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Optimal Integration for Functions of Bounded Variation*

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Abstract. The unique optimal information and the unique optimal linear algorithm are obtained for the integration of functions of bounded variation.

1. Introduction. For a class of real-valued functions, we seek an approximation to the integral of any function in the class, provided that the function values are given at n points. A summary of what is currently known about this problem may be found in [1, Section 6.4].

In this paper, we study the class F of real-valued functions of uniformly bounded variation on the unit interval. Concepts used in this paper are defined for very general settings in [1] and [2]. To aid the reader, they are defined in this paper for the special case of integration. We summarize the results of this paper.

(i) If n function evaluations are used, then the intrinsic uncertainty in the integral is at least $1/2n$, and $\lceil 1/2\varepsilon \rceil$ function evaluations guarantee an ε-approximation.

(ii) The optimal function evaluation points are $(2i-1)/2n$, $i = 1, 2, \ldots, n$, and this optimal information is unique.

(iii) The optimal algorithm using the optimal information is the averaging algorithm: $(1/n)\sum_{i=1}^{n} f((2i-1)/2n)$, and this is the unique optimal linear algorithm.

(iv) The averaging algorithm is within at most one unit of being an optimal complexity algorithm.

(v) The averaging algorithm is only a constant factor better than the composite trapezoidal and Simpson algorithms.

2. Basic Concepts. A function f defined on the unit interval is of *bounded variation* if there exists $M > 0$ such that for any partition π, $0 \leqslant x_0 < x_1 < \cdots < x_n < x_{n+1} \leqslant 1$, $\sum_{i=0}^{n} |f(x_{i+1}) - f(x_i)| < M$. The *total variation* of f is defined as

$$V_f = \sup_{\pi} \sum_{i=0}^{n} |f(x_{i+1}) - f(x_i)|.$$

We say a class F of functions is of *uniformly bounded variation* if $F = \{ f\colon f\colon [0,1] \to \mathbf{R}$ and $V_f \leqslant B\}$, where $B > 0$. Without loss of generality, we take the bound B to be unity.

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We seek an approximation to $\int_0^1 f(x)\,dx$, $\forall f \in F$, given function values at an *n-partition*, that is, at points $0 \leqslant x_1 < x_2 < \cdots < x_n \leqslant 1$. That is, the *information* N is defined as $N\colon F \to \mathbf{R}^n$, and

$$N(f) = [f(x_1), f(x_2), \ldots, f(x_n)] \quad \forall f \in F. \tag{2.1}$$

We denote $x_0 = 0$, $x_{n+1} = 1$, $\Delta_i = x_{i+1} - x_i$ for $i = 0, 1, \ldots, n$, and $\Delta = \max\{2\Delta_0, 2\Delta_n, \Delta_1, \Delta_2, \ldots, \Delta_{n-1}\}$. We have

LEMMA 2.1. (i) $\Delta \geqslant 1/n$; (ii) $\Delta = 1/n$ *iff* $\Delta_0 = \Delta_n = 1/2n$ *and* $\Delta_i = 1/n$ *for* $i = 1, 2, \ldots, n-1$. □

The proof is trivial, and is omitted.

Given information N and $f \in F$, the set of indistinguishable elements from f in F is

$$V(N, f) = \{\tilde{f} \in F\colon \tilde{f}(x_i) = f(x_i), i = 1, 2, \ldots, n\}. \tag{2.2}$$

The following lemma measures the uncertainty in the integral caused by indistinguishable elements.

LEMMA 2.2. *Let N be information corresponding to an n-partition and let $f \in F$. Then,*

$$L \leqslant \int_0^1 \tilde{f}(x)\,dx \leqslant U \quad \text{for all } \tilde{f} \in V(N, f), \tag{2.3}$$

where

$$\begin{aligned} U &= f(x_1)\Delta_0 + f(x_n)\Delta_n \\ &\quad + \sum_{i=1}^{n-1} \max\{f(x_i), f(x_{i+1})\}\Delta_i + \Delta\big(1 - \overline{V}_f\big)/2, \quad \text{and} \\ L &= f(x_1)\Delta_0 + f(x_n)\Delta_n \\ &\quad + \sum_{i=1}^{n-1} \min\{f(x_i), f(x_{i+1})\}\Delta_i - \Delta\big(1 - \overline{V}_f\big)/2, \end{aligned} \tag{2.4}$$

where $\overline{V}_f = \sum_{i=1}^{n-1} |f(x_{i+1}) - f(x_i)|$. Furthermore, there exist $\tilde{f}_L, \tilde{f}_U \in V(N, f)$, such that $\int_0^1 \tilde{f}_L(x)\,dx = L$ and $\int_0^1 \tilde{f}_U(x)\,dx = U$. □

Proof. We first show that for $\tilde{f} \in F$,

$$\begin{aligned} &\Big[\sup_{x_0 \leqslant x \leqslant x_1} \tilde{f}(x) - \tilde{f}(x_1)\Big] + \Big[\sup_{x_n \leqslant x \leqslant x_{n+1}} \tilde{f}(x) - \tilde{f}(x_n)\Big] \\ &\quad + 2\sum_{i=1}^{n-1} \Big\{\sup_{x_i \leqslant x \leqslant x_{i+1}} \tilde{f}(x) - \max\{\tilde{f}(x_i), \tilde{f}(x_{i+1})\}\Big\} \\ &\leqslant 1 - \overline{V}_{\tilde{f}}. \end{aligned} \tag{2.5}$$

For an arbitrary $\delta > 0$, there exists $\xi_i \in [x_i, x_{i+1}]$ such that $\sup_{x_i \leqslant x \leqslant x_{i+1}} \tilde{f}(x) \leqslant \tilde{f}(\xi_i) + \delta$, $i = 0, 1, \ldots, n$. Therefore,

$$\begin{aligned}
&\Big[\sup_{x_0 \leqslant x \leqslant x_1} \tilde{f}(x) - \tilde{f}(x_1)\Big] + \Big[\sup_{x_n \leqslant x \leqslant x_{n+1}} \tilde{f}(x) - \tilde{f}(x_n)\Big] \\
&\quad + 2\sum_{i=1}^{n-1} \Big\{\sup_{x_i \leqslant x \leqslant x_{i+1}} \tilde{f}(x) - \max\{\tilde{f}(x_i), \tilde{f}(x_{i+1})\}\Big\} + \sum_{i=1}^{n-1} |\tilde{f}(x_{i+1}) - \tilde{f}(x_i)| \\
&\leqslant [\tilde{f}(\xi_0) - \tilde{f}(x_1)] + [\tilde{f}(\xi_n) - \tilde{f}(x_n)] \\
&\quad + \sum_{i=1}^{n-1} \big\{2[\tilde{f}(\xi_i) - \max\{\tilde{f}(x_i), \tilde{f}(x_{i+1})\}] + |\tilde{f}(x_{i+1}) - \tilde{f}(x_i)|\big\} + 2n\delta \\
&\leqslant |\tilde{f}(\xi_0) - \tilde{f}(x_1)| + |\tilde{f}(\xi_n) - \tilde{f}(x_n)| \\
&\quad + \sum_{i=1}^{n-1} \big[|\tilde{f}(\xi_i) - \tilde{f}(x_i)| + |\tilde{f}(x_{i+1}) - \tilde{f}(\xi_i)|\big] + 2n\delta \\
&\leqslant V_{\tilde{f}} + 2n\delta \leqslant 1 + 2n\delta.
\end{aligned}$$

Since δ is arbitrary,

$$\begin{aligned}
&\Big[\sup_{x_0 \leqslant x \leqslant x_1} \tilde{f}(x) - \tilde{f}(x_1)\Big] + \Big[\sup_{x_n \leqslant x \leqslant x_{n+1}} \tilde{f}(x) - \tilde{f}(x_n)\Big] \\
&\quad + 2\sum_{i=1}^{n-1} \Big\{\sup_{x_i \leqslant x \leqslant x_{i+1}} \tilde{f}(x) - \max\{\tilde{f}(x_i), \tilde{f}(x_{i+1})\}\Big\} \\
&\quad + \sum_{i=1}^{n-1} |\tilde{f}(x_{i+1}) - \tilde{f}(x_i)| \leqslant 1,
\end{aligned}$$

and (2.5) follows.

Let $\tilde{f} \in V(N, f)$, then

$$\begin{aligned}
\int_0^1 \tilde{f}(x)\,dx &\leqslant \sum_{i=1}^{n} \Big[\sup_{x_i \leqslant x \leqslant x_{i+1}} \tilde{f}(x_i)\Big]\Delta_i \\
&= \tilde{f}(x_1)\Delta_0 + \tilde{f}(x_n)\Delta_n + \sum_{i=1}^{n-1} \max\{\tilde{f}(x_i), \tilde{f}(x_{i+1})\}\Delta_i \\
&\quad + \Big[\sup_{x_0 \leqslant x \leqslant x_1} \tilde{f}(x) - \tilde{f}(x_1)\Big]\Delta_0 + \Big[\sup_{x_n \leqslant x \leqslant x_{n+1}} \tilde{f}(x) - \tilde{f}(x_n)\Big]\Delta_n \\
&\quad + \sum_{i=1}^{n-1} \Big\{\sup_{x_i \leqslant x \leqslant x_{i+1}} \tilde{f}(x) - \max\{\tilde{f}(x_i), \tilde{f}(x_{i+1})\}\Big\}\Delta_i \\
&\leqslant \tilde{f}(x_1)\Delta_0 + \tilde{f}(x_n)\Delta_n + \sum_{i=1}^{n-1} \max\{\tilde{f}(x_i), \tilde{f}(x_{i+1})\}\Delta_i \\
&\quad + \frac{\Delta}{2}\Big\{\Big[\sup_{x_0 \leqslant x \leqslant x_1} \tilde{f}(x) - \tilde{f}(x_1)\Big] + \Big[\sup_{x_n \leqslant x \leqslant x_{n+1}} \tilde{f}(x) - \tilde{f}(x_n)\Big] \\
&\quad + 2\sum_{i=1}^{n-1} \Big\{\sup_{x_i \leqslant x \leqslant x_{i+1}} \tilde{f}(x) - \max\{\tilde{f}(x_i), \tilde{f}(x_{i+1})\}\Big\}\Big\} \\
&\leqslant \tilde{f}(x_1)\Delta_0 + \tilde{f}(x_n)\Delta_n + \sum_{i=1}^{n-1} \max\{\tilde{f}(x_i), \tilde{f}(x_{i+1})\}\Delta_i + \Delta\big(1 - \overline{V}_{\tilde{f}}\big)/2.
\end{aligned}$$

The last step follows from (2.5). Therefore,

$$\int_0^1 \tilde{f}(x)\,dx \leqslant \tilde{f}(x_1)\Delta_0 + \tilde{f}(x_n)\Delta_n + \sum_{i=1}^{n-1} \max\{\tilde{f}(x_i), \tilde{f}(x_{i+1})\}\Delta_i + \Delta\left(1 - \overline{V}_{\tilde{f}}\right)/2$$
$$= f(x_1)\Delta_0 + f(x_n)\Delta_n + \sum_{i=1}^{n-1} \max\{f(x_i), f(x_{i+1})\}\Delta_i + \Delta\left(1 - \overline{V}_f\right)/2,$$

i.e., $\int_0^1 \tilde{f}(x)\,dx \leqslant U$. The result for L follows from that for U by replacing f by $-f$.

Let

$$I = \begin{cases} [0, x_1) & \text{if } 2\Delta_0 = \Delta, \\ (x_i, x_{i+1}) & \text{if } 2\Delta_0 < \Delta \text{ and } i = \min\{j\colon \Delta_j = \Delta \text{ and } 1 \leqslant j \leqslant n-1\}, \\ (x_n, 1] & \text{if } 2\Delta_0 < \Delta,\ \Delta_j < \Delta \text{ for } 1 \leqslant j \leqslant n-1 \text{ and } 2\Delta_n = \Delta. \end{cases}$$

Let

$$f_{\max}(x) = \begin{cases} f(x_1) & \text{if } 0 \leqslant x \leqslant x_1, \\ f(x_n) & \text{if } x_n \leqslant x \leqslant 1, \\ f(x_i) & \text{if } x = x_i,\ i = 2, 3, \ldots, n-1, \\ \max\{f(x_i), f(x_{i+1})\} & \text{if } x_i < x < x_{i+1},\ i = 1, 2, \ldots, n-1. \end{cases}$$

Finally, let

$$\tilde{f}_U(x) = \begin{cases} f_{\max}(x) + \left(1 - \overline{V}_f\right)/C_I & \text{if } x \in I, \\ f_{\max}(x) & \text{otherwise,} \end{cases}$$

where

$$C_I = \begin{cases} 2 & \text{if } I = (x_i, x_{i+1}) \text{ for some } i,\ 1 \leqslant i \leqslant n-1. \\ 1 & \text{if } I = [0, x_1) \text{ or } I = (x_n, 1]. \end{cases}$$

It can be verified that $V_{\tilde{f}_U} = 1$, $\tilde{f}_U \in V(N, f)$, and $\int_0^1 \tilde{f}_U(x)\,dx = U$. An analogous conclusion holds for $\tilde{f}_L$. □

3. Optimal Information. From Lemma 2.2 we know that for all $\tilde{f} \in V(N, f)$, the integral of $\tilde{f}$, $\int_0^1 \tilde{f}(x)\,dx$, is confined to the interval $[L, U]$. We call

(3.1) $$r(N, f) = (U - L)/2,$$

the *local radius of information* N at f. From (2.4) we have

(3.2) $$r(N, f) = \frac{1}{2}\left\{ \sum_{i=1}^{n-1} |f(x_{i+1}) - f(x_i)|\Delta_i + \Delta\left(1 - \overline{V}_f\right)\right\}.$$

We define

(3.3) $$r(N) = \sup_{f \in F} r(N, f)$$

as the *global radius of information*. The quantity $r(N, f)$ measures the intrinsic uncertainty of the integral of f, caused by indistinguishable elements in $V(N, f)$, and $r(N)$ measures that of the worst f in F. We estimate the local and global radii of information in

LEMMA 3.1. *Let N be information corresponding to the n-partition* $0 = x_0 \leqslant x_1 \leqslant x_2 < \cdots < x_n \leqslant x_{n+1} = 1$. *Then,*

(3.4) $$r(N, f) \leqslant \Delta/2 \quad \textit{for all } f \in F,$$

and

$$r(N) = \Delta/2. \quad \square \tag{3.5}$$

Proof. Since $\Delta_i \leqslant \Delta$ for $i = 1, 2, \ldots, n-1$, by (3.2),

$$\begin{aligned} r(N, f) &= \frac{1}{2}\left[\sum_{i=1}^{n-1} |f(x_{i+1}) - f(x_i)|\Delta_i + \Delta\left(1 - \overline{V}_f\right)\right] \\ &\leqslant \frac{\Delta}{2}\left[\sum_{i=1}^{n-1} |f(x_{i+1}) - f(x_i)| + \left(1 - \overline{V}_f\right)\right] = \Delta/2, \end{aligned}$$

i.e., $r(N, f) \leqslant \Delta/2$, proving (3.4). Let $f \equiv 0$. Then by (3.2), $r(N, 0) = \Delta/2$, i.e., $r(N) = \sup_{f \in F} r(N, f) = r(N, 0) = \Delta/2$, proving (3.5). $\square$

The information $N(f)$ in (2.1) is said to be of *cardinality n*. Let $\Psi(n)$ be the class of all information of cardinality n, and let $r(n) = \inf_{N \in \Psi(n)} r(N)$. Then information $N \in \Psi(n)$ is called *nth optimal* if $r(N) = r(n)$. An nth optimal information N has the minimum radius of information, among all information in $\Psi(n)$.

Let N^* be information corresponding to the partition points $x_i = (2i-1)/2n$, where $i = 1, 2, \ldots, n$, and $n \geqslant 2$. We have

THEOREM 3.1. *N^* is the unique nth optimal information with $r(N^*) = r(n) = 1/2n$.* $\square$

Proof. For the information N^*, $\Delta = 1/n$. By (3.5), $r(N^*) = 1/2n$. On the other hand, for an arbitrary $N \in \Psi(n)$, $r(N) = \Delta/2 \geqslant 1/2n = r(N^*)$, by Lemma 2.1(i), and the equality holds iff $N = N^*$, by Lemma 2.1(ii). $\square$

Remark 3.1. (i) If the class of integrands F_1 consists of functions with a uniformly bounded first derivative, then (see [1, Section 6.4]) N^* is an nth optimal information with $r(N^*) = 1/4n$.

(ii) To define information in (2.1), the partition points x_i are independent of function values at the previously chosen partition points. This is *nonadaptive information*. If partition points are chosen sequentially, depending on the function values at the previously chosen partition points, we have *adaptive information*. For integration of functions of bounded variation, the integrand belongs to a balanced convex set. Therefore, one gains nothing by using adaptive information. For the proof, see [1, Section 2.7] or [2, Section 4.3].

4. Optimal Algorithm. Usually, one cannot compute the integral of a function exactly, and instead seeks an approximation to the integral using an *algorithm* φ

$$\varphi\colon N(F) \to \mathbf{R}. \tag{4.1}$$

We define the *local algorithm error* of f as

$$e(\varphi, N, f) = \sup_{\tilde{f} \in V(N, f)} \left|\int_0^1 \tilde{f}(x)\,dx - \varphi(N(f))\right|, \tag{4.2}$$

and the *global algorithm error* as

$$e(\varphi, N) = \sup_{f \in F} e(\varphi, N, f). \tag{4.3}$$

For a given $f \in F$, the integrals of indistinguishable elements $\tilde{f} \in V(N, f)$ are in the interval $[L, U]$, where the sharp bounds L and U are given in (2.4). Therefore, for an arbitrary algorithm φ, $e(\varphi, N, f) \geqslant (U - L)/2$, which by (3.1) is the local radius of information $r(N, f)$. Thus, we have

$$e(\varphi, N, f) \geqslant r(N, f), \tag{4.4}$$

and

$$e(\varphi, N) \geqslant r(N) \quad \text{for all } \varphi. \tag{4.5}$$

Therefore, $r(N, f)$ and $r(N)$ are the lower bounds of local and global algorithm errors, respectively.

We present an algorithm, called the *central algorithm*, by choosing the center of $[L, U]$ as $\varphi^c(N(f))$:

$$\varphi^c(N(f)) = (U + L)/2. \tag{4.6}$$

Then $e(\varphi^c, N, f) = (U - L)/2 = r(N, f)$, and $e(\varphi^c, N) = r(N)$. Since φ^c has the minimal $e(\varphi^c, N)$ among all algorithms, it is called an *optimal error algorithm*.

From (2.4), we have

$$e(\varphi^c, N, f) = \frac{1}{2}\left[\sum_{i=1}^{n-1} |f(x_{i+1}) - f(x_i)|\Delta_i + \Delta\left(1 - \overline{V}_f\right)\right], \tag{4.7}$$

$$e(\varphi^c, N) = \Delta/2, \tag{4.8}$$

$$\varphi^c(N(f)) = f(x_1)\Delta_0 + f(x_n)\Delta_n + \sum_{i=1}^{n-1} \frac{f(x_i) + f(x_{i+1})}{2}\Delta_i, \tag{4.9}$$

or

$$\varphi^c(N(f)) = f(x_1)(\Delta_0 + \Delta_1/2) + \sum_{i=2}^{n-1} f(x_i)\frac{\Delta_{i-1} + \Delta_i}{2} + f(x_n)\left(\Delta_n + \frac{\Delta_{n-1}}{2}\right). \tag{4.10}$$

An *algorithm* is *linear* if it is of the form

$$\varphi(N(f)) = \sum_{i=1}^{n} f(x_i)H_i. \tag{4.11}$$

Therefore, the central algorithm is linear. We summarize the above in

THEOREM 4.1. *Given information N, the central algorithm φ^c is a linear optimal error algorithm, with local and global algorithm error equal to the local and global radius of information, respectively.* □

Remark 4.1. It is true in general (see [2, Sections 1.3 and 1.4]) that the central algorithm is optimal and that the local and global algorithm error of the central algorithm are equal to the local and global radius of information, respectively. □

Given the unique nth optimal information N^*, we compare the algorithm error of the central algorithm with those of other linear algorithms in

THEOREM 4.2. *The central algorithm φ^c using the unique nth optimal information N^* is*

$$\varphi^c(N^*(f)) = \frac{1}{n}\sum_{i=1}^{n} f\left(\frac{2i-1}{2n}\right). \tag{4.12}$$

Furthermore, the linear optimal error algorithm is unique. □

Proof. Let φ be an arbitrary noncentral linear algorithm, with $\varphi(N^*(f)) = \sum_{i=1}^{n} f(x_i)H_i$, and let p be the largest subscript of H such that $H_p \neq 1/n$. Let

$$f_p(x) = \begin{cases} 0 & \text{if } x < x_p, \\ 1 & \text{if } x \geqslant x_p. \end{cases}$$

Then we have

$$\begin{aligned} e\left(\varphi, N^*, f_p\right) &= \sup_{\tilde{f} \in V(N, f_p)} \left| \sum_{i=1}^{n} f_i H_i - \int_0^1 f(x)\,dx \right| \\ &= \max\left\{ \left| H_p + \frac{n-p}{n} - U \right|, \left| H_p + \frac{n-p}{n} - L \right| \right\} \\ &= \max\left\{ \left| H_p - \frac{1}{n} - \frac{1}{2n} \right|, \left| H_p - \frac{1}{n} + \frac{1}{2n} \right| \right\} \\ &\geqslant \left| H_p - \frac{1}{n} \right| + \frac{1}{2n} > \frac{1}{2n} = e\left(\varphi^c, N^*\right). \end{aligned}$$

Therefore, $e(\varphi, N^*) \geqslant e(\varphi, N^*, f_p) > e(\varphi^c, N^*)$ and φ is not optimal. □

5. Complexity. Given information N, we seek an algorithm φ to compute an ε-*approximation* to the integral of any functions in F, with algorithm error $e(\varphi, N) \leqslant \varepsilon$, where $\varepsilon > 0$. We use the nth optimal information N^* and the central algorithm φ^c to obtain an ε-approximation. Then, from Theorem 3.1 and Theorem 4.1, we have $e(\varphi^c, N^*) = r(N^*) = r(n) = 1/2n \leqslant \varepsilon$. Therefore, $n = \lceil 1/2\varepsilon \rceil$. It is obvious that $\lceil 1/2\varepsilon \rceil$ is the minimal number of function evaluations for which we can have an ε-approximation to the integral of any functions in F.

Assume that the cost of each arithmetic operation is 1 and that of each function evaluation is c. We first compute $N(f) = y = (y_1, \ldots, y_n)$ with *information complexity* cn, where n is the cardinality of N. We then compute $\varphi(y)$ with *combinatory complexity* $\text{comp}(\varphi(y))$. The *complexity of algorithm* φ is thus $\text{comp}(\varphi, N) = cn + \sup_{f \in F} \text{comp}(\varphi(y))$. By (4.12) we have

$$\text{(5.1)} \qquad \text{comp}(\varphi^c, N^*) = (c+1)\left\lceil \frac{1}{2\varepsilon} \right\rceil.$$

We define the *problem complexity* of an ε-approximation as

$$\text{(5.2)} \qquad \text{comp}(\varepsilon) = \inf_{\varphi, N} \{\text{comp}(\varphi, N) : e(\varphi, N) \leqslant \varepsilon\},$$

and an *optimal complexity algorithm* φ^* as

$$\text{(5.3)} \qquad \text{comp}\left(\varphi^*, N\right) = \text{comp}(\varepsilon), \quad \text{and} \quad e\left(\varphi^*, N\right) \leqslant \varepsilon \quad \text{for some } N.$$

As noted at the beginning of this section, $n = \lceil 1/2\varepsilon \rceil$ is the minimal number of function evaluations to guarantee an ε-approximation. Thus, the information complexity is no less than $c\lceil 1/2\varepsilon \rceil$, and the combinatory complexity is no less than $\lceil 1/2\varepsilon \rceil - 1$. Therefore $\text{comp}(\varepsilon) \geqslant (c+1)\lceil 1/2\varepsilon \rceil - 1$. Comparing this with (5.1), we notice that the difference between $\text{comp}(\varphi^c, N^*)$ and $\text{comp}(\varepsilon)$ is at most 1. We propose the following

CONJECTURE. *The central algorithm using the optimal information is the optimal complexity algorithm, that is,*

$$\text{(5.4)} \qquad \text{comp}\left(\varphi^c, N^*\right) = \text{comp}(\varepsilon). \quad \square$$

6. Comparison With Other Algorithms. We estimate the global algorithm error and algorithm complexity of some linear algorithms; the proofs are routine and are omitted.

(i) *Another Riemann Sum*. Let the partition points be $x_i = (i-1)/n$, $i = 1, 2, \ldots, n$, and $\Delta = 2/n$.

$$\varphi(N(f)) = \frac{1}{n}\sum_{i=1}^{n} f\left(\frac{i-1}{n}\right). \tag{6.1}$$

$$e(\varphi, N) = \frac{1}{n}. \tag{6.2}$$

The algorithm complexity for an ε-approximation is

$$\operatorname{comp}(\varphi, N) = (c+1)\left\lceil \frac{1}{\varepsilon} \right\rceil. \tag{6.3}$$

(ii) *Composite Trapezoidal Rule*. Let the partition points be $x_i = (i-1)/(n-1)$, $i = 1, 2, \ldots, n$ and $\Delta = 1/(n-1)$.

$$\varphi(N(f)) = \frac{1}{2(n-1)}[f(0) + f(1)] + \frac{1}{n-1}\sum_{i=2}^{n-1} f\left(\frac{i-1}{n-1}\right). \tag{6.4}$$

$$e(N, \varphi) = \frac{1}{n-1}. \tag{6.5}$$

The algorithm complexity for an ε-approximation is

$$\operatorname{comp}(\varphi, N) = (c+1)\left\lceil \frac{1}{\varepsilon} \right\rceil + c + 2. \tag{6.6}$$

(iii) *Composite Simpson's Rule*. Assume that $n = 2m + 1$. Let the partition points be $x_i = (i-1)/2m$, $i = 1, 2, \ldots, 2m+1$, and $\Delta = 1/2m$.

$$\varphi(N(f)) = \frac{1}{6m}[f(0) + f(1)] + \frac{1}{3m}\sum_{i=2}^{m} f\left(\frac{i-1}{m}\right) + \frac{2}{3m}\sum_{i=1}^{m} f\left(\frac{2i-1}{2m}\right), \tag{6.7}$$

$$e(\varphi, N) = \frac{2}{3(n-1)}. \tag{6.8}$$

The algorithm complexity for an ε-approximation is

$$\operatorname{comp}(\varphi, N) = (c+1)\left\lceil \frac{2}{3\varepsilon} \right\rceil + c + 3. \tag{6.9}$$

Observe that the costs of the linear algorithms (i)–(iii) are within a constant factor of $\operatorname{comp}(\varepsilon)$.

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