_j \chi_h - \partial_j u_h, g)$$

$$\leqslant \|\partial_j u - \partial_j \chi_h\|_{L_\infty(\tau_0)} + C \sup(\partial_j u - \partial_j \chi_h, g)$$

(7.3)
$$+ C \sup(\partial_j u - \partial_j u_h, g)$$

$$\leqslant \|\partial_j u - \partial_j \chi_h\|_{L_\infty(\tau_0)} + C \sup \|\partial_j u - \partial_j \chi_h\|_{L_\infty(\tau_0)} \|g\|_{L_1(\tau_0)}$$

$$+ C \sup(\partial_j u - \partial_j u_h, g),$$

where the supremum is taken over all $g \in C_0^\infty(\tau_0)$ with

$$\text{diam}^{N/2}(\tau_0)\|g\|_0 + \text{diam}^{(N+2)/2}(\tau_0)\|g\|_1 \leqslant 1.$$

Since, for $\alpha \geqslant 1/2$, the refinement assumption (RA) implies

$$\text{diam}(\tau_0) \leqslant Ch^{1/(1-\alpha)} \leqslant Ch^2,$$

Proposition 4.2 yields a $\chi_h \in S_h$ such that

(7.4) $$\|\partial_j u - \partial_j \chi_h\|_{L_\infty(\tau_0)} \leqslant C \, \text{diam}^{r-1}(\tau_0)\|u\|_{W_\infty^r} \leqslant Ch^{2r-2}\|u\|_{W_\infty^r}.$$

Moreover, by Schwarz' inequality and the normalization of g, we have

(7.5) $$\|g\|_{L_1(\tau_0)} \leqslant \text{diam}^{N/2}(\tau_0)\|g\|_0 \leqslant 1,$$

and, hence, the first two terms on the last line of (7.3) can be estimated in the desired way. In the last term we first integrate by parts. We then put $g' = -\partial_j g$ and define $G' \in H^1$ and $G_h' \in S_h$ by

(7.6) $$A(v, G') = (v, g') \quad \forall v \in H^1, \qquad A(\chi_h, G' - G_h') = 0 \quad \forall \chi_h \in S_h.$$

Following the proof of Theorem 5.1 with g, G, and G_h replaced by g', G', and G_h', we then obtain

$$(\partial_j u - \partial_j u_h, g) = (u - u_h, -\partial_j g) = (u - u_h, g') = A(u - u_h, G')$$

$$= A(u - u_h, G' - G_h') = A(u - \chi_h, G' - G_h')$$

$$\leqslant C\Big\{ \sum_{j \leqslant J} \|u - \chi_h\|_{W_\infty^1(D_j)}\|G' - G_h'\|_{W_1^1(D_j)}$$

(7.7)
$$+ \|u - \chi_h\|_{W_\infty^1(\Omega_{J+1})}\|G' - G_h'\|_{W_1^1(\Omega_{J+1})}\Big\}$$

$$\leqslant C\|u\|_{W_\infty^r}\Big\{ \sum_{j \leqslant J} h_j^{r-1}\|G' - G_h'\|_{W_1^1(D_j)} + h_J^{r-1}\|G' - G_h'\|_{W_1^1(\Omega_{J+1})}\Big\}.$$

The desired estimate now follows from (7.3)–(7.7) and by Lemma 7.3 below. The independence of x_0 of the constant C is easily checked. This completes the proof of Theorem 7.1. $\square$

LEMMA 7.3. *Let the basic assumptions be as in Theorem 7.1. Then for given α, with $1/2 < \alpha < 1$ and J_1 as before, there is a constant C and an integer $J \leqslant J_1$ such that the following holds: Let $x_0 \in \bar{\tau}_0$ and let $g \in C_0^\infty(\tau_0)$ satisfy*

$$\operatorname{diam}^{N/2}(\tau_0)\|g\|_0 + \operatorname{diam}^{(N+2)/2}(\tau_0)\|g\|_1 \leqslant 1.$$

Set $g' = -\partial_j g$ and define $G' \in H^1$ and $G'_h \in S_h$ as in (7.6). Then for $E' = G' - G'_h$,

$$(7.8) \qquad \sum_{j \leqslant J} h_j^{r-1}\|E'\|_{W_1^1(D_j)} + h_J^{r-1}\|E'\|_{W_1^1(\Omega_{J+1})} \leqslant Ch^{2r-2}.$$

If $\alpha = 1/2$, estimate (7.8) holds with C replaced by $C\ln(h^{-1})$.

Proof. We first estimate the term associated with Ω_{J+1}. As in the proof of Lemma 5.2, we shall choose $J \leqslant J_1$ such that

$$d_J \leqslant C_* h^{1/(1-\alpha)} < 2d_J \quad \text{and} \quad h_J \leqslant C_*^\alpha h^{1/(1-\alpha)} < 2^\alpha h_J,$$

where the constant C_* will be determined later in the proof. Then, by the refinement assumption, h_J, d_J, $h^{1/(1-\alpha)}$, and $\operatorname{diam}(\tau_0)$ are all of the same order. Hence, using Schwarz' inequality, Lemma 6.1, and the bound for g, we have, for $\alpha \geqslant 1/2$,

$$(7.9) \qquad \begin{aligned} h_J^{r-1}\|E'\|_{W_1^1(\Omega_{J+1})} &\leqslant h_J^{r-1}d_J^{N/2}\|E'\|_1 \leqslant Ch_J^{r-1}d_J^{N/2}h_J\|g'\|_0 \\ &\leqslant C(C_*)h^{(r-1)/(1-\alpha)}\operatorname{diam}^{(N+2)/2}(\tau_0)\|g\|_1 \leqslant C(C_*)h^{2r-2}, \end{aligned}$$

which shows the desired estimate for this term.

We now proceed to estimate the sum in (7.8). Setting

$$S = \sum_{j \leqslant J} h_j^{r-1}d_j^{N/2}\|E'\|_{1,D_j},$$

we have, by Schwarz' inequality,

$$(7.10) \qquad \sum_{j \leqslant J} h_j^{r-1}\|E'\|_{W_1^1(D_j)} \leqslant CS.$$

We shall first show that

$$(7.11) \qquad S \leqslant Ch^{2r-2} + C\sum_{j \leqslant J-2} h_j^{2r-2}d_j^{N/2}\|G'\|_{r,D_j}.$$

We shall again use the obvious equalities (6.1), $d_j = 2d_{j+1}$, and $h_j = 2^\alpha h_{j+1}$. In order to estimate the three last terms of S, we shall use part of (7.9), and to the rest of the terms we apply Lemma 6.2. We then have

$$\begin{aligned} S = \sum_{j=J-2}^{J} + \sum_{j \leqslant J-3} &\leqslant 3h_{J-2}^{r-1}d_{J-2}^{N/2}\|E'\|_1 + \sum_{j \leqslant J-3} h_j^{r-1}d_j^{N/2}\|E'\|_{1,D_j} \\ &\leqslant Ch_J^{r-1}d_J^{N/2}\|E'\|_1 + C\sum_{j \leqslant J-3} h_j^{2r-2}d_j^{N/2}\|G'\|_{r,D_j^1} \end{aligned}$$

$$(7.12) \qquad \begin{aligned} &+ C\sum_{j \leqslant J-3} h_j^{r-1}d_j^{N/2-1}\|E'\|_{0,D_j^1} \\ &\leqslant C(C_*)h^{2r-2} + C\sum_{j \leqslant J-2} h_j^{2r-2}d_j^{N/2}\|G'\|_{r,D_j} \\ &+ C_5'\sum_{j \leqslant J-2} h_j^{r-1}d_j^{N/2-1}\|E'\|_{0,D_j}. \end{aligned}$$

In order to estimate the last sum, we proceed, as in Section 6, by first estimating $\|E'\|_{0, D_j}$. Following (6.18)–(6.21), we obtain three terms, I_1', I_2', and I_3', which have to be estimated. The terms I_1' and I_2' are bounded exactly as in Section 6 but for the additional "primes". In order to bound I_3' we use part of (7.9) instead of (6.10). As in Section 6, if C_* is sufficiently large, we finally arrive at the estimate

$$(7.13) \qquad C_5' \sum_{j \leqslant J-2} h_j^{r-1} d_j^{N/2-1} \|E'\|_{0, D_j} \leqslant C(C_*) h^{2r-2} + \frac{1}{2} S.$$

By a simple kickback argument, (7.11) follows from (7.12) and (7.13). It now remains to show

$$(7.14) \qquad \sum_{j \leqslant J-2} h_j^{2r-2} d_j^{N/2} \|G'\|_{r, D_j} \leqslant Ch^{2r-2}.$$

Using the representation (2.5) we have, for G',

$$G'(x) = \int_\Omega \mathscr{G}(x, y) g'(y)\, dy = -\int_\Omega \mathscr{G}(x, y)\, \partial_j g(y)\, dy.$$

Differentiation under the integral sign and integration by parts give

$$D^\beta G'(x) = \int_\Omega \partial_j D_x^\beta \mathscr{G}(x, y) g(y)\, dy.$$

For $j \leqslant J_1$ and any $x \in D_j$ we have that $\mathrm{dist}(x, \mathrm{supp}(g)) \geqslant c d_j$. Thus, Hölder's inequality and estimate (2.6) yield

$$\left| D^\beta G'(x) \right| \leqslant C d_j^{-N+2-|\beta|-1} \|g\|_{L_1(\tau_0)},$$

and, hence,

$$\|G'\|_{W_\infty^r(D_j)} \leqslant C d_j^{-N+1-r} \|g\|_{L_1(\tau_0)}.$$

By Schwarz' inequality and the previously obtained estimate $\|g\|_{L_1(\tau_0)} \leqslant C$, we have

$$\|G'\|_{r, D_j} \leqslant C d_j^{-N/2+1-r}.$$

Inserted in (7.14), with j_1 as before, for fixed $\alpha > 1/2$, this gives

$$\sum_{j \leqslant J-2} h_j^{2r-2} d_j^{N/2} \|G'\|_{r, D_j} \leqslant C \sum_{j=j_1}^J h_j^{2r-2} d_j^{1-r}$$

$$= Ch^{2r-2} \sum_{j=j_1}^J d_j^{\alpha(2r-2)+1-r} \leqslant Ch^{2r-2}.$$

This shows (7.14). In order to complete the proof of Lemma 7.3, we note that in the case $\alpha = 1/2$ all terms in the last sum equal 1. Estimating the number of terms, we obtain the final bound $C \ln(h^{-1}) h^{2r-2}$. □

8. A Negative-Norm Estimate. In this section we shall prove a negative-norm estimate referred to in Sections 9 and 10. The result is well-known from, e.g., Bramble and Osborn [3], but will be proved here for completeness. However, we shall first show the following result on approximation properties of the spaces $S_h(\Omega')$.

PROPOSITION 8.1. *Let the domain Ω, the bilinear form A, and the family of finite-dimensional spaces $S_h = S_h(r, x_0, \alpha)$ be as before. Moreover, let Ω' be a subdomain on Ω with C^∞ boundary $\partial\Omega'$. Then there is a constant C, and for each given $v \in L_1(\Omega')$ a $\chi_h \in S_h(\Omega')$ such that the following holds: If $j \leqslant J_1$, $D' \subset D'' \subset (\Omega' \cap B_{j-1}(x_0))$, $\mathrm{dist}(D', \Omega' \setminus D'') \geqslant c_1 d_j$, $1 \leqslant p \leqslant \infty$, $0 \leqslant l \leqslant 1 \leqslant m \leqslant r$, and if $v \in W_p^m(D'')$, then*

$$(8.1) \qquad \|v - \chi_h\|_{W_p^l(D')} \leqslant C h_j^{m-l} \|v\|_{W_p^m(D'')}.$$

In particular, if $v \in H^r(\Omega')$, then

$$(8.2) \qquad \|v - \chi_h\|_{1,\Omega'} \leqslant C h^{r-1} \|v\|_{r,\Omega'}.$$

Proof. There exist (cf. Eriksson [5]) an extension operator $E: L_1(\Omega') \to L_1(R^N)$ and constants C and c_E such that, for $x \in \overline{\Omega}'$,

$$\|Ev\|_{W_p^m(B_d(x))} \leqslant C\|v\|_{W_p^m(B_{c_E d}(x) \cap \Omega')}.$$

Let χ_h approximate Ev as in Proposition 4.2. Let τ' be an element of the partition of Ω' induced by Δ_h. Following the proof of Proposition 4.2 given in [5], we obtain the estimate

$$(8.3) \qquad \|v - \chi_h\|_{W_p^l(\tau')} \leqslant C \,\mathrm{diam}^{m-l}(\tau) \|v\|_{W_p^m(O_{\tau'})},$$

where $\tau' \subset \tau \in \Delta_h$, and where $O_{\tau'}$ is a neighborhood of τ' in Ω' having diameter of order h_j. Note also that, by the choice of J_1, we can assume that condition (4.7) holds for $O_{\tau'}$. But then estimates (8.1) and (8.2) are easily derived from (8.3) by the same means as we earlier obtained (4.8) and (4.9). $\square$

Proposition 8.1 will be used in Sections 9 and 10. In particular, (8.2) implies the desired negative-norm estimate:

THEOREM 8.2. *Let the domain Ω, the bilinear form A, and the family of finite-dimensional spaces S_h be as before. Moreover, let Ω' be a subdomain of Ω with C^∞ boundary $\partial\Omega'$. Then there is a constant C such that, for $u \in H^r(\Omega')$ and $u_h \in S_h(\Omega')$ its $A_{\Omega'}$-projection*

$$(8.4) \qquad \|\|u - u_h\|\|_{-r+2,\Omega'} \leqslant C h^{2r-2} \|u\|_{r,\Omega'}.$$

Proof. Our task is to show that, for some constant C and any $w \in C^\infty(\Omega')$, we have

$$(8.5) \qquad (u - u_h, w)_{\Omega'} \leqslant C h^{2r-2} \|u\|_{r,\Omega'} \|w\|_{r-2,\Omega'}.$$

Define W by

$$(8.6) \qquad A(v, W)_{\Omega'} = (v, w)_{\Omega'} \quad \forall v \in H^1(\Omega'),$$

and recall from Lemma 2.2 that

$$(8.7) \qquad \|W\|_{r,\Omega'} \leqslant C \|w\|_{r-2,\Omega'}.$$

By the definition of u_h and the continuity of $A_{\Omega'}$, we have, for any $\chi_h \in S_h(\Omega')$,

$$(8.8) \quad A(u - u_h, W)_{\Omega'} = A(u - u_h, W - \chi_h)_{\Omega'} \leqslant C \|u - u_h\|_{1,\Omega'} \|W - \chi_h\|_{1,\Omega'}.$$

By (8.6), (8.8), and the standard $H^2(\Omega')$ estimate

$$\|u - u_h\|_{1,\Omega'} \leqslant C \inf_{\chi_h \in S_h(\Omega')} \|u - \chi_h\|_{1,\Omega'},$$

easily obtained from the coercivity and continuity of $A_{\Omega'}$, and the definition of u_h, we have

$$\begin{aligned}(u - u_h, w)_{\Omega'} &= A(u - u_h, W)_{\Omega'}\\ &\leqslant C \inf_{\chi_h \in S_h(\Omega')} \|u - \chi_h\|_{1,\Omega'} \inf_{\chi_h \in S_h(\Omega')} \|W - \chi_h\|_{1,\Omega'}.\end{aligned}$$

The desired estimate (8.5) then follows by (8.2) of Proposition 8.1 and by the regularity result (8.7). $\square$

9. High-Order Convergence for u Only Locally in W_∞^r. Here we shall relax the regularity assumptions on u away from the point x_0 and show that $O(h^{2r-2})$ convergence of $(u - u_h)(x_0)$ can still hold. This will be done by localizing the arguments used in Sections 5 and 6 to a neighborhood $\Omega^0 \subset \Omega$ of x_0. A pollution term in a so-obtained local estimate will be estimated using Theorem 8.2.

Since we consider also the case when x_0 is close to, or even on the boundary $\partial\Omega$, we shall have to introduce for m a nonnegative integer and for $\Omega' \subset \Omega$, the following special negative norm:

$$\|v\|_{-m,\Omega'}^* = \sup_{\substack{w \in C^\infty(\Omega) \\ \operatorname{supp}(w) \cap \overline{\Omega \setminus \Omega'} = \varnothing}} \frac{(v, w)}{\|w\|_m}.$$

Obviously, for $\Omega' \subset \Omega''$ and $m' \geqslant m''$, we have

$$\|v\|_{-m',\Omega'}^* \leqslant \|v\|_{-m'',\Omega''}^*,$$

and our new norm is related to those defined in Section 1 by

$$\|v\|_{-m,\Omega'} \leqslant \|v\|_{-m,\Omega'}^* \leqslant \interleave v \interleave_{-m,\Omega'}.$$

Besides, if $\overline{\Omega}' \subset \Omega$ or $m = 0$, then

$$\|v\|_{-m,\Omega'}^* = \|v\|_{-m,\Omega'}.$$

We are now ready to state the announced result in precise terms.

THEOREM 9.1. *Let the domain Ω, the bilinear form A, and the family of finite-dimensional spaces $S_h = S_h(r, x_0, \alpha)$ be as before. Let Ω^0 be a subdomain of Ω with $\operatorname{dist}(x_0, \Omega \setminus \Omega^0) = c_0 > 0$, and let $m \geqslant 0$ be an integer. Then, for given α with $(r - 2)/(2r - 2) < \alpha < 1$, there is a constant C such that, if $u \in W_\infty^r(\Omega^0)$ and $u_h \in S_h(\Omega^0)$ satisfy*

$$A(u - u_h, \chi_h) = 0 \quad \forall \chi_h \in S_h^0(\Omega^0),$$

then

$$(9.1) \qquad |(u - u_h)(x_0)| \leqslant Ch^{2r-2}\|u\|_{W_\infty^r(\Omega^0)} + C\|u - u_h\|_{-m,\Omega^0}^*.$$

In particular, if $u \in H^r$ and $u_h \in S_h$ is its A-projection, then

$$(9.2) \qquad |(u - u_h)(x_0)| \leqslant Ch^{2r-2}\big(\|u\|_{W_\infty^r(\Omega^0)} + \|u\|_r\big).$$

If $\alpha = (r - 2)/(2r - 2)$, then (9.1) and (9.2) hold with C replaced by $C\ln(h^{-1})$.

Remark. The constant C in (9.1) and (9.2) depends on c_0, but is, in fact, independent of x_0.

Below we present two lemmas which will be used in the proof of Theorem 9.1. These lemmas are proved at the end of this section. For A_{Ω^0} we shall use the shorter notation A_0. In the proof of Theorem 9.1 we shall estimate the two terms $(U - U_h)(x_0)$ and $(U_h - u_h)(x_0)$, where $U_h \in S_h(\Omega^0)$ is the A_0-projection of $U = u|_{\Omega^0}$. The term $(U - U_h)(x_0)$ will be estimated following essentially the proof of Theorem 5.1. However, for this we shall need the following analog of Lemma 6.2:

LEMMA 9.2. *Let Ω^0 be as in Theorem 9.1 and have a C^∞ boundary $\partial\Omega^0$, and let j_0 be the smallest integer for which $\Omega_{j_0} \subset \Omega^0$, where $\Omega_{j_0} = B_{j_0} \cap \Omega$ as before. Set $D'' = \Omega^0 \setminus \overline{\Omega}_{j_0+2}$ and $D' = \Omega^0 \setminus \overline{\Omega}_{j_0+1}$. Then there is a constant C such that, if $G \in H^r(D'')$ and $G_h \in S_h(\Omega^0)$ satisfies $A_0(\chi_h, G - G_h) = 0 \ \forall \chi_h \in S_h(\Omega^0)$, such that $\chi_h = 0$ on $\Omega^0 \setminus D''$, then*

$$(9.3) \qquad \|G - G_h\|_{1,D'} \leqslant Ch^{r-1}\|G\|_{r,D''} + C\|G - G_h\|_{0,D''}.$$

In order to estimate the term $(U_h - u_h)(x_0)$, we shall use the following result:

LEMMA 9.3. *Let Ω^0 be as in Theorem 9.1, and let $m' \geqslant 0$ be an integer. Then there is a constant C such that, if $v_h \in S_h(\Omega^0)$ satisfies $A(v_h, \chi_h) = 0 \ \forall \chi_h \in S_h^0(\Omega^0)$, then*

$$(9.4) \qquad |v_h(x_0)| \leqslant C\|v_h\|_{-m',\Omega^0}^*.$$

Proof of Theorem 9.1. We may assume, without loss of generality, that $\partial\Omega^0$ is smooth. Set $U = u|_{\Omega^0}$ and let $U_h \in S_h(\Omega^0)$ be its A_0-projection. Then on Ω^0,

$$(9.5) \qquad u - u_h = (u - U) + (U - U_h) + (U_h - u_h) = 0 + e_1 + e_2.$$

We first claim that

$$(9.6) \qquad |e_1(x_0)| = |(U - U_h)(x_0)| \leqslant Ch^{2r-2}\|U\|_{W_\infty^r(\Omega^0)},$$

where C is to be replaced by $C\ln(h^{-1})$ in the case $\alpha = (r-2)/(2r-2)$. This is proved in the same way as Theorem 5.1, but, instead of the subdomains $\Omega_{J+1}, D_J, D_{J-1}, \ldots$ of Ω, we now consider Ω^0 subdivided into the domains $\Omega_{J+1}, D_J, \ldots, D_{j_0+1}$, and D'. The approximation properties needed for the spaces $S_h(\Omega^0)$ are given by Proposition 8.1, and the required counterpart of Lemma 6.2 for the domain D' is given by Lemma 9.2.

We shall next show that

$$(9.7) \qquad |e_2(x_0)| \leqslant Ch^{2r-2}\|U\|_{r,\Omega^0} + C\|u - u_h\|_{-m,\Omega^0}^*.$$

Since $U = u$ on Ω^0, and since $\|\cdot\|_{r,\Omega^0} \leqslant C\|\cdot\|_{W_\infty^r(\Omega^0)}$, (9.7) together with (9.5) and (9.6) will show (9.1). In order to obtain (9.7), set $m' = \max(m, r-2)$ and note that, for any $\chi_h \in S_h^0(\Omega^0)$, we have

$$A(e_2, \chi_h) = A(u - u_h, \chi_h) - A_0(U - U_h, \chi_h) = 0 - 0 = 0.$$

Hence, by Lemma 9.3,

$$|e_2(x_0)| \leqslant C\|e_2\|_{-m',\Omega^0}^*.$$

Using first the triangle inequality and the fact that $U = u$ on Ω^0, and then that $\|\cdot\|_{-m',\Omega^0}^*$ is dominated by both $\|\cdot\|_{-m,\Omega^0}^*$ and $\||\cdot\||_{-r+2,\Omega^0}$, we have

$$\|e_2\|_{-m',\Omega^0}^* \leqslant \|u - u_h\|_{-m,\Omega^0}^* + \||U - U_h\||_{-r+2,\Omega^0}.$$

From here the negative-norm estimate (8.4) settles (9.7) and, thereby, (9.1). In particular, if $u \in H^r$ and $u_h \in S_h$ is its A-projection, then (9.1) with $m = r - 2$, the norm inequality $\| \cdot \|^*_{-m,\Omega^0} \leqslant \||\cdot\||_{-m}$, and the negative-norm estimate (8.4) together show (9.2). This completes the proof. □

Remark 9.1. Since (9.1) is entirely a local estimate, it can be used to obtain estimates similar to (9.2) also in other situations where high-order negative-norm estimates for $u - u_h$ are known, such as, in connection with various procedures for treating the Dirichlet problem.

Proof of Lemma 9.2. We introduce for the proof domains $D^{(j)}$, $j = 0, \ldots, 4$, such that $D' = D^{(0)} \subset D^{(1)} \subset \cdots \subset D^{(4)} = D''$ and with $\mathrm{dist}(D^{(j)}, \Omega^0 \setminus D^{(j+1)}) \geqslant c > 0$. Following the proof of Lemma 6.2 given in [5], we shall first show that the desired estimate (9.3) is implied by the statement: There is a constant C such that, if $v \in H^1(D^{(3)})$ and $v_h \in S_h(\Omega^0)$ satisfy $A_0(\chi_h, v - v_h) = 0 \ \forall \chi_h \in S_h(\Omega^0)$, such that $\chi_h = 0$ on $\Omega^0 \setminus D^{(3)}$, then

$$(9.8) \qquad \|v - v_h\|_{1,D'} \leqslant C\|v\|_{1,D^{(3)}} + C\|v - v_h\|_{0,D^{(3)}}.$$

For, set $v = G - \chi_h$ and $v_h = G_h - \chi_h$, where $\chi_h \in S_h(\Omega^0)$ approximates G. Then by (9.8) and Proposition 8.1,

$$\|G - G_h\|_{1,D'} = \|G - \chi_h - (G_h - \chi_h)\|_{1,D'} \leqslant C\|G - \chi_h\|_{1,D^{(3)}} + C\|G - G_h\|_{0,D^{(3)}}$$

$$\leqslant Ch^{r-1}\|G\|_{r,D''} + C\|G - G_h\|_{0,D''},$$

which shows (9.3).

For the proof of (9.8) we let η be a smooth cutoff function, vanishing in $\Omega^0 \setminus D^{(3)}$, equal to 1 on $D^{(2)}$, and with $\|\eta\|_{W^1_\infty} \leqslant C_\eta$. We use the notation $(\eta v)_h$ for the A_0-projection of ηv. By the triangle inequality, we have

$$\|v - v_h\|_{1,D'} \leqslant \|\eta v - (\eta v)_h\|_{1,D'} + \|(\eta v)_h - v_h\|_{1,D'} = I' + I''.$$

Using the boundedness in $H^1(\Omega^0)$ of the A_0-projection and the properties of η, we have

$$I' \leqslant C\|\eta v\|_{1,\Omega^0} \leqslant C\|v\|_{1,D^{(3)}}.$$

In order to estimate I'', we need the following result: There is a constant C such that, if $w_h \in S_h(\Omega^0)$ satisfies $A_0(\chi_h, w_h) = 0 \ \forall \chi_h \in S_h(\Omega^0)$, such that $\chi_h = 0$ on $\Omega^0 \setminus D^{(2)}$, then

$$(9.9) \qquad \|w_h\|_{1,D'} \leqslant C\|w_h\|_{0,D^{(2)}}.$$

Using (9.9) with $w_h = (\eta v)_h - v_h$, we have

$$I'' \leqslant C\|(\eta v)_h - v_h\|_{0,D^{(2)}} \leqslant C\|v - v_h\|_{0,D^{(3)}} + C\|\eta v - (\eta v)_h\|_{0,\Omega^0},$$

where we have also used the triangle inequality and taken larger domains in the last step. Again using the boundedness of the A_0-projection and the properties of η, we have

$$\|\eta v - (\eta v)_h\|_{0,\Omega^0} \leqslant \|\eta v - (\eta v)_h\|_{1,\Omega^0} \leqslant C\|\eta v\|_{1,\Omega^0} \leqslant C\|v\|_{1,D^{(3)}},$$

which completes the proof of (9.8).

We shall now show (9.9). Let D'_h and D''_h be mesh domains on Ω^0 such that $D' \subset D'_h$, $\Omega^0 \setminus D^{(1)} \subset D''_h$, and $\mathrm{dist}(\hat{D}'_h, \hat{D}''_h) \geqslant c > 0$. Let η be a new smooth cutoff

function equal to 1 on $\hat{D}_h'$, vanishing on $\hat{D}_h''$, and satisfying $\|\eta\|_{W_\infty^r} \leq C_\eta$. As in the proof of Lemma 6.2 given in [5], we have

$$c_A \|w_h\|_{1,D'}^2 \leq c_A \|\eta w_h\|_{1,\Omega^0}^2 \leq A_0(\eta w_h, \eta w_h) = A_0(\eta^2 w_h, w_h) + I,$$

where I can be estimated by

$$\tfrac{1}{2} c_A \|\eta w_h\|_{1,\Omega^0}^2 + C \|w_h\|_{0,D^{(2)}}^2.$$

Hence,

$$\|w_h\|_{1,D'}^2 \leq C A_0(\eta^2 w_h, w_h) + C \|w_h\|_{0,D^{(2)}}^2.$$

Using the discrete A_0^*-harmonicity of w_h and Proposition 4.3, we have, for some $\chi_h \in S_h(\Omega^0)$,

$$A_0(\eta^2 w_h, w_h) = A_0(\eta^2 w_h - \chi_h, w_h) \leq Ch \|w_h\|_{1,D^{(1)}}^2.$$

Now let D_h' and D_h'' be new mesh domains with $D^{(1)} \subset D_h'$, $\Omega^0 \setminus D^{(2)} \subset D_h''$, and $\operatorname{dist}(\hat{D}_h', \hat{D}_h'') \geq c > 0$, and let η be a new smooth cutoff function equal to 1 on $\hat{D}_h'$, vanishing on $\hat{D}_h''$, and satisfying $\|\eta\|_{W_\infty^r} \leq C_\eta$. Set $D_h = \Omega^0 \setminus (\overline{D}_h' \cup \overline{D}_h'')$. Repeating the above arguments then shows that, for any $\chi_h \in S_h(\Omega^0)$ such that $\chi_h = 0$ on $\Omega^0 \setminus D^{(2)}$, we have

$$\|w_h\|_{1,D^{(1)}}^2 \leq C A_0(\eta^2 w_h - \chi_h, w_h) + C \|w_h\|_{0,D^{(2)}}^2.$$

By Proposition 4.3 we have, for a suitable choice of χ_h,

$$A_0(\eta^2 w_h - \chi_h, w_h) \leq Ch \|w_h\|_{1,D_h}^2.$$

Collecting the above estimates and using an inverse estimate on D_h, we have

$$\|w_h\|_{1,D'}^2 \leq Ch^2 \|w_h\|_{1,D_h}^2 + C \|w_h\|_{0,D^{(2)}}^2 \leq C \|w_h\|_{0,D^{(2)}}^2,$$

which shows (9.9) and completes the proof of Lemma 9.2. $\square$

Proof of Lemma 9.3. Given an integer $m' \geq 0$, we shall prove the existence of a constant C such that, for $v_h \in S_h(\Omega^0)$ satisfying $A(v_h, \chi_h) = 0 \ \forall \chi_h \in S_h^0(\Omega^0)$, we have

$$|v_h(x_0)| \leq C \|v_h\|_{-m',\Omega^0}^*.$$

Recall the definition $D_j^k = D_{j-k} \cup \cdots \cup D_{j+k}$. Set $m = \max(m', [N/2] + 1)$, where $[N/2]$ denotes the integral part of $N/2$, and let l be the smallest integer for which $D_l^{m+3} \subset \Omega^0$. With $\tilde{D} = D_l^1$ we shall show the desired estimate in two steps: namely,

(9.10)
$$|v_h(x_0)| \leq C \|v_h\|_{1,\tilde{D}},$$

and

(9.11)
$$\|v_h\|_{1,\tilde{D}} \leq C \|v_h\|_{-m',\Omega^0}^*.$$

Set $\tilde{\tilde{D}} = D_l^{m+3}$. We shall use the fact that $\tilde{\tilde{D}}$ is well separated from the point x_0, so that the mesh size on all of $\tilde{\tilde{D}}$ is of order h, and $d_j^{-1} \leq C$ for each $D_j \subset \tilde{\tilde{D}}$. For the proof of (9.10) and (9.11), we shall use the following proposition:

PROPOSITION 9.4. *Under the assumptions of Theorem* 9.1 *and with l and m as above, there is a constant C such that, if* $v_h \in S_h(\Omega^0)$ *satisfies* $A(v_h, \chi_h) = 0, \chi_h \in S_h^0(D_l^{m+2})$,

then, for $k = 1, \ldots, m + 1,$

$$(9.12) \qquad \|v_h\|_{1, D_l^k} \leq Ch \|v_h\|_{1, D_l^{k+1}} + C \|v_h\|_{-m, D_l^{k+1}}^*.$$

Estimate (9.12) *also holds for* v_h *satisfying* $A(\chi_h, v_h) = 0 \ \forall \chi_h \in S_h^0(D_l^{m+2}).$

Before proving Proposition 9.4, we use it to first prove (9.11) and then (9.10). Repeated use of (9.12) with $k = 1, \ldots, m + 1$ shows that

$$(9.13) \qquad \|v_h\|_{1, D_l^1} \leq Ch^{m+1} \|v_h\|_{1, D_l^{m+2}} + C \|v_h\|_{-m, D_l^{m+2}}^*.$$

Let D_h be a mesh domain such that $D_l^{m+2} \subset D_h \subset \tilde{\tilde{D}}.$ By (4.5) we have

$$\|v_h\|_{1, D_l^{m+2}} \leq Ch^{-m-1} \|v_h\|_{-m, \tilde{\tilde{D}}}^* \leq Ch^{-m-1} \|v_h\|_{-m, \tilde{D}}^*,$$

and, hence, since $m \geq m'$ and $\tilde{\tilde{D}} \subset \Omega^0,$

$$(9.14) \qquad \|v_h\|_{1, \tilde{D}} = \|v_h\|_{1, D_l^1} \leq C \|v_h\|_{-m, \tilde{D}}^* \leq C \|v_h\|_{-m', \Omega^0}^*,$$

which is inequality (9.11).

In order to show (9.10), let $z_h \in S_h^0(\Omega_{l-1})$ be equal to v_h on Ω_l and satisfy $\|z_h\|_{1, \tilde{D}} \leq C \|v_h\|_{1, \tilde{D}}.$ To see that such a z_h exists, let D_h' and D_h'' be mesh domains such that $\Omega_l \subset D_h',$ $\Omega^0 \setminus \Omega_{l-1} \subset D_h'',$ and with $\text{dist}(\hat{D}_h', \hat{D}_h'') \geq c > 0,$ and let η be a smooth cutoff function equal to 1 on $\hat{D}_h',$ vanishing on $\hat{D}_h'',$ and satisfying $\|\eta\|_{W_\infty^r} \leq C_\eta.$ Take z_h as the function χ_h which approximates $\eta v_h,$ as in Proposition 4.3. The triangle inequality and $d_l^{-1} \leq C$ then give

$$\|z_h\|_{1, \tilde{D}} \leq \|\eta v_h - z_h\|_{1, \tilde{D}} + \|\eta v_h\|_{1, \tilde{D}}$$
$$\leq Ch_l d_l^{-1} \|v_h\|_{1, \tilde{D}} + Ch_l d_l^{-2} \|v_h\|_{0, \tilde{D}} + Cd_l^{-1} \|v_h\|_{1, \tilde{D}} \leq C \|v_h\|_{1, \tilde{D}}.$$

As before, let τ_0 be an element for which $x_0 \in \bar{\tau}_0.$ Using the inverse estimate (4.1) and the assumption that $\text{diam}(\tau_0)$ is of order $h^{1/(1-\alpha)},$ we have

$$(9.15) \qquad |v_h(x_0)| = |z_h(x_0)| \leq C \text{diam}^{-N/2}(\tau_0) \|z_h\|_{0, \tau_0} \leq Ch^{-N/(2-2\alpha)} \|z_h\|_{0, \tau_0}.$$

Duality gives

$$(9.16) \qquad \|z_h\|_{0, \tau_0} = \sup(z_h, g) h^{N/(2-2\alpha)},$$

where the supremum is taken over all $g \in C_0^\infty(\tau_0)$ normalized so that $\|g\|_0 = h^{-N/(2-2\alpha)}.$ Let $G \in H^1$ be defined by

$$A(v, G) = (v, g) \quad \forall v \in H^1,$$

and let $G_h \in S_h$ be the A^*-projection of $G.$ Let $\chi_h \in S_h^0(\Omega_l)$ equal G_h on Ω_{l+1} and satisfy

$$\|G_h - \chi_h\|_{1, \tilde{D}} \leq C \|G_h\|_{1, \tilde{D}}.$$

The existence of such a χ_h is an easy consequence of Proposition 4.3. Since v_h is A-harmonic on $\Omega^0,$ and hence z_h on $\Omega_l,$ we have, by the definition of χ_h and $G_h,$

$$A(z_h, G) = A(z_h, G - \chi_h) = A(z_h, G - G_h) + A(z_h, G_h - \chi_h)$$
$$= A(z_h, G_h - \chi_h).$$

Hence, by the definition of G and the estimates on $G_h - \chi_h$ and $z_h,$

$$(9.17) \qquad (z_h, g) = A(z_h, G) = A(z_h, G_h - \chi_h) \leq C \|z_h\|_{1, \tilde{D}} \|G_h - \chi_h\|_{1, \tilde{D}}$$
$$\leq C \|v_h\|_{1, \tilde{D}} \|G_h\|_{1, \tilde{D}},$$

where we also used that either z_h or $G_h - \chi_h$ vanish outside $\tilde{D}$. Now note that G_h satisfies

$$A(\chi_h, G_h) = A(\chi_h, G) = (\chi_h, g) = 0 \quad \forall \chi_h \in S_h^0(\Omega \setminus \tau_0).$$

Hence, Proposition 9.4 applies, and we obtain, as in (9.14),

$$(9.18) \qquad \|G_h\|_{1, \tilde{D}} \le C \|G_h\|_{-m, \tilde{D}}^*.$$

We shall now use the fact that the last negative norm is dominated by the L_1-norm over $\tilde{D}$. Namely, since $m \ge [N/2] + 1$, the Sobolev inequality (cf., e.g., [1])

$$\|w\|_{L_\infty} \le C \|w\|_{[N/2]+1} \le C \|w\|_m$$

yields

$$(9.19) \qquad \begin{aligned} \|G_h\|_{-m, \tilde{D}}^* &= \sup_{\substack{w \in C^\infty(\Omega) \\ \text{supp}(w) \cap \Omega \setminus \tilde{D} = \varnothing}} \frac{(G_h, w)}{\|w\|_m} \le \sup_{w \in C^\infty(\Omega)} \frac{\|G_h\|_{L_1(\tilde{D})} \|w\|_{L_\infty}}{\|w\|_m} \\ &\le C \|G_h\|_{L_1(\tilde{D})} \le C \|G\|_{L_1(\tilde{D})} + C \|G - G_h\|_{W_1^1(\tilde{D})}, \end{aligned}$$

where we have also used the triangle inequality and changed to a sharper norm in the last step. Using Lemma 2.3 and recalling that g, as defined above, has bounded L_1-norm, we at once have

$$\|G\|_{L_1(\tilde{D})} \le C \|g\|_{L_1(\tau_0)} \le C.$$

For the other term we use Lemma 5.2 to obtain

$$\|G - G_h\|_{W_1^1(\tilde{D})} = \|G - G_h\|_{W_1^1(D_l^{m+3})} \le h_{l+m+3}^{-r+1} \left\{ \sum_{j=l-m-3}^{l+m+3} h_j^{r-1} \|G - G_h\|_{W_1^1(D_j)} \right\}$$

$$\le C h^{-r+1} \ln(h^{-1}) h^{2r-2} \le C.$$

Collecting the above inequalities, we have shown that $\|G_h\|_{1, \tilde{D}} \le C$, and, hence, by (9.15)–(9.17), obtained the desired estimate (9.10) and thereby proved Lemma 9.2. □

Proof of Proposition 9.4. We shall prove that if v_h is discrete A-harmonic in D_l^{m+2}, then, for $k = 1, \ldots, m + 1$,

$$\|v_h\|_{1, D_l^k} \le C h \|v_h\|_{1, D_l^{k+1}} + C \|v_h\|_{-m, D_l^{k+1}}^*.$$

Fix k and introduce domains D^j such that

$$D_l^k = D^{-1} \subset D^0 \subset D^1 \subset \cdots \subset D^{m-1} \subset D^m = D_l^{k+1}$$

and with $\text{dist}(D^j, \Omega \setminus D^{j+1}) \ge c > 0$ for $j = -1, \ldots, m - 1$. By Lemma 7 in [5] (with $v \equiv 0$) the H^1-norm of v_h over D_j is dominated by $C d_j^{-1}$ times the L_2-norm of v_h over a slightly larger domain. Using the same arguments step by step, we here obtain

$$\|v_h\|_{1, D_l^k} \le C \|v_h\|_{0, D^0}.$$

We shall next show that, for $j = 0, \ldots, m - 1$, we have

$$(9.20) \qquad \|v_h\|_{-j, D^j}^* \le C h \|v_h\|_{1, D^{j+1}} + C \|v_h\|_{-(j+1), D^{j+1}}^*.$$

Repeated use of (9.20) for these j's yields

$$\bar{\|}v_h\|^*_{0,D^0} \leq Ch\|v_h\|_{1,D^m} + C\|v_h\|^*_{-m,D^m}.$$

Since $\| \cdot \|_{0,D^0} = \| \cdot \|^*_{0,D^0}$, we will then have obtained the desired estimate (9.12).

In order to show (9.20), fix j and consider a domain D with $D^j \subset D \subset D^{j+1}$, dist$(D^j, \Omega \setminus D) \geq c > 0$, and dist$(D, \Omega \setminus D^{j+1}) \geq c > 0$. Let $\eta \in C^\infty(R^N)$ equal 1 on D^j, equal 0 on $\Omega \setminus D$, and satisfy $\|\eta\|_{W^m_\infty} \leq C_\eta$. Moreover, let η satisfy the boundary condition

$$\frac{\partial \eta}{\partial n_c} = \sum_{i,j=1}^N a_{ij} \frac{\partial \eta}{\partial x_j} n_i = 0 \quad \text{on } \partial\Omega,$$

where $n = (n_i)$ is the exterior normal to $\partial\Omega$. That such a function η exists follows from the ellipticity condition (1.3). For let η' satisfy all requirements on η except for the boundary condition, and let ω be a smooth function vanishing on $\partial\Omega$, with nonvanishing gradient on supp(η') and gradient parallel to n on $\partial\Omega \cap$ supp(η'). Then, with $\partial_j = \partial/\partial x_j$, the function

$$\eta = \eta' - \frac{\sum_{i,j=1}^N a_{ij} \partial_i \omega \, \partial_j \eta'}{\sum_{i,j=1}^N a_{ij} \partial_i \omega \, \partial_j \omega} \omega$$

satisfies all requirements, including the boundary condition.

Now, let $w \in C^\infty(\Omega)$ with supp$(w) \cap \overline{\Omega \setminus D^j} = \varnothing$, as in the definition of the negative norm $\| \cdot \|^*_{-j,D^j}$. For such w define W by

$$A(v, W) = (v, w) \quad \forall v \in H^1.$$

We shall integrate by parts and use the boundary condition on η. We have

$$(v_h, w) = (\eta v_h, w) = A(\eta v_h, W)$$

$$= A(v_h, \eta W) + \int_\Omega \sum_{i,j=1}^N a_{ij} \frac{\partial \eta}{\partial x_i} v_h \frac{\partial W}{\partial x_j} dx$$

$$+ \int_\Omega \sum_{i,j=1}^N v_h \frac{\partial}{\partial x_i}\left(a_{ij} \frac{\partial \eta}{\partial x_j} W\right) dx + \int_\Omega \sum_{i=1}^N a_i \frac{\partial \eta}{\partial x_i} v_h W dx$$

$$= A(v_h, \eta W) + I,$$

where

$$I \leq C\|v_h\|^*_{-(j+1),D^{j+1}} \|W\|_{j+2}.$$

Since v_h is discrete A-harmonic, we have, for any $\chi_h \in S_h^0(D^{j+1})$,

$$A(v_h, \eta W) = A(v_h, \eta W - \chi_h) \leq C\|v_h\|_{1,D^{j+1}} \|\eta W - \chi_h\|_{1,D^{j+1}}.$$

By the approximation properties obtained in Section 4, there is such a χ_h with

$$\|\eta W - \chi_h\|_{1,D^{j+1}} \leq Ch\|W\|_2 \leq Ch\|W\|_{j+2}.$$

Since Lemma 2.2 shows that $\|W\|_{j+2} \leq C\|w\|_j$, we have thus obtained

$$(v_h, w) \leq C\left\{h\|v_h\|_{1,D^{j+1}} + \|v_h\|^*_{-(j+1),D^{j+1}}\right\}\|w\|_j,$$

with C independent of h, v_h, and w as above. Since this is the same as (9.20), and since the same estimate could be obtained for v_h being discrete A^*-harmonic in D_l^{m+2}, we have thereby proved Proposition 9.4.   □

10. High-Order Convergence for Derivatives for u Only Locally in W_∞^r. In this section we shall prove a counterpart of the localized estimate for $(u - u_h)(x_0)$ of Section 9 for the gradient and show that $O(h^{2r-2})$ convergence of $\nabla(u - u_h)(x_0)$ can also hold when u is only locally in W_∞^r.

THEOREM 10.1. *Let the domain Ω, the bilinear form A, and the family of finite-dimensional spaces $S_h = S_h(r, x_0, \alpha)$ be as before. Let $\Omega^0 \subset \Omega$ and assume that $\operatorname{dist}(x_0, \Omega \setminus \Omega^0) = c_0 > 0$. Moreover, let $m \geq 0$ be an integer. Then, given α with $1/2 < \alpha < 1$, there is a constant C such that, if $u \in W_\infty^r(\Omega^0)$ and if $u_h \in S_h(\Omega^0)$ satisfies $A(u - u_h, \chi_h) \, \forall \chi_h \in S_h^0(\Omega^0)$, then*

$$(10.1) \qquad \left| \nabla(u - u_h)(x_0) \right| \leq Ch^{2r-2} \|u\|_{W_\infty^r(\Omega^0)} + C\|u - u_h\|_{-m, \Omega^0}^*.$$

In particular, if $u \in H^r$ and $u_h \in S_h$ is the A-projection of u, then

$$(10.2) \qquad \left| \nabla(u - u_h)(x_0) \right| \leq Ch^{2r-2} \left\{ \|u\|_{W_\infty^r(\Omega^0)} + \|u\|_r \right\}.$$

If $\alpha = 1/2$ then (10.1) and (10.2) hold with C replaced by $C\ln(h^{-1})$.

Remark. The constant C in (10.1) and (10.2) depends on c_0, but is independent of x_0.

For the proof of the theorem we shall need the following lemma.

LEMMA 10.2. *Under the assumptions of Theorem 10.1 and given a nonnegative integer m', there is a constant C such that, if $v_h \in S_h(\Omega^0)$ satisfies*

$$(10.3) \qquad A(v_h, \chi_h) = 0 \quad \forall \chi_h \in S_h^0(\Omega^0),$$

then, for any $\partial_j = \partial/\partial x_j$,

$$(10.4) \qquad \left| \partial_j v_h(x_0) \right| \leq C\|v_h\|_{-m', \Omega^0}^*.$$

We give the necessary details of the proof of Lemma 10.2 after having shown (10.1) and (10.2).

Proof of Theorem 10.1. We may assume, without loss of generality, that $\partial\Omega^0$ is smooth. We shall prove (10.1) and (10.2) with ∇ replaced by any ∂_j. Set $U = u|_{\Omega^0}$ and let $U_h \in S_h(\Omega^0)$ be the A_0-projection of U. Then on Ω^0,

$$(10.5) \quad \partial_j(u - u_h) = \partial_j(u - U) + \partial_j(U - U_h) + \partial_j(U_h - u_h) = 0 + \partial_j e_1 + \partial_j e_2.$$

We first claim that

$$(10.6) \qquad \left| \partial_j e_1(x_0) \right| \leq Ch^{2r-2} \|U\|_{W_\infty^r(\Omega^0)},$$

with C replaced by $C\ln(h^{-1})$ in the case $\alpha = 1/2$. This is proved in the same way as Theorem 7.1, but with the modifications described after (9.6). We shall next show that

$$(10.7) \qquad \left| \partial_j e_2(x_0) \right| \leq Ch^{2r-2} \|U\|_{r, \Omega^0} + C\|u - u_h\|_{-m, \Omega^0}^*.$$

Since $U = u$ on Ω^0, and since $\| \cdot \|_{r,\Omega^0} \leqslant C \| \cdot \|_{W^r_\infty(\Omega^0)}$, (10.7) together with (10.5) and (10.6) will show (10.1). By taking $m = r - 2$ in (10.1), using $\| \cdot \|^*_{-m,\Omega^0} \leqslant \||| \cdot \|||_{-m,\Omega}$ and the negative-norm estimate (8.4), we obtain (10.2).

In order to show (10.7) we first note that e_2 satisfies (10.3). Therefore, with $m' = \max(m, r - 2)$, we have

$$\left| \partial_j e_2(x_0) \right| \leqslant C \| e_2 \|^*_{-m',\Omega^0}.$$

Using first the triangle inequality and the fact that $U = u$ on Ω^0, and then using the fact that $\| \cdot \|^*_{-m',\Omega^0}$ is dominated by both $\||| \cdot \||| _{-r+2,\Omega^0}$ and $\| \cdot \|^*_{-m,\Omega^0}$, we have

$$\| e_2 \|^*_{-m',\Omega^0} \leqslant \||| U - U_h \||| _{-r+2,\Omega^0} + \| u - u_h \|^*_{-m,\Omega^0}.$$

From here another application of the negative-norm estimate (8.4) shows (10.7) and completes the proof of Theorem 10.1 $\square$

Remark 10.1. The same comment as in Remark 9.1 applies to estimates (10.1) and (10.2).

Proof of Lemma 10.2. In view of the previously obtained estimate (9.11) for a function v_h satisfying (10.3), it is sufficient to show that

$$\left| \partial_j v_h(x_0) \right| \leqslant C \| v_h \|_{1,\tilde{D}},$$

where $\tilde{D} = D^1_l$ as in the proof of Lemma 9.3. Thus, let $z_h \in S^0_h(\Omega_{l-1})$ equal v_h on Ω_l and satisfy

$$\| z_h \|_{1,\tilde{D}} \leqslant C \| v_h \|_{1,\tilde{D}}.$$

By Lemma 7.2 we have

$$\left| \partial_j v_h(x_0) \right| = \left| \partial_j z_h(x_0) \right| \leqslant C \sup(\partial_j z_h, g) = C \sup(z_h, g'),$$

where the supremum is taken over all $g \in C^\infty_0(\tau_0)$ ($\bar{\tau}_0 \ni x_0$) with

$$\operatorname{diam}^{N/2}(\tau_0) \| g \|_0 + \operatorname{diam}^{(N+2)/2}(\tau_0) \| g \|_1 \leqslant 1,$$

and where we have also integrated by parts and put $g' = -\partial_j g$ in the last step. Define G' by

$$A(v, G') = (v, g') \quad \forall v \in H^1,$$

and let $G'_h \in S_h$ be its A^*-projection. Let $\chi_h \in S^0_h(\Omega_l)$ equal G'_h on Ω_{l+1} and satisfy

$$\left\| G'_h - \chi_h \right\|_{1,\tilde{D}} \leqslant C \left\| G'_h \right\|_{1,\tilde{D}}.$$

Since either z_h or $G'_h - \chi_h$ vanish outside $\tilde{D}$, we have

$$(z_h, g') = A(z_h, G') = A(z_h, G'_h) = A(z_h, G'_h - \chi_h) \leqslant C \| z_h \|_{1,\tilde{D}} \| G'_h \|_{1,\tilde{D}}.$$

It now remains to show that the last factor can be bounded by a constant. By (9.18) and (9.19) we have

$$\left\| G'_h \right\|_{1,\tilde{D}} \leqslant C \| G' \|_{L_1(\tilde{D})} + \left\| G' - G'_h \right\|_{W^1_1(\tilde{D})}.$$

Using the representation

$$G'(x) = \int_\Omega \mathscr{G}(x, y) g'(y) \, dy = \int_\Omega \partial_j \mathscr{G}(x, y) g(y) \, dy$$

and estimates (2.6) and (7.5) for $\mathcal{G}$ and g, respectively, we easily obtain

$$\|G'\|_{L_1(\dot{D})} \leqslant C.$$

Finally, by Lemma 7.3, we have

$$\left\|G' - G'_h\right\|_{W_1^1(\dot{D})} = \left\|G' - G'_h\right\|_{W_1^1(D_l^{m+3})}$$

$$\leqslant h_{l+m+3}^{-r+1} \left(\sum_{j=l-m-3}^{l+m+3} h_j^{r-1} \left\|G' - G'_h\right\|_{W_1^1(D_j)} \right)$$

$$\leqslant Ch^{-r+1}\ln(h^{-1})h^{2r-2} \leqslant C,$$

which completes the proof of Lemma 10.2. $\square$

Remark 10.3. Improved convergence holds also for higher-order derivatives of $u - u_h$ at x_0, provided the mesh is correspondingly further refined.

Remark 10.4. If u is in W_∞^m, $0 < m < r$, the estimate

$$\left|(u - u_h)(x_0)\right| \leqslant Ch^{m+r-2}\|u\|_{W_\infty^m}$$

can be obtained for suitable mesh-refinements.

Remark 10.5. In view of (5.2) and (9.2) and the corresponding more precise estimates (7.1) and (10.2), it is natural to ask if one can further relax the requirement on u and retain the $O(h^{2r-2})$ convergence at x_0. It is clear from approximation theory that we cannot essentially weaken the norm on u near x_0 if these estimates are supposed to hold for *any* $\alpha > (r-2)/(2r-2)$ and $\alpha > 1/2$, respectively. On the other hand, it seems reasonable to believe that by a more sophisticated analysis, one could replace the norm on u by one which changes continuously from a W_∞^r-norm near x_0 to an H^r-norm away from x_0 in some appropriate way. However, we have no proof of this. Neither have we investigated if $O(h^{2r-2})$ convergence can be obtained at x_0 for *some* degree of refinement if u is globally in H^r and locally in W_∞^m with $m < r$.

Department of Mathematics
Chalmers University of Technology
   and the University of Göteborg
S-412 96 Göteborg, Sweden

1. R. A. ADAMS, *Sobolev Spaces*. Academic Press, New York, 1975.

2. J. H. BRAMBLE & S. R. HILBERT, "Estimation of linear functionals on Sobolev spaces with application to Fourier transforms and spline interpolation," *SIAM J. Numer. Anal.*, 7, 1970, pp. 112–124.

3. J. H. BRAMBLE & J. E. OSBORN, "Rate of convergence estimates for nonselfadjoint eigenvalue approximations," *Math. Comp.*, v. 27, 1973, pp. 525–549.

4. J. H. BRAMBLE & A. H. SCHATZ, "Higher order local accuracy by averaging in the finite element method," *Math. Comp.*, v. 31, 1977, pp. 94–111.

5. K. ERIKSSON, "Improved accuracy by adapted mesh-refinements in the finite element method," *Math. Comp.*, v. 44, 1985, pp. 321–343.

6. K. ERIKSSON, "Finite element methods of optimal order for problems with singular data," *Math. Comp.*, v. 44, 1985, pp. 345–360.

7. P. G. CIARLET, *The Finite Element Method for Elliptic Problems*, North-Holland, Amsterdam, 1978.

8. S. R. HILBERT, "A mollifier useful for approximations in Sobolev spaces and some applications to approximating solutions of differential equations," *Math. Comp.*, v. 27, 1973, pp. 81–89.

9. JU. P. KRASOVSKIĬ, "Isolation of singularities of the Green's function," *Math. USSR-Izv.*, v. 1, 1967, pp. 935–966.

10. A. LOUIS & F. NATTERER, *Acceleration of Convergence for Finite Element Solutions of the Poisson Equation on Irregular Meshes*, Preprint, 1977.

11. J. A. NITSCHE, L_∞-Error Analysis for Finite Elements, Proc. 3rd Conf. on Mathematics of Finite Elements and Applications, Brunel University, 1978.

12. J. A. NITSCHE & A. H. SCHATZ, "Interior estimates for Ritz-Galerkin methods," Math. Comp., v. 28, 1974, pp. 937–958.

13. A. H. SCHATZ & L. B. WAHLBIN, "Interior maximum norm estimates for finite element methods," Math. Comp., v. 31, 1977, pp. 414–442.

14. A. H. SCHATZ & L. B. WAHLBIN, "Maximum norm estimates in the finite element method on plane polygonal domains. Part 1," Math. Comp., v. 32, 1978, pp. 73–109.

15. A. H. SCHATZ & L. B. WAHLBIN, "Maximum norm estimates in the finite element method on plane polygonal domains. Part 2: Refinements," Math. Comp., v. 33, 1979, pp. 465–492.

16. A. H. SCHATZ & L. B. WAHLBIN, "On the quasi-optimality in L_∞ of the $\mathring{H}^1$-projection into finite element spaces," Math. Comp., v. 38, 1982, pp. 1–22.

17. M. SCHECHTER, "On L^p estimates and regularity. I." Amer. J. Math., v. 85, 1963, pp. 1–13.

18. R. SCHREIBER, "Finite element methods of high-order accuracy for singular two-point boundary value problems with nonsmooth solutions," SIAM J. Numer. Anal., v. 17, 1980, pp. 547–566.

19. R. SCOTT, "Optimal L_∞ estimates for the finite element method on irregular meshes," Math. Comp., v. 30, 1976, pp. 681–697.

20. V. THOMÉE, "High order local approximations to derivatives in the finite element method," Math. Comp., v. 31, 1977, pp. 652–660.