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Finite Element Technique for Optimal Pressure Recovery from Stream Function Formulation of Viscous Flows*

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Abstract. Following a general analysis of convergence for the finite element solution of the stream function formulation of the Navier-Stokes equation in bounded regions of the plane, an algorithm for pressure recovery is presented. This algorithm, which is easy to implement, is then analyzed and conditions ensuring optimality of the approximation are given. An application is made to a standard conforming cubic macroelement.

1. Introduction. The purpose of this paper is to provide a formulation and analysis of the stream function approach to the solution of the two-dimensional Navier-Stokes equations in polygonal simply connected domains. As with related approaches [2], [3], [4], [9], [10], [11], the pressure must be computed separately if it is required, and we give a natural algorithm for this purpose. In addition, we prove that the computed pressure will be "optimal" in the approximation-theoretic sense, for a particular kind of finite element space. Although specific numerical results are not given, they have been obtained by the authors and confirm the theoretical predictions.

For notation, let Ω be a bounded simply connected domain in $\mathbf{R}^2$. $L^2(\Omega)$ is the Hilbert space of square (Lebesgue) integrable functions with norm $\|\cdot\|_0$ and $L_0^2(\Omega)$ is the subspace of $L^2(\Omega)$ consisting of functions with zero mean. Let $H^m(\Omega)$ be the usual Sobolev space consisting of functions which together with their (distributional) derivatives up through order m are in $L^2(\Omega)$. Denote the norm on $H^m(\Omega)$ by $\|\cdot\|_m$. Let $H_0^m(\Omega)$ be the completion of $C_0^\infty(\Omega)$ under the $\|\cdot\|_m$ norm. We equip $H_0^m(\Omega)$ with the seminorm $|\cdot|_m$, which is a norm equivalent to $\|\cdot\|_m$. Also, the dual of $H_0^m(\Omega)$ is denoted by $H^{-m}(\Omega)$, with norm $\|\cdot\|_{-m}$. Let $\vec{H}^m(\Omega)$, $(\vec{H}_0^m(\Omega))$ be the space $H^m(\Omega)\times H^m(\Omega)$ $(H_0^m(\Omega)\times H_0^m(\Omega))$ equipped with the following norm

$$\|\vec{u}\|_m=\left(\|u_1\|_m^2+\|u_2\|_m^2\right)^{1/2}\quad\left(|\vec{u}|_m=\left(|u_1|_m^2+|u_2|_m^2\right)^{1/2}\right)\quad\text{where } \vec{u}=\begin{pmatrix}u_1\\u_2\end{pmatrix}.$$

Denote the dual of $\vec{H}_0^m(\Omega)$ by $\vec{H}^{-m}(\Omega)$ with norm $\|\cdot\|_{-m}$. For each $\varphi\in H^1(\Omega)$, define

$$\overrightarrow{\operatorname{curl}}\,\varphi=\begin{bmatrix}\varphi_y\\-\varphi_x\end{bmatrix}.$$

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2. Stream Function Formulation. A standard weak form for the stream function formulation for 2-dimensional stationary incompressible fluid flow (on a bounded simply connected domain) is

$$(2.1)\qquad \begin{cases} \text{Find } \psi \in H_0^2(\Omega) \text{ such that for all } \chi \in H_0^2(\Omega) \\ \nu\int_\Omega \Delta\psi\Delta\chi + \int_\Omega \Delta\psi(\psi_y\chi_x - \psi_x\chi_y) = \int_\Omega \vec{f}\cdot \overrightarrow{\operatorname{curl}}\,\chi. \end{cases}$$

For brevity, we use the following notations:

$$a_0(\psi,\chi) = \nu\int_\Omega \Delta\psi\Delta\chi,$$

$$a_1(\zeta;\psi,\chi) = \int_\Omega \Delta\zeta(\psi_y\chi_x - \psi_x\chi_y),$$

$$l(\chi) = \int_\Omega \vec{f}\cdot \overrightarrow{\operatorname{curl}}\,\chi.$$

Note that

$$(2.2)\qquad a_0(\psi,\chi) \leqslant \nu|\psi|_2|\chi|_2 \quad \text{for all } \psi,\chi \in H_0^2(\Omega)$$

and that there exists a $\Gamma_1 > 0$ such that

$$(2.3)\qquad a_1(\zeta;\psi,\chi) \leqslant \Gamma_1|\zeta|_2|\psi|_2|\chi|_2 \quad \text{for all } \zeta,\psi,\chi \in H_0^2(\Omega)$$

and $\|l\|_{-2} \leqslant \|\vec{f}\|_{-1}$. The following theorem can be proved using the method of [8].

THEOREM 2.1. *Let* $\vec{f} \in \vec{H}^{-1}(\Omega)$ *and define*

$$\nu^* = \left(\Gamma_1\|\vec{f}\|_{-1}\right)^{1/2}.$$

Then for any $\nu > \nu^*$, (2.1) *has a unique solution* ψ. *Moreover*, $|\psi|_2 \leqslant \|\vec{f}\|_{-1}/\nu$.

Let $V^h \subset H_0^2(\Omega)$ be a finite element trial space. Then to compute approximations to ψ, we solve for $\psi^h \in V^h$ in the following way:

$$(2.4)\qquad \begin{cases} \text{Find } \psi^h \in V^h \text{ such that for all } \chi^h \in V^h \\ a_0(\psi^h,\chi^h) + a_1(\psi^h;\psi^h,\chi^h) = l(\chi^h). \end{cases}$$

Existence and uniqueness of the solution to (2.4) for $\nu > \nu^*$ follow from the fact that the properties of a_0, a_1 and l used in the proof of Theorem 2.1 are inherited by any closed subspace of $H_0^2(\Omega)$. Moreover, $|\psi^h|_2 \leqslant \|\vec{f}\|_{-1}/\nu$. We shall make use of the following estimate below. From here on, assume $\nu > \nu^*$.

THEOREM 2.2. *Let* ψ *be the solution to* (2.1) *and* ψ^h *the solution to* (2.4). *Then*

$$(2.5)\qquad |\psi - \psi^h|_2 \leqslant c(\nu) \inf_{\chi^h\in V^h} |\psi - \chi^h|_2,$$

where $c(\nu) = (1 + 2\Gamma_1\|\vec{f}\|_{-1}/\nu^2)(1 - \Gamma_1\|\vec{f}\|_{-1}/\nu^2)^{-1} \leqslant c(\nu^*)$.

Proof. Since $V^h \subset H_0^2(\Omega)$, (2.1) holds for all $\chi^h \in V^h$. Subtracting (2.4) from (2.1) gives

$$(2.6)\qquad a_0(\psi - \psi^h,\chi^h) + a_1(\psi;\psi,\chi^h) - a_1(\psi^h;\psi^h,\chi^h) = 0 \quad \text{for all } \chi^h \in V^h.$$

Observe that a_0 is bilinear and a_1 is trilinear. Also, $a_1(\zeta; \psi, \chi) = -a_1(\zeta; \chi, \psi)$ for all $\zeta, \psi, \chi \in H_0^2(\Omega)$. It soon follows that for any $\chi^h \in V^h$,

$$\begin{aligned} &a_0(\psi - \psi^h, \psi - \psi^h) + a_1(\psi^h - \psi; \psi, \psi^h - \psi) \\ &\quad = a_0(\psi - \psi^h, \psi - \chi^h) + a_1(\psi^h; \psi^h - \psi, \chi^h - \psi) + a_1(\psi^h - \psi, \psi, \chi^h - \psi). \end{aligned}$$

Using (2.2), (2.3), and the fact that $a_0(\chi, \chi) = \nu|\chi|_2^2$ for all $\chi \in H_0^2(\Omega)$, we get

$$(\nu - \Gamma_1|\psi|_2)|\psi - \psi^h|_2^2 \leqslant (\nu + \Gamma_1|\psi|_2 + \Gamma_1|\psi^h|_2)|\psi - \psi^h|_2|\psi - \chi^h|_2.$$

Since $|\psi|_2 \leqslant \|\vec{f}\|_{-1}/\nu$ and $|\psi^h|_2 \leqslant \|\vec{f}\|_{-1}/\nu$, the conclusion is immediate. □

Below, we shall provide an algorithm for pressure recovery and an error estimate for the particular space $V^h \subset H_0^2(\Omega)$ consisting of the Clough-Tocher triangles [6]. Let $\{\mathcal{Q}^h\}$ be a regular triangulation of Ω. Each macrotriangle $Q \in \mathcal{Q}^h$ is divided into three subtriangles Q_1, Q_2, Q_3 by joining each vertex of Q to its centroid. For this element, approximation theory shows [6]

$$\text{(2.7)} \qquad \begin{cases} \text{for each } \chi \in H^3(\Omega) \cap H_0^2(\Omega), \text{ there exists a } \chi^h \in V^h \text{ such that} \\ |\chi - \chi^h|_2 \leqslant ch^{2-s}\|\chi\|_{4-s}, \qquad s = 0, 1, \end{cases}$$

where the lower value of s assumes the implied extra regularity of χ.

Although we do not make any further use of the following theorem, it is included here for completeness. Specifically, we can use a duality argument to get estimates for $|\psi - \psi^h|_1$ and $\|\psi - \psi^h\|_0$. Define the (linear) "dual" problem by:

$$\text{(2.8)} \qquad \begin{cases} \text{Find } \zeta \in H_0^2(\Omega) \text{ such that for all } \chi \in H_0^2(\Omega), \\ a_0(\chi, \zeta) + a_1(\psi; \chi, \zeta) + a_1(\chi; \psi, \zeta) = \langle g, \chi \rangle, \end{cases}$$

where ψ is the solution to (2.1) and $\langle \cdot, \cdot \rangle$ is the duality pairing in $L^2(\Omega)$. Since $\nu > \nu^*$, (2.8) is uniquely solvable for $g \in H^{-2}(\Omega)$. Moreover, $|\zeta|_2 \leqslant c\|g\|_{-2}$.

THEOREM 2.3. *Assume that $\psi \in H^4(\Omega) \cap H_0^2(\Omega)$. Moreover, assume that for each $g \in L^2(\Omega)$, the solution ζ to (2.8) satisfies*

$$(*) \qquad \zeta \in H^4(\Omega) \cap H_0^2(\Omega), \qquad \|\zeta\|_4 \leqslant c\|g\|_{-2}.$$

Then there exist positive constants C_0, C_1, C_2 such that

$$\text{(2.9)} \qquad |\psi - \psi^h|_2 \leqslant C_2 h^2,$$

$$\text{(2.10)} \qquad |\psi - \psi^h|_1 \leqslant C_1 h^3,$$

$$\text{(2.11)} \qquad \|\psi - \psi^h\|_0 \leqslant C_0 h^4.$$

Proof. (2.9) follows immediately from (2.7) and Theorem 2.2. Let $\chi = \psi^h - \psi$ in (2.8), choose ζ^h corresponding to ζ as in (2.7), and use (2.6) to get

$$\begin{aligned} \langle g, \psi^h - \psi \rangle &= a_0(\psi^h - \psi, \zeta - \zeta^h) + a_1(\psi; \psi^h - \psi, \zeta - \zeta^h) \\ &\quad + a_1(\psi^h - \psi; \psi^h, \zeta - \zeta^h) + a_1(\psi^h - \psi; \psi - \psi^h, \zeta), \end{aligned}$$

hence

$$\begin{aligned} |\langle g, \psi^h - \psi \rangle| &\leqslant |\psi^h - \psi|_2|\zeta - \zeta^h|_2(\nu + \Gamma_1|\psi|_2 + \Gamma_1|\psi^h|_2) + \Gamma_1|\zeta|_2|\psi - \psi^h|_2^2 \\ \text{(2.12)} \qquad &\leqslant Kh^4(\|\zeta\|_4 + |\zeta|_2) \leqslant K_0 h^4\|g\|_{-2}, \\ &\leqslant K_1 h^4\|g\|_0 \quad \text{for } g \in L^2(\Omega). \end{aligned}$$

Let $g = \Delta(\psi - \psi^h)$; then by (2.12)

$$|\psi - \psi^h|_1^2 = \langle g, \psi^h - \psi \rangle \leqslant K_1 h^4 |\psi^h - \psi|_2 \leqslant K_2 h^6 \text{ by (2.9)},$$

hence

$$|\psi - \psi^h|_1 \leqslant C_1 h^3.$$

Now set $g = \psi^h - \psi$. By (2.12) we get

$$\|\psi - \psi^h\|_0^2 = \langle g, \psi^h - \psi \rangle \leqslant K_1 h^4 \|\psi - \psi^h\|_0,$$

and (2.11) follows. □

We make the remark that (*) will hold if Ω is a polygon with maximum interior vertex angle $\theta < 126°$. See [4] for details.

3. Pressure recovery. We now turn to the important question of pressure recovery. Naturally, the momentum equations are used for this. Unlike other treatments, however, we shall avoid having to specify the pressure boundary conditions. The basic idea is to solve an equation of the form

$$-\int_\Omega p \operatorname{div} \vec{v} = g(\vec{v}),$$

where g is known. For convenience, let $b(\vec{v}, p) = -\int_\Omega p \operatorname{div} \vec{v}$. Note that b is continuous and coercive on $\vec{H}_0^1(\Omega) \times L_0^2(\Omega)$ [8, Theorem 3.7, p. 35]. Thus it follows from Babuška's theorem [1] that for any $g \in [\vec{H}_0^1(\Omega)]'$, a unique $p \in L_0^2(\Omega)$ exists such that the above equation holds for each $\vec{v} \in \vec{H}_0^1(\Omega)$. The term on the right-hand side depends on the solution ψ of (2.1), the data $\vec{f}$ and ν, and is given by

$$\begin{aligned} g(\vec{v}) &= g(\psi; \vec{f}, \nu)(\vec{v}) \\ &= \langle \vec{f}, \vec{v} \rangle - \nu b_0(\overrightarrow{\operatorname{curl}}\, \psi, \vec{v}) - b_1(\overrightarrow{\operatorname{curl}}\, \psi, \overrightarrow{\operatorname{curl}}\, \psi, \vec{v}), \end{aligned} \tag{3.1}$$

where $b_0(\vec{u}, \vec{v}) = \int_\Omega \nabla \vec{u} \colon \nabla \vec{v}$ and $b_1(\vec{u}, \vec{w}, \vec{v}) = \int_\Omega [(\vec{u} \cdot \nabla)\vec{w}] \cdot \vec{v}$. It follows directly from the continuity of b_0 and b_1 (see [8]) that

LEMMA 3.1. *For $\vec{f} \in \vec{H}^{-1}(\Omega)$, $\psi \in H_0^2(\Omega)$, g as defined above is a bounded linear functional on $\vec{H}_0^1(\Omega)$. Moreover, for all ψ, $\chi \in H_0^2(\Omega)$, $\vec{v} \in \vec{H}_0^1(\Omega)$,*

$$|g(\psi; \vec{f}, \nu)(\vec{v}) - g(\chi; \vec{f}, \nu)(\vec{v})| \leqslant (\nu + \Gamma_1 |\psi|_2 + \Gamma_1 |\chi|_2) |\psi - \chi|_2 |\vec{v}|_1.$$

Then we have

THEOREM 3.2. *Given $\psi \in H_0^2(\Omega)$, $\vec{f} \in \vec{H}^{-1}(\Omega)$, there exists a unique $p \in L_0^2(\Omega)$ such that*

$$b(\vec{v}, p) = g(\psi; \vec{f}, \nu)(\vec{v}) \quad \text{for all } \vec{v} \in \vec{H}_0^1(\Omega). \tag{3.2}$$

It is the discretization of (3.2) which must be analyzed. The coercivity condition does not necessarily hold for arbitrary subspaces $X^h \subset \vec{H}_0^1(\Omega)$ and $S^h \subset L_0^2(\Omega)$. Also in discretizing (3.2), not only will p and $\vec{v}$ be discretized, but so will g, i.e., $g(\psi; \vec{f}, \nu)(\vec{v})$ has to be replaced by $g(\psi^h; \vec{f}, \nu)(\vec{v}^h)$. Since the null space of $g(\psi^h; f, \nu)$ does not necessarily coincide with the discretely div-free functions in X^h, the discretized analogue of (3.2) can only hold in some subspaces of X^h. This subspace is generally quite difficult to find. Hence, we introduce the following equivalent problem:

$$(3.3)\quad \begin{cases} \text{Find } \vec{w} \in \vec{H}_0^1(\Omega),\ p \in L_0^2(\Omega) \text{ such that for all } \vec{v} \in \vec{H}_0^1(\Omega),\ q \in L_0^2(\Omega) \\ b_0(\vec{w}, \vec{v}) + b(\vec{v}, p) = g(\psi; \vec{f}, \nu)(\vec{v}) \\ \qquad b(\vec{w}, q) = 0. \end{cases}$$

(3.3) is uniquely solvable, and by Theorem 3.2, $w = 0$. We now discretize (3.3) as follows:

$$(3.4)\quad \begin{cases} \text{Find } \vec{w}^h \in X^h,\ p^h \in S^h \text{ such that for all } \vec{v}^h \in X^h,\ q^h \in S^h \\ b_0(\vec{w}^h, \vec{v}^h) + b(\vec{v}^h, p^h) = g(\psi^h; \vec{f}, \nu)(\vec{v}^h) \\ \qquad b(\vec{w}^h, q^h) = 0. \end{cases}$$

THEOREM 3.3. *Let ψ be the solution to* (2.1) *and ψ^h the solution to* (2.4). *Let the test space $X^h \subset H_0^1(\Omega)$ and the trial space $S^h \subset L_0^2(\Omega)$ be chosen such that*

$$(3.5)\quad \begin{cases} \textit{for every } q^h \in S^h, \textit{ there exists a } \vec{v}^h \neq 0 \in X^h \textit{ such that} \\ b(\vec{v}^h, q^h) \geqslant \beta \| q^h \|_0 | \vec{v}^h |_1 \textit{ for some} \\ \textit{positive constant } \beta, \textit{ independent of } q^h, \vec{v}^h. \end{cases}$$

Then (3.4) *is uniquely solvable and*

$$\|p - p^h\|_0 \leqslant c_1 \inf_{q^h \in S^h} \| p - q^h \|_0 + c_2 |\psi - \psi^h|_2,$$

where c_1, c_2 are positive constants independent of p, p^h, w^h, ψ and ψ^h.

Proof. It follows from primitive variable theory that (3.4) is uniquely solvable.

Subtracting (3.3) from (3.4), we get

$$(*)\qquad b_0(w^h, v^h) = g(\psi^h; \vec{f}, \vec{\nu})(\vec{v}) - g(\psi; \vec{f}, \nu)(\vec{v}^h) - b(\vec{v}^h, p^h - p).$$

Since $b(\vec{w}^h, p^h) = b(\vec{w}^h, q^h) = 0$ for all $q^h \in S^h$,

$$|w^h|_1 \leqslant K_1 |\psi - \psi^h|_2 + \| p - q^h \|_0,$$

where $K_1 = \nu + 2\|\vec{f}\|/\nu^2$.

Hence, for any $\vec{v} \in X^h$,

$$\begin{aligned} b(\vec{v}^h, q^h - p^h) &= b(\vec{v}^h, q^h - p) + b(\vec{v}^h, p - p^h) \\ &= b(\vec{v}^h, q^h - p) + b_0(w^h, v^h) + g(\psi; \vec{f}, \nu)(\vec{v}^h) - g(\psi^h; \vec{f}, \nu)(\vec{v}^h) \quad \text{by } (*). \end{aligned}$$

Using (3.5) we get

$$\begin{aligned} \|q^h - p^h\|_0 &\leqslant \frac{1}{\beta}\left(\|q^h - p\|_0 + |w^h|_1 + K_1|\psi - \psi^h|_2\right) \\ &\leqslant \frac{1}{\beta}\left(2\|q^h - p\|_0 + 2K_1|\psi - \psi^h|_2\right), \end{aligned}$$

hence

$$\begin{aligned} \| p - p^h \|_0 &\leqslant \| p - q^h \|_0 + \| q^h - p^h \|_0 \\ &\leqslant c_1 \inf_{q^h \in S^h} \| p - q^h \|_0 + c_2 |\psi - \psi^h|_2, \end{aligned}$$

where $c_1 = 1 + 2/\beta$, $c_2 = 2K_1/\beta$. □

Let $\{\mathcal{Q}^h\}$ be a uniformly regular family of triangulations of Ω. For each $\mathcal{Q}^h$, associate a triangulation $\mathcal{T}^h$ of Ω consisting of subtriangles constructed as in Section 2. Let

$$(3.6)\quad X^h = \left\{ \vec{v}^h \in \vec{H}_0^1(\Omega) \mid \vec{v}^h \text{ is a continuous piecewise quadratic on } \mathcal{T}^h \right\},$$

$$(3.7)\quad S^h = \left\{ q^h \in L_0^2(\Omega) \mid q^h \text{ is a piecewise linear function on } \mathcal{Q}^h \right\}.$$

To get convergence, it is sufficient to show that the family $\{X^h, S^h\}$ is div-stable, i.e., for all X^h, S^h in the family, (3.5) holds with β independent of h. This can be done using the local test of [5], since the inclusion $S^h \supset$ {piecewise constants on $\mathcal{Q}^h$} is selfevident. Thus, it suffices to show that $\{X^h, S^h\}$ is locally div-stable, i.e., there exists a $\hat{c} \neq \hat{c}(h)$ such that for all $Q \in \mathcal{Q}^h$, and for all $q^h \in S^h \cap L_0^2(Q)$, there exists a $\vec{v}^h \in X^h \cap H_0^1(Q)$ such that

$$(3.8)\qquad \int_Q |\nabla \vec{v}^h|^2 \leqslant \hat{c} \int_Q (q^h)^2 \quad \text{and} \quad \int_Q q^h \operatorname{div} \vec{v}^h \geqslant \int_Q (q^h)^2.$$

THEOREM 3.4. $\{X^h, S^h\}$ *is locally div-stable.*

Proof. We shall use the following inequalities:

$$(3.9)\qquad \int_T |\nabla \vec{v}^h|^2 \leqslant C_1 h^{-2} \int_T |\vec{v}^h|^2 \quad \text{for all } T \in \mathcal{T}^h,\ \vec{v}^h \in X^h,$$

$$(3.10)\qquad \int_Q |\nabla q^h|^2 \leqslant C_1 h^{-2} \int_Q (q^h)^2 \quad \text{for all } Q \in \mathcal{Q}^h,\ q^h \in S^h,$$

$$(3.11)\qquad \int_Q (q^h)^2 \leqslant C_2 h^2 \int_Q |\nabla q^h|^2 \quad \text{for all } Q \in \mathcal{Q}^h,\ q^h \in S^h \cap L_0^2(Q),$$

where C_1 and C_2 are positive constants independent of h.

Now, let $Q \in \mathcal{Q}^h$ and $q^h \in S^h \cap L_0^2(Q)$ be given. Let $\lambda_i, i = 1,2,3$, be the barycentric coordinates associated with Q, and let $Q_1, Q_2, Q_3 \in \mathcal{T}^h$ be such that $\bigcup_{i=1}^3 Q_i = Q$ and one edge of Q_i lies on $\lambda_i = 0$, $i = 1,2,3$. Define $P\colon \Omega \to \mathbf{R}$ by

$$(3.12)\qquad P = \begin{cases} (3\lambda_i)^2 & \text{on } Q_i, \\ 0 & \text{on } \Omega \setminus Q. \end{cases}$$

Then, a direct calculation shows that for all $\alpha, \beta \in \mathbf{R}$,

$$\begin{pmatrix} \alpha P \\ \beta P \end{pmatrix} \in \vec{H}_0^1(Q) \cap X^h, \quad \int_Q P = \frac{1}{6} \operatorname{area} Q, \quad \int_Q P^2 = \frac{1}{15} \operatorname{area} Q.$$

Define $\vec{v}^h = -6C_2 h^2 \nabla q^h P$, where C_2 is given by (3.11) and P is defined by (3.12). Since ∇q^h is constant on Q and $P \in H_0^1(Q)$, it follows that $\vec{v}^h \in X^h \cap \vec{H}_0^1(Q)$. Moreover,

$$\begin{aligned} \int_Q q^h \operatorname{div} \vec{v}^h &= -\int_Q (\nabla q^h) \cdot \vec{v}^h = 6C_2 h^2 |\nabla q^h|^2 \int_Q P \\ &= C_2 h^2 |\nabla q^h|^2 \operatorname{area} Q \geqslant \int_Q (q^h)^2. \end{aligned}$$

Also,

$$\int_Q |\nabla \vec{v}^h|^2 \leqslant C_1 h^{-2} \int_Q |\vec{v}^h|^2 = 36 C_1 C_2^2 h^2 |\nabla q^h|^2 \int_Q P^2$$
$$= \frac{36}{15} C_1 C_2^2 h^2 |\nabla q^h|^2 \text{area } Q \leqslant \frac{36}{15} C_1^2 C_2^2 \int_Q (q^h)^2.$$

Thus, the local stability is proved, and so by [5] global stability in the sense of (3.5) follows. □

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