MATHEMATICS OF COMPUTATION
VOLUME 46, NUMBER 174
APRIL 1986, PAGES 703-714

On Mordell's Equation $y^2 - k = x^3$: A Problem of Stolarsky

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Abstract. On page 1 of his book *Algebraic Numbers and Diophantine Approximation*, K. B. Stolarsky posed the problem of solving the equation $y^2+999 = x^3$ in positive integers. In the present paper we refine some techniques of Ellison and Pethö and show that the complete set of integer solutions of Stolarsky's equation is

$$\begin{aligned} x &= 10, & y &= \pm 1, \\ x &= 12, & y &= \pm 27, \\ x &= 40, & y &= \pm 251, \\ x &= 147, & y &= \pm 1782, \\ x &= 174, & y &= \pm 2295, \end{aligned}$$

and

$$x = 22480, \qquad y = \pm 3370501.$$

1. In his book *Algebraic Numbers and Diophantine Approximation* [10, p. 1] K.B. Stolarsky posed the following problem and called it one which "a fool can ask but a thousand wise men cannot answer": Which integers x, y satisfy the equation

$$y^2 + 999 = x^3? \tag{1}$$

In this paper we shall play the role of "wise man number 1001" and solve Stolarsky's problem completely. We shall prove

THEOREM 1. *The complete set of integer solutions of Eq.* (1) *is given by*

$$\begin{aligned} x &= 10, & y &= \pm 1, \\ x &= 12, & y &= \pm 27, \\ x &= 40, & y &= \pm 251, \\ x &= 147, & y &= \pm 1782, \\ x &= 174, & y &= \pm 2295, \end{aligned}$$

and

$$x = 22480, \qquad y = \pm 3370501.$$

In our solution we shall use some techniques of Ellison [4], [5] and Pethö [8], but by using a recent sharp result of Waldschmidt [12] on linear forms in logarithms of algebraic numbers, we shall show that the bounds of the type derived by Ellison

Received October 9, 1984; revised February 17, 1985 and March 11, 1985.
1980 *Mathematics Subject Classification*. Primary 10B10.
Key words and phrases. Mordell's equation, Ellison's method, Davenport's lemma, linear forms in logarithms, Thue equations, cubic fields.

[4], [5] for the solution of totally real cubic Thue equations can be reduced from $H \leq 10^{700}$ to about $H \leq 10^{28}$ or so. Further, in his use of the Davenport lemma, Ellison had to use numbers with over 1000 decimal places. Our improved bounds enable us to reduce this to about 63 decimal places, thus providing much improved execution time.

We now state without proof the principal theorems used in our work.

THEOREM 2 (HEMER [6, pp. 15-16]). *Let us write the Mordell equation $y^2 - K = x^3$ as $y^2 - kf^2 = x^3$, where k is squarefree and (f, x^3) is cube free. If $2f$ contains r different primes p_i which split in $Q(\sqrt{k})$, i.e., $p_i = P_i P_i'$, and if the class number $h(Q(\sqrt{k}))$ is not divisible by 3, then all the integer solutions of the equation $y^2 - kf^2 = x^3$ can be found by solving the equations*

$$\prod_{i=1}^{r} p_i^{q_i}(\pm y + f\sqrt{k}) = \prod_{i=1}^{r} P_i^{h_i}\alpha^3 = \eta\beta\alpha^3,$$

where $h_i = 0$ or the least positive integer such that $P_i^{h_i}$ is a principal ideal and all combinations of these values are considered. When $h_i = 0$, we put $q_i = 0$, and when $h_i > 0$ (and thus $h \not\equiv 0 \pmod 3$) we put $q_i = h_i - 2$ if $h_i \equiv 1 \pmod 3$ and $q_i = h_i - 1$ if $h_i \equiv 2 \pmod 3$. Here α is an integer in $Q(\sqrt{k})$. Further, if $k > 0$, $\eta = 1$, ε, or ε', where ε is the fundamental unit of $Q(\sqrt{k})$. If $k < 0$ and $k \neq 3$, $\eta = 1$, and if $k = 3$, $\eta = 1$ or $(1 + \sqrt{-3})/2$.

THEOREM 3 (WALDSCHMIDT [12]). *Let $\alpha_1, \ldots, \alpha_n$ be nonzero algebraic numbers and let $b_1, \ldots, b_n$ be rational integers. Let $D = [Q(\alpha_1, \alpha_2, \ldots, \alpha_n) : Q]$, and suppose that α_i has defining equation $a_0x^d + \cdots + a_d = 0$, where $(a_0, \ldots, a_d) = 1$. Define the measure of α_i by*

$$M(\alpha_i) = a_0 \prod_{\sigma} \max(1, |\sigma\alpha_i|),$$

where σ runs through all embeddings of $Q(\alpha_i) \to C$, and the absolute logarithmic height of α_i by

$$h(\alpha_i) = \frac{\log(M(\alpha_i))}{D}.$$

Further, let $V_0 = 1/D$, and $V_j \geq \max(h(\alpha_j), |\log \alpha_j|/D, V_{j-1})$ for $1 \leq j \leq n$. Finally, let E be any number satisfying

$$1 \leq E \leq \min\left\{e^{DV_1}, \min_{1 \leq j \leq n} 4DV_j/|\log \alpha_j|\right\},$$

and let $V_j^+ = \max(V_j, 1)$ for $j = n$ and $n-1$, with $V_0^+ = 1$ if $n = 1$. If the number $\Lambda = b_0 + b_1 \log \alpha_1 + \cdots + b_n \log \alpha_n$ does not vanish then

$$|\Lambda| > \exp[-W(\log H + C)],$$

where

$$W = C(n)D^{n+2}V_1 \cdots V_n(\log EDV_{n-1}^+)(\log E)^{-n-1},$$
$$H = \max_{1 \leq j \leq n} |a_i|, \qquad C = \log(EDV_n^+),$$

and

$$C(1) \leq 2^{35}, \quad C(2) \leq 2^{53}, \quad C(3) \leq 2^{71}, \quad C(n) \leq 2^{8n+51}n^{2n}.$$

Note. The statement of Waldschmidt's theorem has been simplified here to the case when the b_i are rational. The original theorem is stated for the case when b_i are algebraic. See [12, pp. 257-258] for details.

2. We now proceed to the proof of our theorem. We first derive the cubic Thue equations corresponding to (1). The equation

$$y^2 + 999 = x^3 \tag{1}$$

can be written as

$$y^2 + 111 \cdot 3^2 = x^3.$$

Thus $f = 3$, $k = -111$, and since $-111 \equiv 1 \pmod 8$, $(2) = P_2 P_2'$ in $Q(\sqrt{-111})$. Further (see, e.g., Borevich and Shafarevich [2, p. 425]) $h(Q(\sqrt{-111}) = 8$ and

$$2^8 = \left(\frac{5 + 3\sqrt{-111}}{2}\right)\left(\frac{5 - 3\sqrt{-111}}{2}\right)$$

is the least power of 2 dividing into principal ideals. Thus, by Theorem 2, we must consider two cases:

$$\text{(A)} \quad \pm y + 3\sqrt{-111} = \left(\frac{a + b\sqrt{-111}}{2}\right)^3.$$

This yields

$$a^2 b - 37b^3 = 8, \quad a = 12, \quad b = -2,$$
$$x = 147, \qquad y = \pm 1782.$$

$$\text{(B)} \quad 128(y + 3\sqrt{-111}) = \left(\frac{5 + 3\sqrt{-111}}{2}\right)\left(\frac{a + b\sqrt{-111}}{2}\right)^3,$$

which yields

$$a^3 + 5a^2 b - 333ab^2 - 185b^3 = 2048.$$

Let us put $a = x + y$, $b = y$. We get

$$x^3 + 8x^2 y - 320xy^2 - 512y^3 = 2048.$$

This yields $x \equiv 0 \pmod 8$ and if we replace x by $8x$ we get

$$x^3 + x^2 y - 5xy^2 - y^3 = 4.$$

Next, we make the substitution $x \to x + y$, $y \to x$, and get

$$-4x^3 + 4xy^2 + y^3 = 4.$$

This implies y even, and making the substitution $x \to -x$, $y \to 2y$, we finally get

$$x^3 - 4xy^2 + 2y^3 = 1. \tag{2}$$

This equation has the five solutions

$$\begin{aligned} x &= -1, & y &= -1, \\ x &= 1, & y &= 0, \\ x &= 1, & y &= 1, \\ x &= -5, & y &= -3, \end{aligned} \tag{2a}$$

and

$$x = -31, \qquad y = 14.$$

Our goal is now to prove that these are the only integer solutions of (2). Since these are the only solutions with $|y| \le 14$, we assume from now on that $|y| \ge 15$.

3. Equation (2) defines the cubic field $Q(\theta)$, where $\theta^3 - 4\theta + 2 = 0$. As is easily seen, an integral basis of this field is $(1, \theta, \theta^2)$. Further, by Voronoĭ's algorithm (see Delone and Faddeev [3, Chapter 4]) a pair of fundamental units is

$$\varepsilon_1 = 1 - \theta, \qquad \varepsilon_2 = 1 - 2\theta.$$

Then Eq. (2) implies that $N(x - y\theta) = 1$ and thus that

$$x - y\theta = \pm(1-\theta)^{a_1}(1-2\theta)^{a_2}, \tag{3}$$

and we find that

$$\begin{array}{llllll} a_1 = 1, & a_2 = 0 & \Rightarrow & x = -1, & y = 1, \\ a_1 = 0, & a_2 = 1 & \Rightarrow & x = 1, & y = 2, \\ a_1 = 0, & a_2 = 0 & \Rightarrow & x = 1, & y = 0, \\ a_1 = 5, & a_2 = -2 & \Rightarrow & x = -5, & y = 3, \\ a_1 = 8, & a_2 = 1 & \Rightarrow & x = -31, & y = 14. \end{array} \tag{3a}$$

We must now find all units of the form $x - y\theta$, i.e., all *binomial units* in $Q(\theta)$. By direct search we find that the only binomial units with $H \le 8$ are those listed above. Thus from now on we assume $H \ge 9$.

4. In the rest of this paper we shall use the following approximations to θ, ε_1, ε_2:

(A) The roots of $\theta^3 - 4\theta + 2 = 0$ are given by

$$\theta_1 \doteq -2.21431974337753519,$$
$$\theta_2 \doteq 0.53918887281088912,$$
$$\theta_3 \doteq 1.67513087056664607.$$

(B) Let ε_{ij} denote the jth conjugate of ε_i $(i = 1, 2)$. Then $\varepsilon_1 = 1 - \theta$ satisfies the equation $\varepsilon_1^3 - 3\varepsilon_1^2 - \varepsilon_1 + 1 = 0$ and, corresponding to the above designations for θ_1, we find

$$\varepsilon_{11} \doteq 3.21431974337753519,$$
$$\varepsilon_{12} \doteq 0.46081112718911087,$$
$$\varepsilon_{13} \doteq 0.67513087056664607.$$

Further, $\varepsilon_2 = 1 - 2\theta$ satisfies the equation $\varepsilon_2^3 - 3\varepsilon_2^2 - 13\varepsilon_2 - 1 = 0$, and we have

$$\varepsilon_{21} \doteq 5.42863948675507038,$$
$$\varepsilon_{22} \doteq -0.078377745621778233,$$
$$\varepsilon_{23} \doteq -2.35026174113329214.$$

5. The purpose of this section is to establish an inequality of the form $H \le C \log|y|$, where C is a constant which can be computed in terms of the conjugates of ε_1 and ε_2. This will be used to compute an upper bound for H and will also be essential for the application of Davenport's lemma to our problem. In computing C, we will also establish a new lower bound for $|y|$ by using continued fractions. It should be noted that all numerical values given in this section and in those that follow are only given to a few decimal places. They were actually computed on a VAX-780 computer, using the BC multiprecision floating-point package.

First of all, we define

$$\gamma_i = x - \theta_i y \qquad (i = 1, 2, 3)$$

and $|\gamma_k| = \min_i |\gamma_i|$. Then Eq. (3) yields

$$\log|\gamma_i| = a_1 \log|\varepsilon_{1i}| + a_2 \log|\varepsilon_{2i}|,$$
$$\log|\gamma_j| = a_1 \log|\varepsilon_{1j}| + a_2 \log|\varepsilon_{2j}|.$$

Let us consider the matrix

$$U = \begin{pmatrix} \log|\varepsilon_{1i}| & \log|\varepsilon_{2i}| \\ \log|\varepsilon_{1j}| & \log|\varepsilon_{2j}| \end{pmatrix}.$$

Since $\det(U) = \pm R$, where R is the regulator of the field, and $R \neq 0$, this matrix is invertible. Thus, if we set

$$U^{-1} = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22}, \end{pmatrix},$$

we certainly have

$$|a_1| \leq (|v_{11}| + |v_{12}|) \max_i |\log|\gamma_i||,$$
$$|a_2| \leq (|v_{21}| + |v_{22}|) \max_i |\log|\gamma_i||.$$

Thus, on putting $H = \max(|a_1|, |a_2|)$,

$$H \leq N[U^{-1}] \max_i |\log|\gamma_i||, \tag{4}$$

where

$$N[U^{-1}] = \max_i(|v_{i1}| + |v_{i2}|)$$

is the *row norm* of U^{-1}. Finally, we see that the above derivation is independent of which i and j we choose. Thus, we may always take $i = 1$, $j = 2$.

For our problem we get

$$U^{-1} = \begin{pmatrix} 1.5317081730 & 1.0176567656 \\ -0.4660707691 & -0.7023941057 \end{pmatrix}$$

and $N[U^{-1}] \doteq 2.5493649386$. Thus

$$H \leq 2.5494 \max_i |\log|\gamma_i||.$$

Now let us estimate $\max_i |\log|\gamma_i||$. Let $|\bar{\theta}| = \max_i |\theta_i|$. Then

$$\max_i |\log|\gamma_i|| \leq \max_i \big|\log||\theta_i y| + |x|\big|.$$

Since $|\theta_i y| \leq |\bar{\theta}||y|$, and $|x| \leq |x - \theta_k y| + |\theta_k y| \leq |\bar{\theta}||y| + 1$, we have

$$\max_i |\log|\gamma_i|| \leq \log(2|\bar{\theta}||y| + 1),$$

and

$$H \leq 2.5494 \log(2|\bar{\theta}||y| + 1).$$

Next we determine a new lower bound for $|y|$ by estimating γ_k in the manner of Sprindzhuk [9, pp. 88-89]. We have, on denoting the conjugates of θ by θ_k, θ_j, θ_l,

$$\frac{1}{2}|\theta_k - \theta_j| \leq \frac{1}{2}\left|\frac{x}{y} - \theta_j\right| + \frac{1}{2}\left|\frac{x}{y} - \theta_k\right|.$$

Since $|\gamma_k| = \min_i |\gamma_i|$, we have

$$\left|\frac{x}{y} - \theta_k\right| = \min_i \left|\frac{x}{y} - \theta_i\right|.$$

Thus

$$\frac{1}{2}|\theta_k - \theta_j| \le \left|\frac{x}{y} - \theta_j\right| \tag{5}$$

for $j \ne k$. Also, by Eq. (2) factored over $Q(\theta)$,

$$\begin{aligned} 1 &= |y|^3 \prod_{i=1}^{3} \left|\frac{x}{y} - \theta_i\right| \ge |y|^3 \left|\frac{x}{y} - \theta_k\right| \cdot \frac{1}{4}|\theta_k - \theta_j||\theta_k - \theta_l| \\ &= |y|^3 \frac{|\gamma_k|}{|y|} \cdot \frac{1}{4}|f'(\theta_k)|. \end{aligned}$$

Thus

$$|\gamma_k| \le \frac{4}{\min_i |f'(\theta_i)|}|y|^{-2}.$$

Here $f'(x) = 3x^2 - 4$, so

$$|\gamma_k| \le 1.27885|y|^{-2} \tag{6}$$

and

$$\left|\frac{x}{y} - \theta_k\right| \le \frac{1.27885}{|y|^3}.$$

Further, if $|y| \ge 2$,

$$\frac{1.27885}{|y|^3} < \frac{1}{2y^2},$$

so if (x, y) is a solution of $|f(x, y)| = 1$ with $|y| \ge 3$, $|x/y|$ must be a convergent in the continued fraction expansion of one of the $|\theta_i|$. Using a computer program written in the BC multiprecision language, we examined all convergents of the $|\theta_i|$ with $|y| \le 10^{30}$, and found no solutions except those listed in (3a). Thus $|y| \ge 10^{30}$. Finally, we establish the desired inequality. We are searching for K satisfying $\log(2|\bar{\theta}||y| + 1) \le K \log |y|$, i.e.,

$$2|\bar{\theta}||y| + 1 \le |y|^K,$$

where $|\bar{\theta}| \doteq 2.2143\ldots$. For this it suffices that $|y|^K \ge 4.4286|y| + 1$, i.e., $|y|^{K-1} \ge 4.4286$. Since $|y| \ge 10^{30}$ and this relation must hold for *all* such y, we can take K to satisfy $(10^{30})^{K-1} \ge 4.4286$, which yields $K \ge 1.0215$ and

$$H \le 2.6043 \log |y|. \tag{7}$$

6. In this section we reduce our problem to consideration of an inequality in linear forms in logarithms of algebraic numbers, using some ideas of Sprindzhuk [9] and Ellison [5]. We also establish an upper bound in terms of H for the absolute value of this linear form.

First, we eliminate x and y from two of the γ_i. We have

$$x - \theta_k y = \gamma_k, \qquad x - \theta_j y = \gamma_j,$$

and thus

$$x = \frac{\gamma_j \theta_k - \gamma_k \theta_j}{\theta_k - \theta_j}, \qquad y = \frac{\gamma_j - \gamma_k}{\theta_k - \theta_j}.$$

Substituting into $\gamma_l = x - \theta_l y$, and simplifying, we get

$$(\theta_k - \theta_j)\gamma_l - (\theta_k - \theta_l)\gamma_j = (\theta_l - \theta_j)\gamma_k.$$

Dividing by $(\theta_k - \theta_l)\gamma_j$, we get

$$\delta_0 \delta_1^{a_1} \delta_2^{a_2} - 1 = \frac{\theta_l - \theta_j}{\theta_k - \theta_l} \frac{\gamma_k}{\gamma_j}, \tag{8}$$

where

$$\delta_0 = \pm \frac{\theta_k - \theta_j}{\theta_k - \theta_l}, \qquad \delta_1 = \frac{\varepsilon_{1l}}{\varepsilon_{1j}}, \qquad \delta_2 = \frac{\varepsilon_{2l}}{\varepsilon_{2j}}.$$

Next, we use this result to derive an upper estimate for $|\delta_0 \delta_1^{a_1} \delta_2^{a_2} - 1|$. We have

$$E_1 = \frac{\theta_1 - \theta_2}{\theta_3 - \theta_1} \doteq -0.70794281, \qquad \frac{1}{E_1} \doteq -1.412543469,$$
$$E_2 = \frac{\theta_1 - \theta_2}{\theta_2 - \theta_3} \doteq 2.42398698, \qquad \frac{1}{E_2} \doteq 0.41254347,$$
$$E_3 = \frac{\theta_3 - \theta_1}{\theta_2 - \theta_3} \doteq -3.42398698, \qquad \frac{1}{E_3} \doteq -0.292057185.$$

Thus $(\theta_l - \theta_j)/(\theta_k - \theta_l) \leq 3.42398698$. Also, Eq. (5) yields $\frac{1}{2}|\theta_k - \theta_j| \leq |\gamma_j|/|y|$. Thus

$$|\gamma_j| \geq \frac{|y|}{2} \min |\theta_k - \theta_j| \geq 0.5679709985|y|.$$

So

$$|\gamma_j|^{-1} \leq 1.7606|y|^{-1}.$$

Combining this result with Eq. (6), we get

$$|\delta_0 \delta_1^{a_1} \delta_2^{a_2} - 1| \leq 7.71|y|^{-3}. \tag{9}$$

Let us now reduce (9) to an inequality in the linear forms of the logarithms of the $|\delta_i|$. First, we note that (7) yields

$$|y|^{-3} < e^{-1.1363H}.$$

Further, we can write (8) as

$$\delta_1^{a_1} \delta_2^{a_2} + \delta_3 = \pm \frac{\theta_j - \theta_l}{\theta_k - \theta_j} \frac{\gamma_k}{\gamma_j} = w,$$

where $\delta_3 = -1/\delta_0$. This equation may be written

$$|\delta_1^{a_1} \delta_2^{a_2}| = |-\delta_3 + w|,$$

which yields

$$a_1 \log|\delta_1| + a_2 \log|\delta_2| = \log|\delta_3||1 - w/\delta_3| = \log|\delta_3| + \log|1 - w/\delta_3|.$$

Thus

$$|a_1 \log|\delta_1| + a_2 \log|\delta_2| - \log|\delta_3|| = |\log|1 - w/\delta_3||.$$

Now we estimate $|\log|1 - w/\delta_3||$. We have

$$|\log|1 - w/\delta_3|| = \left| w/\delta_3 + \frac{1}{2} w^2/\delta_3^2 + \ldots \right| \leq |w/\delta_3| \left| \frac{1}{1 - w/\delta_3} \right|,$$

the series expansion being justified because

$$|w/\delta_3| = \left| \frac{\theta_j - \theta_l}{\theta_l - \theta_k} \right| \left| \frac{\gamma_k}{\gamma_j} \right| \leq 7.71|y|^{-3} \leq 7.71 \cdot 10^{-90}.$$

Thus

$$\frac{1}{1-|w/\delta_3|} \le \frac{1}{1-7.71\cdot 10^{-90}} \le 1+10^{-88}.$$

This yields

$$\begin{aligned}|\log|1-w/\delta_3|| &\le (1+10^{-88})\exp(2.04251819-1.1363H)\\ &\le \exp(-0.925H), \quad \text{provided } H \ge 9.\end{aligned}$$

Thus our final inequality is

$$|\Lambda| = |a_1\log|\delta_1| + a_2\log|\delta_2| - \log|\delta_3|| \le \exp(-tH) \quad \text{for } H \ge 9, \tag{10}$$

where $t = 0.925$. Now let us show that there are essentially only three inequalities here. This will be of importance when we apply Davenport's lemma to our problem in Section 8. Then in all future calculations we will choose δ_2 so that $|\delta_2| > 1$, i.e., $\log|\delta_2| > 0$. First, suppose that $k = 1$. Then either $j = 2$, $l = 3$ or $j = 3$, $l = 2$. If $j = 2$, $l = 3$, we have

$$\delta_1 = \frac{\varepsilon_{13}}{\varepsilon_{12}}, \qquad \delta_2 = \frac{\varepsilon_{23}}{\varepsilon_{22}}, \qquad |\delta_3| = \frac{\theta_3-\theta_1}{\theta_1-\theta_2}.$$

If $k = 1$, $j = 3$, $l = 2$, we have

$$\delta_1 = \frac{\varepsilon_{12}}{\varepsilon_{13}}, \qquad \delta_2 = \frac{\varepsilon_{22}}{\varepsilon_{23}}, \qquad |\delta_3| = \frac{\theta_2-\theta_1}{\theta_1-\theta_3},$$

and each quantity in the second case is the reciprocal of the corresponding quantity in the first case. So when we take logarithms all signs change and the corresponding inequalities are the same. The same result holds if $k = 2$ or $k = 3$.

7. Next, we apply Waldschmidt's theorem to derive a lower bound for $|\Lambda|$ in terms of H. By comparing this lower bound with the upper bound obtained in (10), we will derive an inequality for H, which will yield an upper bound for H by solving the corresponding equation. To this end, we first note that the left side of (10) is not zero, since if it were, we would have $|\delta_0\delta_1^{a_1}\delta_2^{a_2}| = 1$. This would yield $\delta_0\delta_1^{a_1}\delta_2^{a_2} = \pm 1$. The upper sign is clearly impossible by Eq. (8), while the lower sign is impossible by Eq. (9) and the fact that $|y| \ge 10^{30}$. Thus we can apply Waldschmidt's theorem to our problem.

Next, by symmetric functions, we find that the equation with roots E_1 to E_6, i.e., that satisfied by all conjugates of δ_3 is

$$37x^6 + 111x^5 - 210x^2 - 605x^3 - 210x^4 + 111x + 37 = 0.$$

Further, we have the following results:

(a) $n = 3$, $D = 6$.

(b) The leading coefficient of the defining equation for δ_3 is 37 and the leading coefficients of the defining equations for δ_1 and δ_2 are both 1 since they are units in $Q(\delta_0, \delta_1, \delta_2)$.

(c) The absolute values of the ε_{ij} are

$$\left|\frac{\varepsilon_{11}}{\varepsilon_{12}}\right| \doteq 6.975318388, \qquad \left|\frac{\varepsilon_{11}}{\varepsilon_{13}}\right| \doteq 4.761032095,$$

$$\left|\frac{\varepsilon_{13}}{\varepsilon_{12}}\right| \doteq 1.465092379, \qquad \left|\frac{\varepsilon_{21}}{\varepsilon_{22}}\right| \doteq 69.26251125,$$

$$\left|\frac{\varepsilon_{21}}{\varepsilon_{23}}\right| \doteq 2.309802092, \qquad \left|\frac{\varepsilon_{23}}{\varepsilon_{22}}\right| \doteq 29.98634016.$$

This yields

$$\begin{aligned} M(\delta_1) &\doteq 48.65533275, & h(\delta_1) &\doteq 0.6474609, \\ M(\delta_2) &\doteq 4797.295465, & h(\delta_2) &\doteq 1.412635, \\ M(\delta_3) &\doteq 433.7764127, & h(\delta_3) &\doteq 1.012089. \end{aligned}$$

We now calculate V_i and E. At this point, we must break the calculations into three separate cases, depending on the value of k. This is because the δ_1, δ_2, δ_3 go together in *triples*; once one is selected, the other two are uniquely determined. Since δ_i and $1/\delta_i$ have the same absolute logarithm, we always choose that pair of subscripts which make $|\delta_2| > 1$. We have

$$\delta_1 = \frac{\varepsilon_{1l}}{\varepsilon_{1j}}, \qquad \delta_2 = \frac{\varepsilon_{2l}}{\varepsilon_{2j}}, \qquad \delta_3 = \pm\frac{\theta_k - \theta_l}{\theta_k - \theta_j}.$$

Let us now state the results for each case:

Case 1: $k = 1,\ j = 2,\ l = 3$,

$$\begin{aligned} &\delta_1 \doteq -1.465092379, && \delta_2 \doteq 29.98634017, && \delta_3 \doteq \pm 1.412543469, \\ &v_1 \doteq 0.6474609, && v_2 \doteq 1.412634599, && v_3 \doteq 1.412634599, \\ & && E \doteq 9.96936282. \end{aligned}$$

Case 2: $k = 2,\ j = 3,\ l = 1$,

$$\begin{aligned} &\delta_1 \doteq -4.761032095, && \delta_2 \doteq -2.309802092, && \delta_3 \doteq \pm 2.42398969, \\ &v_1 \doteq 0.6474609, && v_2 \doteq 1.412634599, && v_3 \doteq 1.412634599, \\ & && E \doteq 9.95797241. \end{aligned}$$

Case 3: $k = 3,\ j = 2,\ l = 1$,

$$\begin{aligned} &\delta_1 \doteq 6.9753518388, && \delta_2 \doteq -69.26251125, && \delta_3 \doteq \pm 3.42398699, \\ &v_1 \doteq 0.6474609, && v_2 \doteq 1.412634599, && v_3 \doteq 1.412634599, \\ & && E \doteq 8.00000000. \end{aligned}$$

Thus the conclusion of Waldschmidt's theorem yields

(A) $k = 1$

$$\begin{aligned} W &\doteq 37642303239098061754906 09.75753769859021343187, \\ C &\doteq 4.43673261256286621140. \end{aligned}$$

(B) $k = 2$

$$\begin{aligned} W &\doteq 37707532765723089403769 86.45810716684223861847, \\ C &\doteq 4.43558941750240920667. \end{aligned}$$

(C) $k = 3$

$$\begin{aligned} W &\doteq 53498381794859368655251 16.77553401405923592710, \\ C &\doteq 4.21665748159514026327, \end{aligned}$$

and, in all cases,

$$tH < W(\log H + C),$$

i.e.,

$$H - \frac{W}{t}\log H - \frac{C}{t} < 0. \tag{11}$$

Now we are searching for an *upper limit* to the values of H satisfying (11). With this in mind, let us choose $H > W/t$. Then the left side of (11) will be an increasing function of H and thus (11) can hold only for $H \le x$, where

$$x - \frac{W}{t}\log x - \frac{C}{t} = 0.$$

If we solve this equation by Newton's method, we find for $k = 1, 2, 3$, respectively, that

$$\begin{aligned} &(\mathrm{A}) \quad H \le 2.66 \cdot 10^{26}, \\ &(\mathrm{B}) \quad H \le 2.67 \cdot 10^{26}, \\ &(\mathrm{C}) \quad H \le 3.79 \cdot 10^{26}. \end{aligned}$$

Thus, in all cases we have $H \le 3.79 \cdot 10^{26}$.

8. We now apply Davenport's lemma to lower the bound for H. First, let us state and prove this lemma.

LEMMA 1 (DAVENPORT). *Suppose θ, β are given real numbers, M and B are rational integers with $B > 6$, and p, q are rational integers satisfying $1 \le q \le MB$, $|\theta q - p| < 2/MB$. Let $H = \max(|b_1|, |b_2|)$. Then, if $\|q\beta\| \ge 3/B$, there is no solution of the inequality*

$$|b_1\theta + b_2 - \beta| \le K^{-H} \tag{12}$$

in rational integers b_1, b_2 with $\log(B^2M)/\log K \le H \le M$, where $\|x\|$ denotes the distance of x to the nearest integer.

Proof (Ellison [4]). Let $\theta - p/q = w$, with $|w| \le 2/qMB$. Then

$$|b_1 q\theta + b_2 q - q\beta| < qK^{-H} \le MBK^{-H}.$$

Now $q\theta = p + wq$, so

$$|b_1 p + b_1 qw + b_2 q - q\beta| < MBK^{-H}.$$

Since $\|q\beta\| \ge 3/B$ and $|b_1 qw| \le 2Mq/MqB = 2/B$, we have

$$\|b_1 qw - q\beta\| \ge 1/B.$$

Thus

$$1/B \le |b_1 p + b_2 q + b_1 qw - q\beta| < MBK^{-H},$$

which yields

$$H \le \frac{\log(B^2M)}{\log K}.$$

To apply the lemma to our problem we must:

(1) Reduce each inequality(10) to one of type (12).

(2) Compute a rational approximation θ_0 to θ such that

$$|\theta - \theta_0| < \frac{1}{(MB)^2}.$$

Now let p/q be the convergent with maximal q in the continued fraction expansion of θ_0 such that $q \le MB$, and let p_{n+1}/q_{n+1} be the next convergent after p/q. Then

$$|q\theta_0 - p| < \frac{1}{q_{n+1}},$$

and

$$|q\theta - p| \le q|\theta - \theta_0| + |q\theta_0 - p| < \frac{q}{(MB)^2} + \frac{1}{q_{n+1}} < \frac{2}{MB}.$$

So we must compute convergents p/q of the continued fraction of θ_0 for each case, find the largest q satisfying $q \le MB$, and check whether $\|q\beta\| \ge 3/B$ holds. If it does, we have a new, substantially lower bound for H. If necessary, we can repeat the lemma again to lower the bound still more.

Note. If $\|q\beta\| < 3/B$ we may either raise B or use the procedure of page 11-07 of [4].

Let us now return to (10). We get

$$\left| a_1 \frac{\log|\delta_1|}{\log|\delta_2|} + a_2 - \frac{\log|\delta_3|}{\log|\delta_2|} \right| < \exp(-0.925H) \quad \text{for } H \ge 9$$

if $k = 1$ and $k = 3$, while, if $k = 2$,

$$\left| a_1 \frac{\log|\delta_1|}{\log|\delta_2|} + a_2 - \frac{\log|\delta_3|}{\log|\delta_2|} \right| < 1.1946 \exp(-0.925H) \le \exp(0.1777738 - 0.925H)$$
$$\le \exp(-0.9052H) \quad \text{for } H \ge 9.$$

Thus, a single inequality which covers all cases is

$$\left| a_1 \frac{\log|\delta_1|}{\log|\delta_2|} + a_2 - \frac{\log|\delta_3|}{\log|\delta_2|} \right| < \exp(-0.9052H) \quad \text{for } H \ge 9. \tag{13}$$

Now we apply Davenport's lemma. We wrote a computer program, using the BC multiprecision language, to expand $\log|\delta_1|/\log|\delta_2|$ in a continued fraction. We used the values of δ_i found in the calculations of Waldschmidt's theorem and

$$M = 3.79 \cdot 10^{26}, \quad B = 100, \quad K = \exp(0.9052),$$

and obtained the following results:

(A) $k = 1$

$$q = 188349937547091612304178180 03, \qquad \|q\beta\| > 0.435 > 0.03.$$

(B) $k = 2$

$$q = 161865729550520164758346434 10, \qquad \|q\beta\| > 0.226 > 0.03.$$

(C) $k = 3$

$$q = 357604185395900095592132048 86, \qquad \|q\beta\| > 0.194 > 0.03.$$

Thus we now have $H \le \log(B^2 M)/\log K \le 77$ for all cases.

We repeated the lemma a second time with $M = 77$, $B = 500$, and found that

$$\begin{array}{llll} \text{(A)} & k = 1, & q = 37897, & \|q\beta\| > 0.040202 > 0.006, \\ \text{(B)} & k = 2, & q = 35991, & \|q\beta\| > 0.428503 > 0.006, \\ \text{(C)} & k = 3, & q = 17371, & \|q\beta\| > 0.340584 > 0.006. \end{array}$$

This yields $H \le 18$ in all cases.

Finally, we searched the range $9 \le H \le 18$ for solutions of (10) and found no further solutions in this range. Thus the only remaining possibility is $H \le 8$. For this case the only solutions are listed in (3a). Thus the equation $y^2 + 999 = x^3$ has exactly the solutions given in Theorem 1.

9. In conclusion, we note that the method used in the proof of Theorem 1 is perfectly general and applies to solving any cubic Thue equation with positive discriminant. Indeed, it is even possible to automate the entire solution process. In a future article, we shall generalize this method to solving totally real quartic Thue equations.

If we combine the results of this paper with those of Baulin [1] and Ljunggren [7] (see also Tzanakis [11]), we now have all integer solutions of the three totally real cubic equations with smallest positive discriminant. Let us list these here.

D	Equation	Solutions
49	$x^3 + x^2y - 2xy^2 - y^3 = 1$	$(1,0),(0,1),(-1,1),(5,4),(4,-9),(-9,5),$ $(2,-1),(-1,-1),(-1,2)$
81	$x^3 - 3xy^2 + y^3 = 1$	$(1,0),(0,-1),(-1,1),(2,1),(-3,2),(1,-3)$
148	$x^3 - 4xy^2 + 2y^3 = 1$	$(-1,-1),(1,0),(1,2),(-5,3),(-31,14)$

Finally, the author wishes to offer his sincere thanks to Professors Josef Blass, Andrew Glass and David Meronk, who pointed out a flaw in an earlier version of this paper and also showed how the constants computed in Section 5 of that paper could be vastly improved.

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