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On Weighted Chebyshev-Type Quadrature Formulas

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Abstract. A weighted quadrature formula is of Chebyshev type if it has equal coefficients and real (but not necessarily distinct) nodes. For a given weight function we study the set $T(n, d)$ consisting of all Chebyshev-type formulas with n nodes and at least degree d. It is shown that in nonempty $T(n, d)$ there exist two special formulas having "extremal" properties. This result is used to prove uniqueness and further results for E-optimal Chebyshev-type formulas. For the weight function $w \equiv 1$, numerical investigations are carried out for $n \leqslant 25$.

1. Introduction. Let w be a nonnegative weight function on the interval (a, b), $-\infty \leqslant a < b \leqslant \infty$, admitting moments m_j of all order

$$m_j = \int_a^b x^j w(x)\,dx, \qquad j = 0,1,2,\ldots,\ m_0 > 0. \tag{1.1}$$

We consider (weighted) Chebyshev-type quadrature formulas [7]. These are quadrature formulas Q_n with equal coefficients and real (but not necessarily distinct) nodes:

$$\begin{gathered} Q_n[f] := c\sum_{i=1}^{n} f(x_i), \qquad -\infty < x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n < \infty, \\ \int_a^b f(x)w(x)\,dx = Q_n[f] + R_n[f]. \end{gathered} \tag{1.2}$$

By this definition, it is possible that some nodes x_i are not contained in the interval (a, b). Q_n has at least degree d (of exactness) if

$$R_n[p_i] = 0, \qquad i = 0,1,\ldots,d, \tag{1.3}$$

where p_i, here and throughout this paper, denotes the monomial $p_i(x) := x^i$. If $d \geqslant 0$, the coefficient c in (1.2) is determined by (1.3):

$$c = m_0/n. \tag{1.4}$$

The maximal possible degree of a Chebyshev-type quadrature formula with n nodes is denoted by d_n.

Let, in the following, $T(n, d)$ be the set of all Chebyshev-type quadrature formulas with n nodes and at least degree d. One has $T(n, d+1) \subseteq T(n, d)$ and $T(n, d) \subseteq T(kn, d)$ for every $k \in \mathbf{N}$. For $n > 2$, a simple calculation shows that

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each set $T(n, 2)$ contains an infinite number of elements. In case of $d_n \geqslant n$ the set $T(n, d_n)$ contains only one element, the so-called "Chebyshev quadrature formula in the strict sense". Weight functions w which allow such formulas for every $n \in \mathbf{N}$ are rare [7, p. 109]. In case of $d_n < n$ the set $T(n, d_n)$ possibly contains more than one element. To select some of these formulas several criteria can be found in the literature.

From a historical point of view it may be obvious to consider such quadrature formulas $Q_n^{\text{opt}} \in T(n, d_n)$, which minimize $|R_n[p_{d_n+1}]|$ among all $Q_n \in T(n, d_n)$. Such quadrature formulas are called E-optimal [9], [2], [7]. In case of weight functions w, which are symmetric with respect to the (finite) interval (a, b), several authors have distinguished between symmetric, i.e., $x_i - a = b - x_{n-i+1}$ for $i = 1, \dots, n$, and unsymmetric formulas with regard to E-optimality [10], [9], [2]. There is computational evidence that E-optimal formulas for symmetric weight functions are indeed symmetric [7, p. 113]. Gautschi and Yanagiwara [9] have shown that symmetry would follow, if E-optimal formulas are unique in $T(n, d_n)$. One aim of this paper is to prove the uniqueness of E-optimal formulas in general.

Several authors have proposed other criteria to select special Chebyshev-type formulas—necessarily not contained in $T(n, d_n)$ (see, e.g., [7]). Therefore, it may be of interest to study the set $T(n, d)$ in general. If $T(n, d)$ contains more than one element, we show that there exists in $T(n, d)$ an infinite number of formulas which have pairwise distinct nodes. In this case, there also exists in $T(n, d)$ an infinite number of interpolatory quadrature formulas (for definition, see, e.g., [3]). Among these interpolatory quadrature formulas there are two unique formulas which have several "extremal" properties with respect to all other formulas in $T(n, d)$. By proving that the E-optimal formula Q_n^{opt} is one of the two extremal formulas in $T(n, d_n)$, we can show various properties of E-optimal formulas.

The proofs of all theorems can be found in the supplements section of this issue.

2. *E*-Extremal Formulas. We call a (Chebyshev-type quadrature) formula $Q_n \in T(n, d)$ E-minimal in $T(n, d)$ and denote it by $Q_{n,d}^{\min}$ if

$$R_{n,d}^{\min}[p_{d+1}] = \min\{R_n[p_{d+1}] \mid Q_n \in T(n, d)\}. \tag{2.1}$$

Correspondingly, we define E-maximal formulas $Q_{n,d}^{\max} \in T(n, d)$ by

$$R_{n,d}^{\max}[p_{d+1}] = \max\{R_n[p_{d+1}] \mid Q_n \in T(n, d)\}. \tag{2.2}$$

Therefore, the following inequalities are valid for every $Q_n \in T(n, d)$:

$$Q_{n,d}^{\max}[p_{d+1}] \leqslant Q_n[p_{d+1}] \leqslant Q_{n,d}^{\min}[p_{d+1}].$$

Formulas with property (2.1) or (2.2) we call E-extremal. According to the arguments of Gautschi and Yanagiwara [9] for the existence of E-optimal formulas there exist E-minimal and E-maximal formulas in $T(n, d)$ for all d with

$$1 < d \leqslant d_n. \tag{2.3}$$

Remark. In the following we require for d the validity of (2.3), unless noted otherwise.

Our first result is the uniqueness of E-extremal formulas in $T(n, d)$.

THEOREM 1. *In $T(n, d)$ there exists only one E-minimal formula $Q_{n,d}^{\min}$ and only one E-maximal formula $Q_{n,d}^{\max}$.*

Definition (1.2) allows the possibility that some of the nodes coincide. It can be shown that E-extremal formulas have multiple nodes. Moreover, the two E-extremal formulas can be characterized by a special arrangement of these multiple nodes. To describe this arrangement we define for every Chebyshev-type quadrature formula Q_n the sequence $S(Q_n) := (s_i(Q_n))_{i=1}^{n-1}$ as follows:

$$s_i(Q_n) = \begin{cases} 0, & \text{if } x_{n+1-i} \neq x_{n-i}, \\ 1, & \text{if } x_{n+1-i} = x_{n-i} \text{ and } i \text{ odd}, \\ -1, & \text{if } x_{n+1-i} = x_{n-i} \text{ and } i \text{ even}. \end{cases} \tag{2.4}$$

We speak of a change of sign of the sequence $S(Q_n)$ (between $s_i(Q_n)$ and $s_{i+l}(Q_n)$) if

$$\begin{aligned} &\operatorname{sign}(s_i(Q_n)) = -\operatorname{sign}(s_{i+l}(Q_n)) \neq 0, \\ &s_{i+1}(Q_n) = s_{i+2}(Q_n) = \cdots = s_{i+l-1}(Q_n) = 0. \end{aligned} \tag{2.5}$$

THEOREM 2. *Let Q_n be E-extremal in $T(n, d)$. Then Q_n has at most d distinct nodes. Moreover,*

(i) *Let $d < n - 1$. A formula Q_n is E-extremal in $T(n, d)$ if and only if $S(Q_n)$ has at least $(n - d - 1)$ changes of sign. In this case the following holds: If the first nonzero term of $S(Q_n)$ is negative, then Q_n is E-minimal. If this term is positive, then Q_n is E-maximal. If $S(Q_n)$ has more than $(n - d - 1)$ changes of sign, then Q_n is E-minimal as well as E-maximal and $T(n, d)$ contains only Q_n.*

(ii) *Let $d = n - 1$. A formula Q_n is E-extremal in $T(n, d)$ if and only if $S(Q_n)$ has at least one nonzero term. If this term is negative, then Q_n is E-minimal. If this term is positive, then Q_n is E-maximal. If $S(Q_n)$ has at least one change of sign, then Q_n is E-minimal as well as E-maximal and $T(n, d)$ contains only Q_n.*

Theorem 2 shows that E-extremal formulas are interpolatory quadrature formulas (for definition see, e.g., [3]). The following theorem answers the question for other interpolatory quadrature formulas in $T(n, d)$ and for formulas with pairwise distinct nodes.

THEOREM 3. *Let $Q_{n,d}^{\min}$ and $Q_{n,d}^{\max}$ be the E-minimal and the E-maximal formula in $T(n, d)$. Let*

$$r \in \left(R_{n,d}^{\min}[p_{d+1}],\ R_{n,d}^{\max}[p_{d+1}]\right).$$

Then there exist formulas $\tilde{Q}_n$ and $\overline{Q}_n$ in $T(n, d)$ with $\tilde{R}_n[p_{d+1}] = \overline{R}_n[p_{d+1}] = r$ and

(i) *$\tilde{Q}_n$ has pairwise distinct nodes,*

(ii) *$\overline{Q}_n$ has at most $(d + 1)$ distinct nodes.*

In the case of $d < n - 1$, there exists for each such r even an infinite number of formulas with property (i). In the case of $d = n - 1$ there exists for each such r only one formula $Q_n \in T(n, d)$ with $R_n[p_{d+1}] = r$ and this formula has pairwise distinct nodes.

A first justification for the consideration of E-extremal formulas is the fact that their first and their nth node have extremal properties with respect to all $Q_n \in T(n, d)$.

THEOREM 4. *Let x_i be the nodes of a formula $Q_n \in T(n, d)$, which is not E-extremal. Let $x_i^{\min}$ and $x_i^{\max}$ be the nodes of the E-extremal formulas $Q_{n,d}^{\min}$ and $Q_{n,d}^{\max}$ in $T(n, d)$. Then*

(i) $x_n^{\min} > x_n > x_n^{\max}$,

(ii) $(-1)^d x_1^{\min} > (-1)^d x_1 > (-1)^d x_1^{\max}$.

Therefore, it is also possible to characterize the E-minimal (E-maximal) formula in $T(n, d)$ to be that formula, whose nth node has the largest (smallest) value.

Furthermore, Theorem 4 may be helpful for the investigation of the question of whether all nodes of a formula $Q_n \in T(n, d)$ are contained in the interval $[a, b]$.

The formulas $Q_{n,d}^{\min}$ and $Q_{n,d}^{\max}$ are defined by the extremal property (2.1) and (2.2) of their remainder with respect to only one function, the monomial p_{d+1}. The following theorem shows that these extremal properties remain valid for a wide class of functions, which contains especially all monomials p_{d+2k-1} for all $k \in \mathbf{N}$.

THEOREM 5. *Let $Q_{n,d}^{\min}$ and $Q_{n,d}^{\max}$ be the E-minimal and the E-maximal formula in $T(n, d)$. Then, for all $f \in C^{d+1}$, $f^{(d+1)} \geqslant 0$, there hold*

(i) $R_{n,d}^{\min}[f] = \min\{R_n[f] \mid Q_n \in T(n, d)\}$,

(ii) $R_{n,d}^{\max}[f] = \max\{R_n[f] \mid Q_n \in T(n, d)\}$.

Another interpretation of Theorem 5 may be of interest:

Let K_{d+1} denote the Peano kernel of degree $d + 1$ ([7, p. 112], [3, p. 39]), of a formula Q_n in $T(n, d)$ and $K_{d+1}^{\min}$, resp. $K_{d+1}^{\max}$, the Peano kernels of the same degree of the E-extremal formulas $Q_n^{\min}$ or $Q_n^{\max}$ in $T(n, d)$. Theorem 5 implies the inequalities

$$K_{d+1}^{\min}(x) \leqslant K_{d+1}(x) \leqslant K_{d+1}^{\max}(x)$$

for all $x \in \mathbf{R}$.

3. E-Optimal Formulas. Our basic result for E-optimal formulas is given in the following theorem.

THEOREM 6. *Let $n \in \mathbf{N}$ and let Q_n^{opt} be E-optimal. Then Q_n^{opt} is E-extremal in $T(n, d_n)$.*

An E-optimal formula is therefore E-minimal or E-maximal in $T(n, d_n)$ and has the corresponding properties given in Section 2.

The first part of Theorem 2 has been proven for E-optimal formulas by Anderson and Gautschi [2]. The second part of Theorem 2 reduces all the remaining cases to only two formulas, characterized also by the value of the nth node according to

Theorem 4. Theorem 3 shows, in particular, the impossibility that different E-extremal formulas in $T(n, d_n)$ are both E-optimal. This answers the question of the uniqueness of E-optimal formulas [2], [7].

COROLLARY 1. *Let $n \in \mathbf{N}$. Then there exists one and only one E-optimal formula Q_n^{opt}.*

Therefore, by the result of Gautschi and Yanagiwara [9] mentioned above, it follows from Corollary 1 that

COROLLARY 2. *Let the weight function w be symmetric with respect to (a, b) and let $n \in \mathbf{N}$. Then d_n is odd and the E-optimal formula is symmetric.*

Rabinowitz and Richter [11] have shown that E-optimal rules minimize $|R_n|$ for formulas (1.2) in special function spaces. With the help of Theorem 5, a different justification for the consideration of E-optimal formulas is given by the following theorem.

THEOREM 7. *Let $n \in \mathbf{N}$ and Q_n^{opt} be the E-optimal formula and let $f \in C^{d_n+1}$, $f^{(d_n+1)} \geqslant 0$. If*

$$\operatorname{sign}\left(R_n^{\text{opt}}\left[p_{d_n+1}\right]\right) = \operatorname{sign}\left(R_n^{\text{opt}}[f]\right),$$

then

$$|R_n^{\text{opt}}[f]| = \min\left\{|R_n[f]| \,\middle|\, Q_n \in T(n, d_n)\right\}.$$

4. Numerical Results for the Weight Function $w \equiv 1$. For the weight function $w \equiv 1$ there exist Chebyshev formulas in the strict sense for $n = 1, 2, \ldots, 7$ and $n = 9$—see, e.g., [7]. The E-optimal formulas have been computed by Gautschi and Yanagiwara [9] for $n = 8, 10, 11, 13$ and by Anderson and Gautschi [2] for $n = 12, 14, 15, 16, 17$. Anderson [1] has shown that these formulas, except for $n = 12$, are definite, i.e., there exists a representation of their remainder term of the form

$$R_n[f] = \frac{R_n\left[p_{d_n+1}\right]}{(d_n + 1)!} f^{(d_n+1)}(\xi) \tag{4.1}$$

for every $f \in C^{d_n+1}$.

The present authors have computed the E-extremal formulas in $T(n, d_n)$ for $n \leqslant 25$ by a different method with the help of Theorem 2 resp. Theorem 3 [6]. The E-optimal formulas for $n = 18, \ldots, 25$ are given at the end of this section. These formulas are all definite. Theorem 5 implies that every $Q_n \in T(n, d_n)$ is also definite for $n \leqslant 25$, $n \neq 12$; for $n = 8, 10, 11, 13$ see Förster [4]. In case of definiteness, the comparison of the coefficients of $f^{(d_n+1)}(\xi)$ in (4.1) between the E-minimal and the E-maximal formula gives information as to how useful the choice of the E-optimal formula is in $T(n, d_n)$. These coefficients are listed in Table 1. In every case, the E-minimal formula is E-optimal. The numerical results correspond to the interval of integration $[-1, 1]$.

A conclusion of the above theorems is that the results of Gautschi and Monegato [8] and Förster [4] for $n = 8, 10, 11, 13$ remain valid for all $n \leqslant 25$, $n \neq 12$.

TABLE 1

n	d_n	$R^{\min}_{n,d}[p_{d_n+1}]$	$R^{\max}_{n,d}[p_{d_n+1}]$	$\dfrac{R^{\max}_{n,d}[p_{d_n+1}]}{R^{\min}_{n,d}[p_{d_n+1}]}$	
1	1	0.667 E 0	0.667 E 0	1	definite
2	3	0.178 E 0	0.178 E 0	1	definite
3	3	0.667 E-1	0.667 E-1	1	definite
4	5	0.339 E-1	0.339 E-1	1	definite
5	5	0.172 E-1	0.172 E-1	1	definite
6	7	0.102 E-1	0.102 E-1	1	definite
7	7	0.578 E-2	0.578 E-2	1	definite
8	7	0.2o2 E-2	0.541 E-2	2.68	definite
9	9	0.221 E-2	0.221 E-2	1	definite
10	9	0.119 E-2	0.153 E-2	1.29	definite
11	9	0.573 E-3	0.155 E-2	2.71	definite
12	9	0.663 E-4	0.121 E-2	18.25	$R^{\min}_{12}$ not definite $R^{\max}_{12}$ definite
13	11	0.384 E-3	0.440 E-3	1.15	definite
14	11	0.218 E-3	0.464 E-3	2.13	definite
15	11	0.102 E-3	0.384 E-3	3.76	definite
16	11	0.407 E-4	0.352 E-3	8.65	definite
17	13	0.105 E-3	0.117 E-3	1.11	definite
18	13	0.656 E-4	0.115 E-3	1.75	definite
19	13	0.399 E-4	0.108 E-3	2.71	definite
20	13	0.198 E-4	0.101 E-3	5.10	definite
21	13	0.613 E-5	0.860 E-4	14.03	definite
22	15	0.242 E-4	0.319 E-4	1.32	definite
23	15	0.159 E-4	0.273 E-4	1.72	definite
24	15	0.102 E-4	0.298 E-4	2.92	definite
25	15	0.594 E-5	0.262 E-4	4.41	definite

COROLLARY 3. *Let* $n \leqslant 25$ *and* $w \equiv 1$. *Let* Q_n^{opt} *be the E-optimal formula and* $Q_n \in T(n, d_n)$.

(a) *If in* (1.1) $b = -a$, *then for every* $m \in \mathbf{N}$,

$$0 \leqslant R_n^{\mathrm{opt}}[p_m] \leqslant R_n[p_m].$$

(b) *If* $n \neq 12$ *and* $f \in C^{d_n+1}$, $f^{(d_n+1)} \geqslant 0$, *then*

$$0 \leqslant R_n^{\mathrm{opt}}[f] \leqslant R_n[f].$$

Therefore, these E-optimal formulas satisfy also every optimality criterion of the form

$$\min\left\{ \sum_{i=d_n+1}^{\infty} a_i (R_n[p_i])^2 \,\middle|\, Q_n \in T(n, d_n) \right\}$$

with any $a_i \geqslant 0$ [7, p. 113]. They are, in particular for $n \neq 12$, also optimal in the sense of Sard [7, p. 112], [4].

The E-Optimal Formulas for $w \equiv 1 \qquad 18 \leqslant n \leqslant 25$

$\underline{n = 18}$

$-x_1 =$	0.95611589370931681977	$= x_{18}$
$-x_2 = -x_3 =$	0.78339593833119703042	$= x_{16} = x_{17}$
$-x_4 =$	0.58679047283945639018	$= x_{15}$
$-x_5 = -x_6 =$	0.45756408008040941541	$= x_{13} = x_{14}$
$-x_7 =$	0.25737493728377540704	$= x_{12}$
$-x_8 = -x_9 =$	0.12068411927871514185	$= x_{10} = x_{11}$

$\underline{n = 19}$

$-x_1 =$	0.95841522638659246454	$= x_{19}$
$-x_2 = -x_3 =$	0.79485226355878236323	$= x_{17} = x_{18}$
$-x_4 =$	0.60772484959475892451	$= x_{16}$
$-x_5 = -x_6 =$	0.48688511013054279206	$= x_{14} = x_{15}$
$-x_7 =$	0.29638895564058655907	$= x_{13}$
$-x_8 = -x_9 =$	0.16315108328419371742	$= x_{11} = x_{12}$
$-x_{10} =$	0.0	

$\underline{n = 20}$

$-x_1 =$	0.96051482286129288228	$= x_{20}$
$-x_2 = -x_3 =$	0.80496515092537905967	$= x_{18} = x_{19}$
$-x_4 =$	0.63049631592920524269	$= x_{17}$
$-x_5 = -x_6 =$	0.50749481899047359478	$= x_{15} = x_{16}$
$-x_7 =$	0.35906562874648327105	$= x_{14}$
$-x_8 = -x_9 = -x_{10}$	0.15625951409613565727	$= x_{11} = x_{12} = x_{13}$

$\underline{n = 21}$

$-x_1 =$	0.96243015157286074846	$= x_{21}$
$-x_2 = -x_3 =$	0.81403490074542027161	$= x_{19} = x_{20}$
$-x_4 =$	0.65167313907372323093	$= x_{18}$
$-x_5 = -x_6 =$	0.52555764207596964732	$= x_{16} = x_{17}$
$-x_7 =$	0.40559995128245393129	$= x_{15}$
$-x_8 = -x_9 = -x_{10} =$	0.18868995126113857640	$= x_{12} = x_{13} = x_{14}$
$-x_{11} =$	0.0	

(continues)

(continued)

$n = 22$

$-x_1 =$	0.9641571029955698317 1	$= x_{22}$
$-x_2 = -x_3 =$	0.82238727412825985167	$= x_{20} = x_{21}$
$-x_4 =$	0.66864696018494187221	$= x_{19}$
$-x_5 = -x_6 =$	0.54600146439908270396	$= x_{17} = x_{18}$
$-x_7 =$	0.43289922578951637757	$= x_{16}$
$-x_8 = -x_9 =$	0.24362435512429351622	$= x_{14} = x_{15}$
$-x_{10} =$	0.18706881618423297318	$= x_{13}$
$-x_{11} =$	0.0	$= x_{12}$

$n = 23$

$-x_1 =$	0.96570343338357257096	$= x_{23}$
$-x_2 = -x_3 =$	0.83018849753913168834	$= x_{21} = x_{22}$
$-x_4 =$	0.68178827221824105045	$= x_{20}$
$-x_5 = -x_6 =$	0.56773078130524428871	$= x_{18} = x_{19}$
$-x_7 =$	0.45254730818175202350	$= x_{17}$
$-x_8 = -x_9 =$	0.26810913164371012869	$= x_{15} = x_{16}$
$-x_{10} =$	0.25355181302970919482	$= x_{14}$
$-x_{11} = -x_{12} =$	0.0	$= x_{13}$

$n = 24$

$-x_1 =$	0.96712730714333769553	$= x_{24}$
$-x_2 = -x_3 =$	0.83729311756137103729	$= x_{22} = x_{23}$
$-x_4 =$	0.69467063974654513014	$= x_{21}$
$-x_5 = -x_6 =$	0.58616217620434405885	$= x_{19} = x_{20}$
$-x_7 =$	0.47487624160088429065	$= x_{18}$
$-x_8 = -x_9 = -x_{10} =$	0.29353907385470281834	$= x_{15} = x_{16} = x_{17}$
$-x_{11} =$	0.09382293173785193807	$= x_{14}$
$-x_{12} =$	0.0	$= x_{13}$

$n = 25$

$-x_1 =$	0.96844773854353010676	$= x_{25}$
$-x_2 = -x_3 =$	0.84375871505247479493	$= x_{23} = x_{24}$
$-x_4 =$	0.70773522849837207585	$= x_{22}$
$-x_5 = -x_6 =$	0.60110284438058970914	$= x_{20} = x_{21}$
$-x_7 =$	0.50135993977793685911	$= x_{19}$
$-x_8 = -x_9 = -x_{10} =$	0.31836145542090472915	$= x_{16} = x_{17} = x_{18}$
$-x_{11} =$	0.16110820932771201152	$= x_{15}$
$-x_{12} = -x_{13} =$	0.0	$= x_{14}$

5. Examples. Table 1 shows that in case of $w \equiv 1$ the sets $T(n, d_n)$ for $n \leqslant 25$ and $d_n < n$ contain an infinite number of elements (see Theorem 3). The same is true for the examples computed by Anderson and Gautschi [2] in case of other weight functions. The following example shows with the help of Theorem 2 the possibility that for $d_n < n$ the set $T(n, d_n)$ contains only one element.

Let the weight function w be given by $w(x) := \sqrt{1 - x^2}$. The corresponding Gauss-formula G_5 with 5 nodes and therefore degree 9 is given by (see Szegö [12, p. 344])

$$G_5[f] = \frac{\pi}{24}\left\{f\left(-\frac{\sqrt{3}}{2}\right) + 3f\left(-\frac{1}{2}\right) + 4f(0) + 3f\left(\frac{1}{2}\right) + f\left(\frac{\sqrt{3}}{2}\right)\right\}.$$

Because of $m_0 = \pi/2$ the formula G_5 is a Chebyshev-type quadrature formula (1.2) with the twelve nodes

$$x_1 = -\frac{\sqrt{3}}{2} = -x_{12},$$

$$x_2 = x_3 = x_4 = -\frac{1}{2} = -x_{11} = -x_{10} = -x_9,$$

$$x_5 = x_6 = 0 = -x_8 = -x_7.$$

So G_5 is an element of $T(12, 9)$. By (2.4) the sequence $S(G_5)$ is given by

$$S(G_5) = (0, 1, -1, 0, -1, 1, -1, 0, -1, 1, 0)$$

and has four changes of sign; see (2.5). Theorem 2(i) shows that $T(12, 9)$ contains only the element G_5. Furthermore, G_5 is also the only element of $T(12, 8)$, and G_5 is the E-maximal formula $Q_{12,7}^{\max}$ in $T(12, 7)$.

In the case of $w \equiv 1$ and $n \leqslant 25$ the nodes of the E-minimal formula $Q_{n,d_n}^{\min}$ are contained in the interval $(-1, 1)$. Therefore, in these cases, using Theorem 4, the nodes of every formula $Q_n \in T(n, d_n)$ are also contained in $(-1, 1)$. But this is not so, in general, for every weight function w and every $n \in \mathbf{N}$. The following example shows that there exist even Chebyshev quadrature formulas in the strict sense, i.e., $d_n \geqslant n$, with nodes not all contained in $[a, b]$:

Let w be a weight function on $(-1, 1)$ with $w(x) := (1 - x^2)^{-4/5}$. A simple calculation with the help of Newton's identities (see [7, p. 104]) shows that for $n = 3$, 4, 6, 7 the Chebyshev quadrature formulas in the strict sense exist and that their first and last nodes are not contained in $[-1, 1]$.

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1. L. A. Anderson, *Optimal Chebyshev-Type Quadrature Formulas for Various Weight Functions*, Ph. D. Thesis, Purdue University, 1974.

2. L. A. Anderson & W. Gautschi, "Optimal weighted Chebyshev-type quadrature formulas," *Calcolo*, v. 12, 1975, pp. 211–248.

3. H. Brass, *Quadraturverfahren*, Vandenhoeck & Ruprecht, Göttingen, 1977.

4. K.-J. Förster, "Bemerkungen zur optimalen Tschebyscheff-Typ Quadratur," *Numer. Math.*, v. 38, 1982, pp. 421–425.

5. K.-J. Förster, "A comparison theorem for linear functionals and its application in quadrature," in *Numerical Integration* (G. Hämmerlin, ed.), (ISNM 57), Birkhäuser Verlag, Berlin, 1982, pp. 66–76.

6. K.-J. Förster & G.-P. Ostermeyer, *Zur Optimalität und Berechnung von Tschebyscheff-Typ-Quadraturformeln*, Bericht 1/1984, Institut für Technische Mechanik der Technischen Universität Braunschweig, 1983.

7. W. Gautschi, *Advances in Chebyshev Quadrature*, Lecture Notes in Math., vol. 506, Springer-Verlag, Berlin and New York, 1977, pp. 100–121.

8. W. Gautschi & G. Monegato, "On optimal Chebyshev-type quadratures," *Numer. Math.*, v. 28, 1977, pp. 59–67.

9. W. Gautschi & H. Yanagiwara, "On Chebyshev-type quadratures," *Math. Comp.*, v. 28, 1974, pp. 125–134.

10. D. K. Kahaner, "Chebyshev-type quadrature formulas," *Math. Comp.*, v. 25, 1970, pp. 571–574.

11. P. Rabinowitz & N. Richter, "Chebyshev-type integration rules of minimum norm," *Math. Comp.*, v. 24, 1970, pp. 831–845.

12. G. Szegö, *Orthogonal Polynomials*, Amer. Math. Soc. Colloq. Publ., vol. 23, Amer. Math. Soc., Providence, R.I., 1939.

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Supplement to
On Weighted Chebyshev-Type Quadrature Formulas

By Klaus-Jürgen Förster and Georg-Peter Ostermeyer

6. PROOFS OF THE THEOREMS

Fundamental for our considerations are Newton's well-known identities [7, p. 104] :

Let $t_1,\dots,t_n$ be a solution of

$$\sum_{i=1}^{n} t_i^{\,j} = u_j \,, \quad j = 1,2,\dots,n \,, \tag{6.1}$$

for given values $u_j \in \mathbb{R}$. Let

$$g(t) := \prod_{i=1}^{n} (t-t_i) := t^n + \sum_{i=1}^{n} c_i \, t^{n-i} \,. \tag{6.2}$$

Then

$$\begin{aligned}
&c_1 = - u_1 \\
&c_1u_1 + 2\,c_2 = - u_2 \\
&c_1u_2 + c_2u_1 + 3\,c_3 = - u_3 \\
&\vdots \\
&c_1u_{n-1} + c_2u_{n-2} + \dots + c_{n-1}u_1 + n\,c_n = - u_n \,.
\end{aligned} \tag{6.3}$$

If, conversely, the coefficients c_i are a solution of the system (6.3) then the roots t_i of the corresponding function $g(t)$ in (6.2) satisfy (6.1).

With respect to (1.3) the following holds for all $Q_n \in T(n,d)$ and $j = 1,\dots,d$

$$\sum_{i=1}^{n} x_i^{\,j} = \frac{n\,m_j}{m_o} \,. \tag{6.4}$$

Let $P_n(Q_n)$ be that polynomial of degree n whose roots are the nodes of Q_n

(6.5) $$P_n(Q_n)(x) := \prod_{i=1}^{n} (x-x_i)$$

$$:= x^n + \sum_{i=1}^{d} a_i x^{n-i} + \sum_{i=d+1}^{n} b_i x^{n-i} .$$

It follows from (6.3) and (6.4) that the coefficients a_i in (6.5) of all $P_n(Q_n)$ with $Q_n \in T(n,d)$ are identical. Conversely, every P_n of type (6.5) with these coefficients a_i and only real roots represents a formula $Q_n \in T(n,d)$.

Let H_n be that polynomial of type (6.2) whose coefficients (6.3) are uniquely given by (6.4) with $j=1,\dots,n$:

(6.6) $$H_n(x) := x^n + \sum_{i=1}^{n} e_i x^{n-i} .$$

It follows from (6.3) for $Q_n \in T(n,d)$ and the corresponding $P_n(Q_n)$

$$R_n [p_{d+1}] = \int_a^b w(x)\, x^{d+1}\, dx - \frac{m_o}{n} \sum_{i=1}^{n} x_i^{d+1}$$

(6.7) $$= m_{d+1} + \frac{m_o}{n} \{(d+1)\, b_{d+1} + \sum_{i=1}^{d} a_i \frac{n}{m_o} m_{d+1-i}\}$$

$$= \frac{m_o}{n} (d+1)\, (b_{d+1} - e_{d+1}) .$$

$Q_n \in T(n,d)$ is therefore E-minimal (E-maximal) in $T(n,d)$ if and only if the coefficient b_{d+1} of $P_n(Q_n)$ is minimal (maximal) under the restriction of only real roots of $P_n(Q_n)$ [11].

Corresponding to the sequence $S(Q_n)$ (see (2.4)) we define a sequence $V(Q_n)$ as follows. Let k be the number of sign changes (2.5) of $S(Q_n)$. If there is a sign change between $s_i(Q_n)$ and $s_{i+\ell}(Q_n)$, $0<i<i+\ell<r$, the pair (u,v) is given by

$$(u,v) := (x_{n+1-i-\ell}, x_{n-i}) .$$

The sequence $V(Q_n) := \{(u_i,v_i)\}_{i=1}^{k}$ consists of all these pairs ordered by $u_i \geq u_{i+1}$. By this definition, it is possible that $u_i = v_i = u_{i+1} = v_{i+1}$ and for $k = 0$ that $V(Q_n)$ is empty. We define $K(Q_n) \in \{0,1\}$ by

(6.8) $$K(Q_n) := \begin{cases} 0, & \text{if all elements of } S(Q_n) \text{ are zero} \\ 1, & \text{if an element of } S(Q_n) \text{ is different from zero.} \end{cases}$$

<u>Lemma 1 :</u> Let $Q_n \in T(n,d)$. Let k be the number of sign changes of $S(Q_n)$. If

(6.9) $$k + K(Q_n) < n-d,$$

then Q_n is not E-extremal in $T(n,d)$.

Proof: We have to show the existence of $\bar{Q}_n, \check{Q}_n \in T(n,d)$ with

(6.10) $$\bar{R}_n[p_{d+1}] < R_n[p_{d+1}] < \check{R}_n[p_{d+1}] .$$

One has $k = 0$ for $d = n-1$ and $k \leq n-d-2$ for $d < n-1$. If $d = n-1$ we choose two real numbers $y_o > x_n$ and $y_1 < x_1$. If $d < n-1$ we choose $(n-d)$ real numbers $y_o \geq y_1 \geq \dots \geq y_{n-d-1}$ by means of the sequence $V(Q_n) = \{(u_i,v_i)\}_{i=1}^{k}$ as follows

$$\begin{aligned} &y_i \in (u_i,v_i) \text{ and } y_i \neq x_j \ (j=1,\dots,n), && \text{for } u_i \neq v_i \\ &y_i = u_i && \text{for } u_i = v_i \\ &y_o > x_n && \\ &y_{n-d-1} < \dots < y_{k+1} < x_1 . && \end{aligned}$$

Let α be the first nonzero element of $S(Q_n)$. If all elements of $S(Q_n)$ are zero we define $\alpha := -1$. We consider polynomials $\bar{h}, \check{h}$ of degree $n-d-1$ given by

$$\bar{h}(x) := -\alpha \sum_{i=0}^{n-d-2} (x-y_i),$$

(6.12)

$$h(x) := \alpha \sum_{i=1}^{n-d-1} (x-y_i).$$

For $d = n-1$ we obtain $\bar{h} \equiv -\alpha$ and $\tilde{h} \equiv \alpha$. First, we consider the case $\alpha = -1$. In every interval (y_{2i+1}, y_{2i}), $i = 0,1,\ldots$, by definition of $S(Q_n)$ and $V(Q_n)$ all relative minima of $P_n(Q_n)$ are negative. Let M_1 be the (negative) maximum of all these minima. A similar argument yields the positivity of all relative maxima in every interval (y_{2i}, y_{2i-1}). We denote by M_2 the (positive) minimum of all these maxima. Therefore the following numbers $\bar{m}$, $\tilde{m}$ are positive,

$$\bar{m} := \frac{\min\{|M_1|, M_2\}}{\max\limits_{x\in[y_{n-d-1}, y_o]} |\bar{h}(x)|} ,$$

(6.13)

$$\tilde{m} := \frac{\min\{|M_1|, M_2\}}{\max\limits_{x\in[y_{n-d-1}, y_o]} |h(x)|} .$$

The polynomials $\bar{P}_n := P_n(Q_n) - \bar{m}\,\bar{h}$ and $\tilde{P}_n := P_n(Q_n) - \tilde{m}\,\tilde{h}$ have only real roots. We proof this for $\bar{P}_n$; for $\tilde{P}_n$ the proof is similar. We distinguish two cases.

1) We consider an interval (y_{r+1}, y_r). First we assume that neither y_r nor y_{r+1} coincides with a root of $P_n(Q_n)$. Let ℓ be the number of roots of $P_n(Q_n)$ in this interval. It follows from (6.12) that $\bar{h}$ is of constant sign in (y_{r+1}, y_r). This sign is negative for r even or zero, otherwise it is positive. In (y_{r+1}, y_r), $P_n(Q_n)$ has only roots of maximal multiplicity two by definition of $S(Q_n)$. The existence of a root with higher multiplicity would imply that y_r or y_{r+1} coincides with this root, contrary to our assumption. In the case that two roots of P_n coincide in $y \in (y_{r+1}, y_r)$ then $P_n(Q_n)$ has in y a relative minimum for odd r and a relative maximum otherwise. By choice of $\bar{m}$ the polynomials $P_n(Q_n)$ and $\bar{m}\,\bar{h}$ have ℓ points of intersection.

2) We now consider the case, that y_r coincides with a root of $P_n(Q_n)$. By (6.11) this is possible only for $k > 0$ and $n-d > 2$. The interval (y_{r+i}, y_{r-j}) is defined as follows.
Each $y_p \in (y_{r+i}, y_{r-j})$ is a root of $P_n(Q_n)$, and y_{r+i} as well as y_{r-j} are not roots of $P_n(Q_n)$. Especially y_o and y_{n-d-1} are not roots of $P_n(Q_n)$ by (6.11).
Let ℓ be the number of distinct y_p in $[y_{r+i}, y_{r-j}]$, i.e., the number of elements of the set $\{y_{r+i}, y_{r+i-1}, \ldots, y_{r-j+1}, y_{r-j}\}$. We denote these elements by z_p, $z_1 > z_2 > \ldots > z_\ell$. Let Z_p be the multiplicity of the roots of $\bar{h}$ at z_p and let W_p be the number of roots of $P_n(Q_n)$ in the interval (z_{p+1}, z_p) for $p = 1,\ldots,\ell-1$. We remark that $P_n(Q_n)$ has for $1 < p < \ell$ in z_p a root of multiplicity Z_p+2. $P_n(Q_n) - \bar{m}\,\bar{h}$ has in (y_{r+i}, y_{r-j}) the same number of roots as $P_n(Q_n)$ if $P_n(Q_n) - \bar{m}\,\bar{h}$ has in (z_{p+1}, z_p) for $p \neq 1$, $p \neq \ell-1$ at least W_p+2 roots and for $p = 1$ and $p = \ell-1$ at least W_p+1 roots.
This can be shown by similar argumentation as in case 1), considering the fact, that on the one hand for $1 < p < \ell-1$ the multiplicity of the root z_p of $P_n(Q_n)$ is equal to the multiplicity of the root of $\bar{h}$ in z_p plus two and that on the other hand there exists a $\varepsilon > 0$ so that $\bar{h}$ and $P_n(Q_n)$ are of same constant sign in $z_p-\varepsilon$ as well as in $z_p+\varepsilon$.

By these two cases all roots of $P_n(Q_n)$ have been taken into account once. So $P_n(Q_n)$ and $P_n(Q_n) - \bar{m}\,\bar{h}$ have the same number of real roots. $\bar{h}$ is a polynomial of degree n-d-1. Therefore the formula $\bar{Q}_n$ resulting from $\bar{P}_n = P_n(Q_n) - \bar{m}\,\bar{h}$ is in T(n,d). The leading coefficient of $\bar{m}\,\bar{h}$ is positive by virtue of (6.12). Therefore (6.7) implies the left part of (6.10). For $\alpha = 1$ the argumentation is similar. $\tilde{h}$ and $\bar{h}$ have to be changed.

<u>Remark 1:</u> The proof remains valid if in (6.13) the value $\bar{m}$ ($\tilde{m}$) is replaced by $\bar{m} := c\,\bar{m}$ ($\tilde{m} := c\,\tilde{m}$) with $c \in (0,1)$. Therefore (6.10) implies the fact that under the assumptions of Lemma 1 there is a positive $s \in \mathbb{R}$ so that for any r with

$$r \in [R_n[p_{d+1}] - s\ ,\ R_n[p_{d+1}] + s]$$

a formula $\tilde{Q}_n$ exists in T(n,d) with $\tilde{R}_n[p_{d+1}] = r$.

<u>Remark 2:</u> If $P_n(Q_n)$ has roots of multiplicity higher than one, then the maximal multiplicity of the roots of $P_n(Q_n) - \bar{m}\,\bar{h}$ or $P_n(Q_n) - \tilde{m}\,\tilde{h}$ with any $\bar{m}$ or $\tilde{m}$ chosen as in Remark 1) is lower than the maximal multiplicity before. For these formulas $\tilde{Q}_n$ and $\bar{Q}_n$, Equ. (6.9) is also valid. So if a formula $Q_n \in T(n,d)$ has property (6.9), then there exists an infinite number of formulas in T(n,d) with pairwise distinct nodes.

Let $\bar{Q}_n$ be an E-extremal formula in T(n,d). Lemma 1 implies that $S(\bar{Q}_n)$ has for d = n-1 at least one nonzero element and for d < n-1 at least n-d-1 changes of sign. With definition (2.4), this yields the proof for the first part of Theorem 2.

The next lemma follows from Theorem 1 of [5] together with Remark 1, there.

<u>Lemma 2:</u> Let Q_n and $\tilde{Q}_n$ be in T(n,d). Let $L := Q_n - Q_n$ with

$$L[f] := \sum_{i=1}^{m} A_i\ f(B_i),\ i=1,\dots,m,\ B_1 < B_2 < \dots < B_m\ . \tag{6.14}$$

Let q be defined by $B_q := \max\{B_i | A_i \neq 0,\ i=1,..,m\}$ and let A be the following sequence

$$A := \{ \sum_{i=1}^{k} A_i \}_{k=1}^{m} \ .$$

Then

1) A has at least d changes of sign.
2) If A has at most d changes of sign, then for every

$$f \in C^{d+1}[B_1,B_m],\ f^{(d+1)} \geq 0\ ,$$

$$\mathrm{sign}\,(A_q)\ \tilde{R}_n[f] \ \geq\ \mathrm{sign}\,(A_1)\ \ R_n[f]\ .$$

In the following we first consider the case d < n-1. Let $\bar{x}_i$ be the nodes of an E-extremal formula $\bar{Q}_n$ in T(n,d). Lemma 1 and (2.4) imply that there exist n-d nodes $\bar{x}_{g(i)}$ with $g(i) < g(i+1)$ for i=1,...,n-d-1 so that

$$\begin{aligned} &\bar{x}_{g(i)} = \bar{x}_{g(i)+1}\ , \quad i=1,\dots,n-d\ , \\ &[g(i) + g(i+1)] \text{ is odd for } i=1,\dots,n-d-1\ . \end{aligned} \tag{6.15}$$

Let $\bar{x}_{g(i)+1} \neq \bar{x}_{g(i+1)}$. Then the number of nodes between $\bar{x}_{g(i)+1}$ and $\bar{x}_{g(i+1)}$ is equal to $g(i+1)-g(i)-2$. Eq. (6.15) shows that this number is odd. Let this number be $2\ell(i)-1$. In case of $\bar{x}_{g(i)+1} = \bar{x}_{g(i+1)}$ - for avoiding multiple counting of the same node - we define $2\ell(i)-1 := -1$. Let $\ell(0) := g(1)-1$ and $\ell(n-d) := n-g(n-d)-1$. There follows

$$\begin{aligned} n &= \ell(0) + \ell(n-d) + 2(n-d) + \sum_{i=1}^{n-d-1} (2\ell(i)-1)\ , \\ d &= 1 + \ell(0) + \ell(n-d) + 2 \sum_{i=1}^{n-d-1} \ell(i)\ . \end{aligned} \tag{6.16}$$

In Lemma 2 let for $Q_n \in T(n,d)$

$$L := Q_n - \bar{Q}_n ,$$

$$L[f] := \sum_{i=1}^{m} A_i f(B_i) .$$

For the nodes B_i we require

$$B_1 < B_2 < \dots < B_m ,$$
$$\{B_1, B_2, \dots, B_m\} = \{x_1, \dots, x_n, \bar{x}_1, \dots, \bar{x}_n\} . \qquad (6.17)$$

We see that some coefficients A_i may be zero. Let $A_{r,s}$ be the sequence

$$A_{r,s} := \{ \sum_{j=1}^{k} A_j \}_{k=r}^{s} \qquad 1 \le r \le s \le m \qquad (6.18)$$

and by (6.17)

$$B_{\bar{g}(i)} = \bar{x}_{g(i)} .$$

$\bar{x}_{g(i)} = \bar{x}_{g(i+1)}$, implies $\bar{g}(i) = \bar{g}(i+1)$ and therefore implies that $A_{\bar{g}(i),\bar{g}(i+1)}$ has no sign change, i.e., by the above definition of $\ell(i)$ this sequence has $2\ell(i)$ sign changes. In the case $\bar{g}(i) \neq \bar{g}(i+1)$ the number of nodes in the interval $(B_{\bar{g}(i)}, B_{\bar{g}(i+1)})$ is at least $2\ell(i)-1$. For an estimation of the number of sign changes of $A_{\bar{g}(i),\bar{g}(i+1)}$ we remark, that every element of the sequence A is of the type $c_j c$ with $c_j \in \mathbb{Z}$ and c defined in (1.4). The case $A_{r,r}$ positive and $A_{r+s,r+s}$ negative implies therefore that in the interval $(B_r, B_{r+s}]$ there are at least two nodes of $\bar{Q}_n$. This means that the sequence $A_{\bar{g}(i),\bar{g}(i+1)}$ has at most $2\ell(i)$ changes of sign. This number is reduced to $2\ell(i)-1$ if $A_{\bar{g}(i),\bar{g}(i)}$ or $A_{\bar{g}(i+1),\bar{g}(i+1)}$ is not negative. In the following this last remark is of importance in the case $\ell(n-d) = 0$ and $\bar{x}_n \ge x_n$, because $A_{\bar{g}(n-d),\bar{g}(n-d)} = A_{m,m} = 0$. So we have shown that the number of sign changes of the sequence $A_{\bar{g}(1),\bar{g}(n-d)}$ is at most

$$2 \sum_{i=1}^{n-d-1} \ell(i) \qquad (6.19)$$

or even

$$\{ 2 \sum_{i=1}^{n-d-1} \ell(i) \} - 1 \quad \text{if } \ell(n-d) = 0 \text{ and } \bar{x}_n \ge x_n \text{ and } \bar{g}(1) \neq \bar{g}(n-d) . \qquad (6.20)$$

We now consider the number of sign changes of the sequences $A_{1,\bar{g}(1)}$ and $A_{\bar{g}(n-d),m}$. We distinguish between the following cases.

	I)	Ia) $\ell(0)$ even or $\ell(0)=0$	II)	IIa) $\ell(n-d)$ even or $\ell(n-d)=0$	
(6.21)		Ib) $\ell(o)$ odd		IIb) $\ell(n-d)$ odd	
	1)	1a) $x_1 < \bar{x}_1$	2)	2a) $x_n < \bar{x}_n$	
(6.22)		1b) $x_1 > \bar{x}_1$		2b) $x_n > \bar{x}_n$	
		1c) $x_1 = \bar{x}_1$		2c) $x_n = \bar{x}_n$	

By the same argumentation as above we get the following results. The number of sign changes of $A_{1,\bar{g}(1)}$ is at most

$$\ell(0)+1 \quad \text{for Ia1a) or Ib1b) if } A_{\bar{g}(1),\bar{g}(1)} < 0 . \qquad (6.23)$$

In case of every other combination of I) and 1) the number of sign changes of $A_{1,\bar{g}(1)}$ is at most $\ell(0)$.

Because of $A_{m,m} = 0$ the number of sign changes of $A_{\bar{g}(n-d),m}$ is at most

$$\ell(n-d) \quad \text{for i) IIa2b) or IIa2a) ii) } \ell(n-d) = 0 . \qquad (6.24)$$

For any other combination of II) and 2) there are at most $\ell(n-d)-1$ sign changes of this sequence.

By virtue of (6.16) and (6.19), (6.20), (6.23), (6.24) the sequence A has at most d changes of sign. This number can only be achieved in the following cases

$$(6.24)\qquad \begin{array}{ll} \text{A)} & x_n < \bar{x}_n,\ x_1 > \bar{x}_1,\ \ell(n-d) \text{ odd},\ \ell(0) \text{ odd}, \\ \text{B)} & x_n < \bar{x}_n,\ x_1 < \bar{x}_1,\ \ell(n-d) \text{ odd},\ \ell(0) \text{ even or } \ell(0)=0, \\ \text{C)} & x_n > \bar{x}_n,\ x_1 > \bar{x}_1,\ \ell(n-d) \text{ even or } \ell(n-d)=0,\ \ell(0) \text{ odd}, \\ \text{D)} & x_n > \bar{x}_n,\ x_1 < \bar{x}_1,\ \ell(n-d) \text{ even or } \ell(n-d)=0,\ \ell(0) \text{ even or } \ell(0)=0. \end{array}$$

So we have shown for every Q_n in $T(n,d)$ and for every $f \in C^{d+1}$, $f^{(d+1)} \geq 0$, with the help of Lemma 2, the following result

$$(6.25)\qquad \begin{array}{ll} R_n[f] \geq \bar{R}_n[f] & \text{for } \ell(n-d) \text{ odd}, \\ R_n[f] \leq \bar{R}_n[f] & \text{for } \ell(n-d) \text{ even or } \ell(n-d)=0. \end{array}$$

The number $\ell(n-d)$ resp. $\ell(0)$ is independent of the choice of Q_n in $T(n,d)$. Therefore, $\bar{Q}_n$ is E-minimal for odd $\ell(n-d)$, and E-maximal for even $\ell(n-d)$, or $\ell(n-d)=0$. This proves Theorem 5.

(6.24) resp. (6.25) follows only from the fact that $S(\bar{Q}_n)$ has at least n-d-1 changes of sign. So, by means of Lemma 1, we have proven the first part of Theorem 2i. The second part of Theorem 2i follows from the definition of $\ell(n-d)$ and (2.4).

In the cases A and D in (6.24), d is odd; in the other two cases d is even. This proves Theorem 4 and therefore also Theorem 1.

In the case d = n-1, these theorems can be proven in the same way or with the help of the methods given in[4]. We remark that by Lemma 1 an E-extremal formula has at least one multiple node $\bar{x}_{g(1)}$.

Theorem 3i follows immediately from Theorem 2 and Remarks 1 and 2. To prove Theorem 3ii we consider the corresponding formula Q_n in Theorem 3i and ask for "E-extremal" formulas under the additional assumption $R_n[p_{d+1}] = r$. By the methods given above, it follows that these "E-extremal" formulas have at most d+1 distinct nodes (see Theorem 2).

Let $n > 2$ and therefore $d_n > 1$. Let $Q^{min}_{n,d}$ and $Q^{max}_{n,d}$ be the E-minimal and the E-maximal formula in $T(n,d_n)$. It follows from the definition of d_n and from Theorem 3 that

$$(6.26)\qquad 0 \in [R^{min}_{n,d}[p_{d_n+1}],\ R^{max}_{n,d}[p_{d_n+1}]] \ .$$

By definition, the E-optimal formula Q^{opt}_n must be either E-minimal or E-maximal. This proves Theorem 6 for $n > 2$.

For n = 1, T(1,1) has only one element, so the conclusion is valid. For n = 2, there remains the case $d_2=1$. For every Q_2 in T(2,1) $P(Q_2)$ is a parabola with positive leading coefficient. So there exists a unique E-maximal formula in T(2.1). This is E-optimal by the same arguments as above. An E-minimal formula doesn't exist.

For the proof of Theorem 7, we first consider the case that Q^{opt}_n is the E-minimal formula in $T(n,d_n)$. It follows from (6.26) and Theorem 5 that

$$(6.27)\qquad R^{opt}_n[p_{d_n+1}] > 0 \ ,$$

$$(6.28)\qquad R^{opt}_n[f] \leq R_n[f]$$

for every Q_n in $T(n,d)$ and every $f \in C^{d_n+1}$, $f^{(d_n+1)} \geq 0$. (6.27) and (6.28) imply the assertion. For E-maximal Q^{opt}_n the inequality signs in (6.27) and (6.28) have to be reversed.

It remains to prove Corollary 3a for n = 12. One has $d_{12}=9$ and $R^{opt}_{12}[p_{10}] > 0$. By Theorem 5 and the symmetry of Q^{opt}_{12} we have to show

$$R_{12}^{opt}\,[p_{2i+10}] \geq 0 \tag{6.29}$$

for every $i \in \mathbb{N}$. With Peano's represention of the remainder term - cf. [3,p.39] - there follows

$$\frac{1}{2}\,\frac{(2i)!}{(2i+10)!}\,R_{12}^{opt}\,[p_{2i+10}] = \int_a^b x^{2i}\,K_{10}^{opt}(x)\,dx\ . \tag{6.30}$$

K_{10}^{opt} denotes the Peano kernel of highest degree with respect to Q_{12}^{opt}. K_{10}^{opt} has in (0,b) only one change of sign and is negative at the orign - see [1,p.65]. The positivity of R_{12}^{opt} therefore implies

$$\int_0^b K_{10}^{opt}(x)\,dx > 0\ . \tag{6.31}$$

From (6.30) and (6.31) the inequality (6.29) follows in view of the monotonicity of p_{2i} in (0,b).