

Rates of Convergence of Gauss, Lobatto, and Radau Integration Rules for Singular Integrands

By Philip Rabinowitz

Abstract. Rates of convergence (or divergence) are obtained in the application of Gauss, Lobatto, and Radau integration rules to functions with an algebraic or logarithmic singularity inside, or at an endpoint of, the interval of integration. A typical result is the following: For a generalized Jacobi weight function on $[-1, 1]$, the error in applying an n -point rule to $f(x) = |x - y|^{-\delta}$ is $O(n^{-2+2\delta})$, if $y = \pm 1$ and $O(n^{-1+\delta})$ if $y \in (-1, 1)$, provided we avoid the singularity. If we ignore the singularity y , the error is $O(n^{-1+2\delta}(\log n)^\delta(\log \log n)^{\delta(1+\epsilon)})$ for almost all choices of y . These assertions are sharp with respect to order.

1. Introduction. This paper is a sequel to that of Lubinsky and myself [2] on the rates of convergence of Gauss integration rules for singular integrands. In this paper we extend the results in [2] in several directions. On the one hand, we extend the results for Gauss integration rules to results for Lobatto and Radau rules. This extension follows easily from the generalized Markov-Stieltjes inequality for Lobatto and Radau rules given in Lemma 3.2 and from a representation of the coefficients (or weights or Cotes numbers) in the Lobatto and Radau rules in terms of the coefficients in the Gauss rule with respect to a related weight function. A second extension is to the special case of Gauss-Jacobi integration rules as well as Lobatto-Jacobi and Radau-Jacobi rules for certain values of the parameters defining the Jacobi weight function

$$(1.1) \quad \omega^{(\alpha, \beta)}(x) = (1 - x)^\alpha (1 + x)^\beta, \quad \alpha, \beta > -1.$$

In [2], there were some results for $\alpha, \beta = \pm \frac{1}{2}$. In this paper, there are similar results for

$$(1.2) \quad -1 < \alpha = \beta \quad \text{and} \quad -\frac{1}{2} \leq \alpha, \beta \leq \frac{1}{2}$$

in the Gauss case, and for other ranges of α and β in the Lobatto and Radau cases. These results are based on the results in [5], where convergence and divergence theorems were proved for Gauss-Jacobi rules but no stress was placed on the rates of convergence as in this paper and in [2]. Finally, in dealing with endpoint singularities, we generalize the weight functions considered in [2] to the generalized Jacobi weight functions studied by Nevai and others [3].

Since this paper follows closely on [2], we shall only state the theorems generalizing those given therein but shall not, in general, give proofs, since they are almost identical word-for-word with those in [2], the major difference being that we shall

Received September 3, 1985.
1980 *Mathematics Subject Classification.* Primary 65D32.

use $\omega(x) dx$ in place of $d\alpha(x)$ used in [2]. Of course, where there are some differences in the proofs, we shall indicate this. In Section 2, we shall establish our notation and introduce the integration rules. In Section 3, we shall prove the generalized Markov-Stieltjes inequality for Lobatto and Radau rules (Lemmas 3.2 and 3.3), and an additional basic lemma. In Sections 4 and 5, we study rates of convergence of our rules for functions with an interior singularity which satisfy certain monotonicity conditions, in particular, the functions $|x - y|^{-\delta}$, $0 < \delta < 1$, and $-\log|x - y|$. In Section 6, we do the same for functions with endpoint singularities. Finally, in Sections 7 and 8, we generalize the results of Sections 4–6 to more general functions. This is the same structure as in [2], and will facilitate reference to that paper.

2. Notation. Let $[a, b]$ be a finite interval and $r, s \in \{0, 1\}$. We shall be concerned with the approximation of the integral

$$(2.1) \quad I[f] = \int_a^b f(x)\omega(x) dx$$

by numerical integration rules of the form

$$(2.2) \quad I_n[f] = \sum_{i=1-r}^{n+s} \lambda_{ni} f(x_{ni}).$$

Here, $\omega(x)$ is a nonnegative weight function which is positive over a subinterval of $[a, b]$ (usually almost everywhere) and in $L_1(a, b)$, and is such that $I[f]$ is properly or improperly Riemann-integrable. The points x_{ni} are the zeros of

$$(1+x)^r(1-x)^s p_n(x),$$

where $p_n(x) = k_n x^n + \dots$, $k_n > 0$, belongs to the family of orthonormal polynomials with respect to the weight function

$$(2.3) \quad \bar{\omega}(x) = (1+x)^r(1-x)^s \omega(x),$$

and are ordered as follows:

$$a = x_{n0} < x_{n1} < \dots < x_{nn} < x_{n,n+1} = b.$$

The coefficients λ_{ni} are interpolatory and are given explicitly in terms of the $p_n(x)$ by

$$(2.4) \quad \lambda_{ni} = \bar{\lambda}_{ni} (1+x_{ni})^{-r} (1-x_{ni})^{-s}, \quad i = 1, \dots, n,$$

where the $\bar{\lambda}_{ni}$ are the Christoffel numbers

$$(2.5) \quad \bar{\lambda}_{ni} = \left(\sum_{k=0}^{n-1} p_k^2(x_{ni}) \right)^{-1}, \quad i = 1, \dots, n,$$

while if $r = 1$, $(b-a)^s \lambda_{n0} = I((b-x)^s) - \sum_{j=1}^n \lambda_{nj} (b-x_{nj})^s$, and if $s = 1$, $(b-a)^r \lambda_{n,n+1} = I((x-a)^r) - \sum_{j=1}^n \lambda_{nj} (x_{nj}-a)^r$.

For $r = s = 0$, (2.3) reduces to the Gauss rule; for $r = 1, s = 0$, to the left Radau rule; for $r = 0, s = 1$, to the right Radau rule; for $r = s = 1$, to the Lobatto rule. If we define

$$E_n[f] = I[f] - I_n[f],$$

then $E_n[f] = 0$ whenever f is a polynomial of degree less than $2n + r + s$.

We shall frequently need to consider some fixed point $y \in (a, b)$ at which $f(x)$ may, or may not, have a singularity. Throughout $x_{c(n)}, x_{l(n)}, x_{r(n)}$ denote the points from $\{x_{n0}, x_{n1}, \dots, x_{nn}, x_{n,n+1}\}$ which are, respectively, the closest to y , the closest from the left to y , and the closest from the right to y . More precisely,

$$\begin{aligned} |x_{c(n)} - y| &= \min\{|x_{nj} - y|: j = 0, 1, \dots, n + 1\}, \\ y - x_{l(n)} &= \min\{y - x_{nj}: x_{nj} \leq y\}, \\ x_{r(n)} - y &= \min\{x_{nj} - y: x_{nj} > y\}. \end{aligned}$$

When $x_{c(n)}$ is not uniquely defined by the above, which is the case only when y is midway between $x_{l(n)}$ and $x_{r(n)}$, we take $x_{c(n)} = x_{l(n)}$. We let

$$I_n^*[f] = \sum_{\substack{j=1 \\ j \neq c(n)}}^n \lambda_{nj} f(x_{nj}),$$

so that I_n^* avoids the singularity by omitting the closest abscissa to it. Further, we let

$$E_n^*[f] = I[f] - I_n^*[f].$$

Similarly, we define

$$I_n^{**}[f] = \sum_{\substack{j=1 \\ j \neq l(n), r(n)}}^n \lambda_{nj} f(x_{nj}),$$

so that I_n^{**} avoids the singularity by omitting the closest abscissas from the left and right to y . Further,

$$E_n^{**}[f] = I[f] - I_n^{**}[f].$$

We let $\lambda_{c(n)}, \lambda_{l(n)}, \lambda_{r(n)}$ denote the coefficients corresponding to $x_{c(n)}, x_{l(n)}, x_{r(n)}$, respectively. Similarly, $\lambda_{c(n) \pm 1}, \lambda_{l(n) \pm 1}$ denote $\lambda_{n, c(n) \pm 1}$ and $\lambda_{n, l(n) \pm 1}$ and so on. Note that $x_{r(n)} = x_{l(n)+1}$.

Definition 2.1. We shall say that the integral of $\omega(x)$ is bounded above and below near y , $\omega \in IB(y)$, if there exist positive constants m and M such that

$$(2.6) \quad m \leq (x_2 - x_1)^{-1} \int_{x_1}^{x_2} \omega(x) dx \leq M$$

for all x_1, x_2 in a neighborhood of y .

Thus the Jacobi weight $\omega^{(\alpha, \beta)}(x) \in IB(y)$ for all $y \in (-1, 1)$.

The usual symbols O, o, \sim, \cong will be used to compare sequences and functions. For example, if $(c_n), (d_n)$ are sequences of real numbers,

$$\begin{aligned} c_n = O(d_n) &\Leftrightarrow \limsup_{n \rightarrow \infty} |c_n/d_n| < \infty, \\ c_n = o(d_n) &\Leftrightarrow \lim_{n \rightarrow \infty} c_n/d_n = 0, \\ c_n \cong d_n &\Leftrightarrow \lim_{n \rightarrow \infty} c_n/d_n = 1, \\ c_n \sim d_n &\Leftrightarrow K_1 \leq c_n/d_n \leq K_2, \end{aligned}$$

for all large enough n , where K_1 and K_2 are positive constants.

Definition 2.2. Let \mathcal{G} be a finite real closed interval. If $f \in C[\mathcal{G}]$, the modulus of continuity of f in \mathcal{G} is

$$\omega_f(\mathcal{G}; \epsilon) = \max\{|f(x_1) - f(x_2)| : |x_1 - x_2| \leq \epsilon; x_1, x_2 \in \mathcal{G}\} \quad \text{for any } \epsilon > 0.$$

We say $f \in \text{Lip}(\theta)$ in \mathcal{G} where $0 < \theta \leq 1$ if $\omega_f(\mathcal{G}; \epsilon) = O(\epsilon^\theta)$, and we say $f \in \text{Lip}(\theta; \eta)$ in \mathcal{G} where $\theta \geq 0$ and η is real if $\omega_f(\mathcal{G}; \epsilon) = O(\epsilon^\theta |\log \epsilon|^{-\eta})$.

Definition 2.3. Let \mathcal{G} be a real interval. Let k be a positive integer. We shall say $f: \mathcal{G} \rightarrow \mathbf{R}$ is k -absolutely monotone in \mathcal{G} (k -completely monotone in \mathcal{G}) if

$$(2.7) \quad \begin{aligned} f^{(j)}(x) &\geq 0, & x \in \mathcal{G}, j = 0, 1, 2, \dots, k, \\ ((-1)^j f^{(j)}(x) &\geq 0, & x \in \mathcal{G}, j = 0, 1, 2, \dots, k). \end{aligned}$$

If f is k -absolutely monotone in \mathcal{G} (k -completely monotone in \mathcal{G}) for all positive integers k , we shall say f is absolutely monotone in \mathcal{G} (completely monotone in \mathcal{G}).

3. Basic Lemmas. The Markov-Stieltjes inequalities in Lemmas 3.2 and 3.3 depend on the following fundamental lemma:

LEMMA 3.1. *Let f be $(m + 1)$ -absolutely monotone in $[a, \xi]$ with strict inequality holding in (2.7). Let $P(x)$ be a polynomial of degree at most m . Let*

$$m_1 = \text{total multiplicity of zeros of } f - P \text{ in } [a, \xi],$$

$$m_2 = \text{total multiplicity of zeros of } P \text{ in } [\xi, b].$$

Then $m_1 + m_2 \leq m + 1$.

Proof. Freud [1, Lemma I.5.3] gives a proof for the case $(-\infty, \xi]$ and $[\xi, \infty)$. By substituting $[a, \xi]$ and $[\xi, b]$ for $(-\infty, \xi]$ and $[\xi, \infty)$ respectively throughout his proof, we see that our statement is also true. \square

LEMMA 3.2. *Let $f(x)$ be $(2n - 1 + s)$ -absolutely monotone in $[a, x_{nk})$ for some $n \geq 1, 1 - r \leq k \leq n + s$. Then*

(i)

$$\sum_{j=1-r}^{k-1} \lambda_{nj} f(x_{nj}) \leq \int_a^{x_{nk}} f(x) \omega(x) dx.$$

(ii) *If, in addition, $f(x)$ is $(2n - 1 + s)$ -absolutely monotone in $[a, x_{nk}]$, then*

$$\sum_{j=1-r}^k \lambda_{nj} f(x_{nj}) \geq \int_a^{x_{nk}} f(x) \omega(x) dx.$$

In particular,

$$\sum_{j=1-r}^{k-1} \lambda_{nj} \leq \int_a^{x_{nk}} \omega(x) dx \leq \sum_{j=1-r}^k \lambda_{nj}, \quad 1 \leq k \leq n + s.$$

Proof. (i) Define a polynomial $p(x)$ of degree $\leq 2n - 2 + s$ by the $2n - 1 + s$ interpolation conditions

$$(3.1A) \quad p(x_{nj}) = \begin{cases} f(x_{nj}), & j = 1, 2, \dots, k - 1, \\ 0, & j = k, k + 1, \dots, n + s; \end{cases}$$

$$(3.1B) \quad p'(x_{nj}) = \begin{cases} f'(x_{nj}), & j = 1, 2, \dots, k - 1, \\ 0, & j = k + 1, k + 2, \dots, n. \end{cases}$$

We shall assume initially that strict inequality holds in (2.7). Let $\xi \in (x_{n,k-1}, x_{nk})$. Then by (3.1A, B), $f - p$ has $m_1 \geq 2k - 2$ zeros in $[a, \xi]$ and p has $m_2 \geq 2n - 2k + 1 + s$ zeros in $[\xi, b]$. Thus $m_1 + m_2 \geq 2n - 1 + s = \deg(p) + 1$. By Lemma 3.1, we have $m_1 + m_2 \leq 2n - 1 + s$. Thus, $m_1 = 2k - 2$ and $m_2 = 2n - 2k + 1 + s$, and the only zeros of $f - p$ and p in $[a, \xi]$ and $[\xi, b]$, respectively, are already listed in (3.1A, B). As all zeros of $f - p$ in $(a, \xi]$ are double zeros, it follows that $f - p$ does not change sign in $(a, \xi]$ for any $\xi < x_{nk}$, and hence $f - p$ does not change sign in (a, x_{nk}) . As $p(x_{nk}) = 0$, we deduce

$$(3.2) \quad f(x) \geq p(x), \quad x \in [a, x_{nk}).$$

Next, as $\xi > x_{n,k-1}$ was arbitrary, it follows $p(x)$ has $2n - 2k + 1 + s$ zeros in $(x_{n,k-1}, b]$, these being listed in (3.1A, B). Since $p(x_{n,k-1}) = f(x_{n,k-1}) > 0$ and as $p(x)$ has a simple zero at x_{nk} and double zeros at x_{nj} , $j = k + 1, k + 2, \dots, n$, it follows that $p(x)$ changes sign at x_{nk} and

$$(3.3) \quad 0 \geq p(x), \quad x \in [x_{nk}, b].$$

Then by (3.2) and (3.3), and by (3.1A),

$$\int_a^{x_{nk}} f(x)\omega(x) dx \geq \int_a^b p(x)\omega(x) dx = \sum_{j=1-r}^{n+s} \lambda_{nj}p(x_{nj}) = \sum_{j=1-r}^{k-1} \lambda_{nj}f(x_{nj}).$$

Finally, if strict inequality does not hold in (2.7), $f_\epsilon(x) = f(x) + \epsilon e^x$ satisfies (2.7) with strict inequality for any $\epsilon > 0$. Applying the above inequality to f_ϵ , and letting $\epsilon \rightarrow 0 +$, we obtain the more general inequality.

(ii) is similar: One defines a polynomial $P(x)$ of degree $\leq 2n - 2 + s$ by

$$P(x_{nj}) = \begin{cases} f(x_{nj}), & j = 1, 2, \dots, k, \\ 0, & j = k + 1, k + 2, \dots, n + s; \end{cases}$$

$$P'(x_{nj}) = \begin{cases} f'(x_{nj}), & j = 1, 2, \dots, k - 1, \\ 0, & j = k + 1, k + 2, \dots, n, \end{cases}$$

and uses Lemma 3.1 to deduce

$$f(x) \leq P(x), \quad x \in [a, x_{nk}],$$

$$0 \leq P(x), \quad x \in [x_{nk}, b]. \quad \square$$

For $(2n - 1 + r)$ -completely monotone functions, there is the following corollary.

LEMMA 3.3. *Let $f(x)$ be $(2n - 1 + r)$ -completely monotone in $(x_{nk}, b]$ for some $n \geq 1, 1 \leq k \leq n$. Then*

(i)

$$\sum_{j=k+1}^{n+s} \lambda_{nj}f(x_{nj}) \leq \int_{x_{nk}}^b f(x)\omega(x) dx.$$

(ii) *If, in addition, $f(x)$ is $(2n - 1 + r)$ -completely monotone in $[x_{nk}, b]$, then*

$$\sum_{j=k}^{n+s} \lambda_{nj}f(x_{nj}) \geq \int_{x_{nk}}^b f(x)\omega(x) dx.$$

Proof. See [2].

The following lemma on the asymptotic behavior of coefficients and abscissas will be useful in the sequel.

LEMMA 3.4. Assume that $\omega(x) \in IB(y)$ for $y \in (a, b)$. Then there exist positive constants c_1, c_2, c_3, c_4 and a neighborhood \mathcal{G} of y such that for all n and $j, 1 \leq j \leq n$,

$$(3.4) \quad \begin{aligned} (i) \quad & x_{nj} \in \mathcal{G} \Rightarrow c_1/n \leq x_{n,j+1} - x_{nj} \leq c_2/n, \\ (ii) \quad & x_{nj} \in \mathcal{G} \Rightarrow c_3/n \leq \lambda_{nj} \leq c_4/n. \end{aligned}$$

Proof. (i) See [2].

(ii) By Lemma 3.4(ii) in [2], the Christoffel numbers $\bar{\lambda}_{nj}$ satisfy (ii). Hence, by (2.4), (ii) holds. \square

4. Interior Singularities, Part 1. In this section, we investigate the asymptotic behavior of $E_n[f]$, where $f(x) = |x - y|^{-\delta}$ or $-\log|x - y|$. First, however, we establish our basic error estimate which may be applied to functions with a singularity on either one or both sides of y .

LEMMA 4.1. Let $f(x)$ be $(2n - 1 + s)$ -absolutely monotone in $[a, y)$ and $(2n - 1 + r)$ -completely monotone in $(y, b]$. Then

$$(4.1) \quad (i) \quad \int_{x_{l(n)}}^{x_{r(n)}} f(x)\omega(x) dx \leq E_n^{**}[f] \leq \int_{x_{l(n)-1}}^{x_{r(n)+1}} f(x)\omega(x) dx.$$

(ii) If $y \neq x_{c(n)}$,

$$(4.2) \quad \int_{x_{l(n)}}^{x_{r(n)}} f(x)\omega(x) dx - \sum_{j=l(n)}^{r(n)} \lambda_{nj}f(x_{nj}) \leq E_n[f] \leq \int_{x_{l(n)}}^{x_{r(n)}} f(x)\omega(x) dx.$$

(iii) If j is the integer such that $j \in \{l(n), r(n)\} \setminus \{c(n)\}$, then

$$(4.3) \quad E_n^*[f] = E_n^{**}[f] - \lambda_{nj}f(x_{nj}).$$

(iv) If $y = x_{c(n)}$,

$$(4.4) \quad 0 \leq E_n^*[f] \leq \int_{x_{l(n)-1}}^{x_{r(n)}} f(x)\omega(x) dx.$$

Proof. See [2].

We can now prove a general theorem for “2-sided” singularities:

THEOREM 4.2. Let (a, b) be a finite interval and $y \in (a, b)$. Let $\omega(x) \in IB(y)$. Let $f(x)$ be absolutely monotone in $[a, y)$, completely monotone in $(y, b]$, and let $f(y) = 0$. Further assume $f(x)$ grows at roughly the same rate on both sides of y as $x \rightarrow y$, that is

$$(4.5) \quad f(y - u) \sim f(y + u) \quad \text{as } u \rightarrow 0 + .$$

Let $\mu_n = \int_{y-1/n}^y f(x) dx, n = 1, 2, 3, \dots$. Then

- (i) $E_n^{**}[f] \sim \mu_n$,
 - (ii) $E_n^*[f] = O(\mu_n)$,
 - (iii) $E_n[f] = O(\mu_n) - \lambda_{c(n)}f(x_{c(n)})$,
- and $\lambda_{c(n)} \sim n^{-1}$.

Proof. See proof of Theorem 4.3 in [2].

Thus the rate of convergence to 0 of the error, where the singularity is avoided using I_n^* or I_n^{**} , is determined by the asymptotic behavior of μ_n . As a first

corollary, we have

COROLLARY 4.3. *Let (a, b) be a finite interval and $y \in (a, b)$. Let*

$$f(x) = \begin{cases} |x - y|^{-\delta}, & x \in (a, b) \setminus \{y\}, \\ 0, & x = y, \end{cases}$$

where $0 < \delta < 1$. Assume $\omega \in IB(y)$. Then

- (i) $E_n^{**}[f] \sim n^{-1+\delta}$,
- (ii) $E_n^*[f] = O(n^{-1+\delta})$,

and there exists $\delta_0 \in (0, 1)$ such that, whenever $\delta \in (\delta_0, 1)$, we have

$$(4.6) \quad E_n^*[f] \sim n^{-1+\delta}.$$

- (iii) For those positive integers n for which $y \neq x_{c(n)}$,

$$(4.7) \quad E_n[f] = -\lambda_{c(n)}|x_{c(n)} - y|^{-\delta} + O(n^{-1+\delta}) = O(n^{-1}|x_{c(n)} - y|^{-\delta}),$$

where $\lambda_{c(n)} \sim n^{-1}$.

Proof. See proof of Corollary 4.4 in [2].

Next, we have a corollary for logarithmic singularities.

COROLLARY 4.4. *Let (a, b) be a finite interval and $y \in (a, b)$. Let*

$$f(x) = \begin{cases} -\log|x - y|, & x \in (a, b) \setminus \{y\}, \\ 0, & x = y. \end{cases}$$

Assume $\omega \in IB(y)$. Then

- (i) $E_n^{**}[f] \sim n^{-1} \log n$.
- (ii) $E_n^*[f] = O(n^{-1} \log n)$.
- (iii) For those positive integers n for which $y \neq x_{c(n)}$,

$$E_n[f] = -\lambda_{c(n)} \log|x_{c(n)} - y| + O(n^{-1} \log n) = O(n^{-1} \log|x_{c(n)} - y|),$$

where $\lambda_{c(n)} \sim n^{-1}$.

Proof. See proof of Corollary 4.5 in [2].

In our next corollary, we have the following analogue of Theorem 2 in Rabinowitz [4], for the case where $y = \cos(\pi p/q)$ with p/q a rational number.

COROLLARY 4.5. *Let $(a, b) = (-1, 1)$ and let $\omega(x)$ be the Jacobi weight function $\omega^{(\alpha, \beta)}(x)$, where $\alpha + s, \beta + r = \pm 1/2$ and $\alpha, \beta > -1$. Let $y = \cos(\pi p/q)$, where p/q is a rational number in $(0, 1)$.*

- (i) If

$$f(x) = \begin{cases} |x - y|^{-\delta}, & x \in (-1, 1) \setminus \{y\}, \\ 0, & x = y, \end{cases}$$

where $0 < \delta < 1$, then $E_n[f] = O(n^{-1+\delta})$.

- (ii) If

$$f(x) = \begin{cases} -\log|x - y|, & x \in (-1, 1) \setminus \{y\}, \\ 0, & x = y, \end{cases}$$

then $E_n[f] = O(n^{-1} \log n)$.

Proof. See proof of Corollary 4.6 in [2].

In our final corollary, we extend the results of Corollary 4.5 for more general Jacobi weights.

COROLLARY 4.6. *Let $(a, b) = (-1, 1)$ and let $\omega(x)$ be the Jacobi weight function $\omega^{(\alpha, \beta)}(x)$, where α and β satisfy the following conditions in addition to the conditions $\alpha, \beta > -1$:*

$$(4.8) \quad -1 < \alpha + s = \beta + r \quad \text{or} \quad -\frac{1}{2} \leq \alpha + s, \beta + r \leq \frac{1}{2}.$$

Let $y = \cos(\pi p/q)$, where p/q is a rational number in $(0, 1)$. Then the conclusions of Corollary 4.5 hold.

Proof. In [5], it was shown that $|y - x_{c(n)}| \geq c/n$ for large n such that $y \neq x_{c(n)}$. The rest of the proof is the same as that of Corollary 4.6 in [2]. \square

5. Interior Singularities, Part 2. We now prove results of a different character to those of Section 4. For example, we show that, for almost all choices of y ,

$$E_n \left[|x - y|^{-\delta} \right] = O \left(n^{-1+2\delta} (\log n)^\delta (\log \log n)^{\varepsilon\delta} \right),$$

where $\varepsilon > 1$, and that this result is substantially the best possible. This is the analogue of Theorem 3 in Rabinowitz [4].

THEOREM 5.1. (i) *Assume $\omega(x) \in IB(y)$ for each y interior to the interval (a, b) . Then, given $\varepsilon > 1$, there is a set \mathcal{E}_ε in (a, b) of linear Lebesgue measure zero with the following property:*

$$(5.1) \quad E_n \left[|x - y|^{-\delta} \right] = O \left(n^{-1+2\delta} (\log n)^\delta (\log \log n)^{\varepsilon\delta} \right)$$

for all $0 < \delta < 1$, whenever $y \notin \mathcal{E}_\varepsilon$.

Hence, if $\delta < \frac{1}{2}$, $E_n[|x - y|^{-\delta}] \rightarrow 0$ as $n \rightarrow \infty$ for almost all $y \in (a, b)$.

(ii) *Assume $(a, b) = (-1, 1)$ and that $\omega(x)$ is the Jacobi weight function $\omega^{(\alpha, \beta)}(x)$, where $\alpha + s, \beta + r = \pm \frac{1}{2}$ and $\alpha, \beta > -1$. Then there is a set \mathcal{E} in $(-1, 1)$ of linear Lebesgue measure zero with the following property:*

$$E_n \left[|x - y|^{-\delta} \right] \geq cn^{-1+2\delta} (\log n)^\delta (\log \log n)^\delta$$

for infinitely many integers n and for all $0 < \delta < 1$, whenever $y \notin \mathcal{E}$. Here c is a positive constant independent of n, y and δ .

Hence, if $\delta \geq \frac{1}{2}$, $E_n[|x - y|^{-\delta}] \not\rightarrow 0$ as $n \rightarrow \infty$ for almost all $y \in (-1, 1)$.

(iii) *Assume $(a, b) = (-1, 1)$, and that $\omega(x)$ is the Jacobi weight function $\omega^{(\alpha, \beta)}(x)$, where α and β satisfy (4.8). Then, if $y = \cos \pi \xi$, where ξ is an irrational number, then*

$$\left| E_n \left[|x - y|^{-\delta} \right] \right| \geq cn^{-1+2\delta}$$

for infinitely many integers and all $0 < \delta < 1$.

Proof. (i) See [2].

(ii) See [2] with ν, β replaced by $\alpha + s, \beta + r$, respectively.

(iii) In [5], it was shown that $|y - x_{c(n)}| \leq c_1/n^2$ for infinitely many integers n . Applying Corollary 4.3(iii) and Lemma 3.4(ii), we have that

$$\left| E_n \left[|x - y|^{-\delta} \right] \right| = \lambda_{c(n)} |x_{c(n)} - y|^{-\delta} + O(n^{-1+\delta}) \geq \frac{c_3}{n} \left(\frac{c_1}{n^2} \right)^{-\delta} = cn^{-1+2\delta}$$

for infinitely many n and all $0 < \delta < 1$. \square

Note that $(\log \log n)^{\varepsilon\delta}$ in (5.1) may be replaced by $(\log \log n)^\delta(\log \log \log n)^{\varepsilon\delta}$, and so on. Similar remarks apply to part (ii) of the above theorem. The proof of the following result is similar to that of Theorem 5.1.

THEOREM 5.2. *Assume $\omega \in IB(y)$ for each y interior to the finite interval (a, b) . Then there is a set \mathcal{E} of linear Lebesgue measure zero (even further of Hausdorff dimension zero) such that $E_n[-\log|x - y|] = O(n^{-1} \log n)$ whenever $y \notin \mathcal{E}$.*

The results in Theorems 5.1(i) and 5.2 are the best possible in that they cannot be improved to include all real numbers in (a, b) . This is shown by Example 1 in [5], where a number \hat{y} and a subsequence $I_{n_k}[f]$ of the sequence of Gauss rules ($r = s = 0$) are constructed for which $I_{n_k}[\log|x - \hat{y}|]$ does not converge to $I[\log|x - \hat{y}|]$.

6. Endpoint Singularities. We discuss here the case where the singularity is at the right endpoint b . All the results can be easily rewritten for the case when the singularity is at a . If we define the value of the integrand at b to be $f(b) = 0$, we see that in all cases,

$$I_n[f] = \sum_{i=1-r}^n \lambda_{ni} f(x_{ni}).$$

Nevertheless, in our next lemma, we shall have to distinguish between the cases $s = 0$ and $s = 1$. Even though the end result is the same, the proofs are different. Before stating our lemma, we note that for endpoint singularities, there is no need to omit any abscissas, so that we can restrict ourselves to the study of $E_n[f]$.

LEMMA 6.1. (a) *Let $f(x)$ be $(2n + r + s)$ -absolutely monotone in $[a, b)$ such that $\lim_{x \rightarrow b} f(x) = \infty$ and define $f(b) = 0$. Then*

$$(6.1) \quad \max\left\{ \int_{x_{nn}}^b f(x)\omega(x) dx - \lambda_{nn}f(x_{nn}), 0 \right\} \leq E_n[f] \leq \int_{x_{nn}}^b f(x)\omega(x) dx.$$

(b) *Let $f(x)$ be $(2n + r + s)$ -completely monotone in $(a, b]$ such that $\lim_{x \rightarrow a} f(x) = \infty$ and define $f(a) = 0$. Then*

$$(6.2) \quad \max\left\{ \int_a^{x_{n1}} f(x)\omega(x) dx - \lambda_{n1}f(x_{n1}), 0 \right\} \leq E_n[f] \leq \int_a^{x_{n1}} f(x)\omega(x) dx.$$

Proof. (a) By Lemmas 3.2(i) and (ii),

$$\begin{aligned} \sum_{j=1}^{n-1} \lambda_{nj} f(x_{nj}) &\leq \int_a^{x_{nn}} f(x)\omega(x) dx \leq \sum_{j=1}^n \lambda_{nj} f(x_{nj}) \\ &\Rightarrow I_n[f] - \lambda_{nn}f(x_{nn}) \leq I[f] - \int_{x_{nn}}^b f(x)\omega(x) dx \leq I_n[f]. \end{aligned}$$

It remains to show that $I_n[f] \leq I[f]$. When $s = 1$, this follows from Lemma 3.2(i) with $k = n + 1$ since $I_n[f] = \sum_{j=1-r}^n \lambda_{nj} f(x_{nj})$. When $s = 0$, we shall prove that $I_n[f] \leq I[f]$ using the ideas in Freud's proof of Lemma III.1.5 in [1], which appears to be faulty.

Let $\varepsilon > 0$ be arbitrary and define $f_\varepsilon = f(x) + \varepsilon x^{2n+r}$. Let $H(x)$ be the polynomial of degree $2n + r - 1$ satisfying

$$\begin{aligned} H(x_{nj}) &= f_\varepsilon(x_{nj}), & j &= 1 - r(1)n, \\ H'(x_{nj}) &= f'_\varepsilon(x_{nj}), & j &= 1(1)n. \end{aligned}$$

As is well known, $I_n[f_\epsilon] = I_n[H] = I[H]$. Hence, it suffices to show that $G(x) \geq 0$ in (a, b) , where $G(x) = f_\epsilon(x) - H(x)$, since this implies that $I_n[f_\epsilon] \leq I[f_\epsilon]$, and by letting ϵ tend to zero, we get that $I_n[f] \leq I[f]$. Now, since $f_\epsilon(x) \rightarrow \infty$ as $x \rightarrow b$, $G(x) > 0$ in a neighborhood of b . Furthermore, $G(x)$ has n double zeros in (a, b) , and when $r = 1$, it has a simple zero at $x = a$ for a total of $2n + r$ zeros. Now, if $G(x_0) < 0$ for some $x_0 \in (a, b)$, then $G(x)$ would have a zero of odd multiplicity at some point in (a, b) and the total number of zeros would be at least $2n + r + 1$. By Rolle's theorem, there would exist a point $\xi \in (a, b)$ such that $G^{(2n+r)}(\xi) = 0$. But $H^{(2n+r)}(x) \equiv 0$ and $f_\epsilon^{(2n+r)}(\xi) = f^{(2n+r)}(\xi) + \epsilon(2n+r)! > 0$, since $f^{(2n+r)}(x) \geq 0$ in (a, b) . This contradiction proves our result.

(b) is similar. \square

Unfortunately, the behavior of λ_{nn} , $b - x_{nn}$, $x_{nn} - x_{n,n-1}$, and so on, have not been thoroughly investigated for general weights and there seems to be no analogue of Lemma 3.4. Thus we are not able to prove results as general as those in Sections 4 and 5, but can prove results for the generalized Jacobi weight functions. These weight functions were studied by Nevai and others and are defined as follows:

Definition 6.2 [3]. ω is a generalized Jacobi weight function ($\omega \in GJ$) if ω can be written in the form

$$(6.3) \quad \omega(x) = \psi(x)(1-x)^\alpha \prod_{k=1}^m |t_k - x|^{\gamma_k} (1+x)^\beta$$

for $-1 \leq x \leq 1$, where $-1 < t_m < t_{m-1} < \dots < t_1 < 1$, $m \geq 0$, $\alpha, \beta, \gamma_k > -1$, $k = 1, \dots, m$ and $\psi \geq 0$ satisfies $\psi^{\pm 1} \in L^\infty[-1, 1]$.

Note that if $\omega \in GJ$, then for any $y \in (-1, 1)$ such that $y \neq t_k$, $k = 1, \dots, m$, $\omega \in IB(y)$. Hence, the results of Sections 4 and 5, in particular Corollaries 4.3 and 4.4 and Theorems 5.1(i) and 5.2, hold for generalized Jacobi weight functions.

The two basic properties of GJ weights that we need are given in [3, p. 673]. Taking into account that if $\omega \in GJ$, then $\bar{\omega} \in GJ$, these properties are

$$(6.4) \quad 1 - c_1/n^2 < x_{nn} < 1 - c_2/n^2,$$

$$(6.5) \quad c_3(1 - x_{nn})^{\alpha+1/2+s}/n < \bar{\lambda}_{nn} < c_4(1 - x_{nn})^{\alpha+1/2+s}/n,$$

where, as before, x_{nn} is the n th root of the polynomial of degree n orthogonal with respect to $\bar{\omega}(x)$, and $\bar{\lambda}_{nn}$ is the Christoffel number corresponding to x_{nn} . The positive constants c_1, c_2, c_3 and c_4 are independent of n .

THEOREM 6.3. *Let $(a, b) = (-1, 1)$ and let $\omega(x)$ be a generalized Jacobi weight function as given by (6.3). Then*

(a) $E_n[(1-x)^{-\delta}] = O(n^{-2\alpha-2+2\delta})$ if $\alpha - \delta > -1$ and $\delta > 0$. Further, there exists a positive η such that

$$E_n[(1-x)^{-\delta}] \sim n^{-2\alpha-2+2\delta} \text{ whenever } \delta \in (1 + \alpha - \eta, 1 + \alpha).$$

(b) $E_n[-\log(1-x)] = O(n^{-2\alpha-2} \log n)$.

Proof. (a) When $m \geq 1$ in (6.3) we choose n sufficiently large so that $x_{nn} > t_1$. Then, by (6.1) and (6.4),

$$0 \leq E_n[(1-x)^{-\delta}] \leq c(1 + \alpha - \delta)^{-1}(1 - x_{nn})^{1+\alpha-\delta} = O(n^{-2\alpha-2+2\delta})$$

for some positive constant c .

Further, by (6.5),

$$\lambda_{nn}f(x_{nn}) = \frac{\bar{\lambda}_{nn}(1-x_{nn})^{-\delta}}{(1+x_{nn})^r(1-x_{nn})^s} \geq \frac{c_3}{2n}(1-x_{nn})^{\alpha+1/2-\delta}.$$

Hence, by (6.1),

$$\begin{aligned} E_n[(1-x)^{-\delta}] &\geq \int_{x_{nn}}^1 \omega(x)(1-x)^{-\delta} dx - \frac{c_3}{2n}(1-x_{nn})^{\alpha+1/2-\delta} \\ &\geq \hat{c} \frac{(1-x_{nn})^{-\delta+\alpha+1}}{(1+\alpha-\delta)} - \frac{c_3}{2n}(1-x_{nn})^{\alpha+1/2-\delta} \\ &= (1-x_{nn})^{-\delta+\alpha+1} \left[\frac{\hat{c}}{1+\alpha-\delta} - \frac{c_3}{2n}(1-x_{nn})^{-1/2} \right] \\ &\geq (1-x_{nn})^{-\delta+\alpha+1} \left[\frac{\hat{c}}{1+\alpha-\delta} - \frac{c_3}{2} \cdot c_1^{-1/2} \right]. \end{aligned}$$

Hence

$$\exists \eta > 0 \ni E_n[(1-x)^{-\delta}] \geq c(1-x_{nn})^{\alpha+1-\delta}$$

whenever $\delta \in (1 + \alpha - \eta, 1 + \alpha)$; i.e.,

$$E_n[(1-x)^{-\delta}] \sim n^{-2\alpha-2+2\delta}.$$

(b) is similar to the first part of (a). \square

For Jacobi weights, we obtain the following more precise result.

THEOREM 6.4. *Let $(a, b) = (-1, 1)$, and let $\omega(x)$ be the Jacobi weight function $\omega^{(\alpha, \beta)}(x)$, $\alpha, \beta > -1$. Let $J_\nu(x)$ be the Bessel function of the first kind of order ν and $j_{1\nu}$ its first positive zero, where $\nu = \alpha + s$.*

(a) *Let $0 < \delta < 1 + \alpha$ and*

$$s_n = 2^{-\beta} (j_{1\nu}^2/2)^{-1-\nu+\delta} n^{2\alpha+2-2\delta} (1+\alpha-\delta) E_n[(1-x)^{-\delta}].$$

Then

$$(6.6) \quad \max\{0, 1 - c_0^2(\nu)(1 + \alpha - \delta)\} \leq \liminf_{n \rightarrow \infty} s_n \leq \limsup_{n \rightarrow \infty} s_n \leq 1,$$

where

$$(6.7) \quad c_0(\nu) = 2/(j_{1\nu} J'_\nu(j_{1\nu})).$$

(b) *Let*

$$t_n = 2^{-\beta-1} (j_{1\nu}^2/2)^{-1-\nu} n^{2\alpha+2} (\log n)^{-1} (1 + \alpha) E_n[-\log(1-x)].$$

Then

$$(6.8) \quad \max\{0, 1 - c_0^2(\nu)(1 + \alpha)\} \leq \liminf_{n \rightarrow \infty} t_n \leq \limsup_{n \rightarrow \infty} t_n \leq 1.$$

Proof. For the Jacobi weight function we have that

$$\bar{\omega}(x) = (1-x)^\nu (1+x)^{\beta+r}.$$

Now, by Theorem 8.1.2 in Szegő [6] (cf. [2, p. 233]),

$$(6.9) \quad \lim_{n \rightarrow \infty} n^2(1 - x_{nn}) = j_{1\nu}^2/2.$$

Hence,

$$(6.10) \quad \int_{x_{nn}}^1 (1-x)^{-\delta} \omega(x) dx \cong 2^\beta (1-x_{nn})^{1+\alpha-\delta} / (1+\alpha-\delta) \\ \cong 2^\beta (j_{1\nu}^2/(2n^2))^{1+\alpha-\delta} / (1+\alpha-\delta),$$

by (6.9). Further, by (15.3.11) in Szegő [6, p. 350],

$$\bar{\lambda}_{nn} \cong 2^{\nu+\beta+r+1} (j_{1\nu}/2)^{2\nu} \{J_r'(j_{1\nu})\}^{-2} n^{-2\nu-2},$$

so that

$$(6.11) \quad \lambda_{nn}(1-x_{nn})^{-\delta} = \bar{\lambda}_{nn}(1-x_{nn})^{-\delta-s}(1+x_{nn})^{-r} \\ \cong 2^{\nu+\beta+1} (j_{1\nu}/2)^{2\nu} \{J_\nu'(j_{1\nu})\}^{-2} n^{-2\nu-2} (j_{1\nu}^2/(2n^2))^{-\delta-s}.$$

Then (6.6) follows easily from (6.1), (6.10), (6.11), and (6.7).

(b) is similar. \square

7. Interior Singularities for More General Functions. We now extend the results of Sections 4 and 5 to the function $f(x) = \phi(x)g(x)$, where $g(x)$ is smooth and $\phi(x) = |x-y|^{-\delta}$ or $\phi(x) = -\log|x-y|$. Throughout, without further mention, we assume $y \in (a, b)$.

We state first our result on avoiding the singularity.

THEOREM 7.1. *Assume $\omega(x) \in IB(y)$. Assume $g \in C[a, b]$.*

(i) *Let $f(x) = |x-y|^{-\delta}g(x)$, $x \in [a, b] \setminus \{y\}$, where $0 < \delta < 1$.*

(a) *If $g \in \text{Lip}(1-\delta)$ in $[a, b]$ and $g \in \text{Lip}(1)$ near y , then*

$$E_n^{**}[f] = O(n^{-1+\delta}), \quad E_n^*[f] = O(n^{-1+\delta}).$$

(b) *If, further, there exists $0 < \varepsilon < \delta$ such that $g \in \text{Lip}(1-\varepsilon)$ in $[a, b]$ and $g' \in \text{Lip}(\delta-\varepsilon)$ near y , and if $g(y) \neq 0$, then*

$$E_n^{**}[f] \sim g(y)n^{-1+\delta}.$$

Further, $E_n^[f] \sim g(y)n^{-1+\delta}$ if δ is close enough to 1.*

(ii) *Let $f(x) = (-\log|x-y|)g(x)$, $x \in [a, b] \setminus \{y\}$.*

(a) *If $g \in \text{Lip}(1; -1)$ in $[a, b]$ and $g \in \text{Lip}(1)$ near y , then*

$$E_n^{**}[f] = O(n^{-1} \log n), \quad E_n^*[f] = O(n^{-1} \log n).$$

(b) *If, further, there exists $0 < \eta < 1$ such that $g \in \text{Lip}(1; -1+\eta)$ in $[a, b]$ and $g' \in \text{Lip}(0; \eta)$ near y , and if $g(y) \neq 0$, then*

$$E_n^{**}[f] \sim g(y)n^{-1} \log n.$$

Proof. See the proof of Theorem 7.5 in [2].

The following result analyzes the error when the singularity is ignored.

THEOREM 7.2. (i) Assume $\omega(x) \in IB(y)$ for each y interior to (a, b) . Then, given $\epsilon > 1$, there is a set \mathcal{E}_ϵ in (a, b) of linear Lebesgue measure zero with the following property: If $g \in \text{Lip}(1)$ in $[a, b]$, then

$$E_n \left[|x - y|^{-\delta} g \right] = O \left(n^{-1+2\delta} (\log n)^\delta (\log \log n)^{\epsilon\delta} \right)$$

for all $0 < \delta < 1$, whenever $y \notin \mathcal{E}_\epsilon$.

Hence, if $\delta < \frac{1}{2}$, $E_n[|x - y|^{-\delta} g] \rightarrow 0$ as $n \rightarrow \infty$ for almost all $y \in (a, b)$.

(ii) Assume $(a, b) = (-1, 1)$, and $\omega(x)$ is the Jacobi weight function $\omega^{(\alpha, \beta)}(x)$, where $\alpha + s, \beta + r = \pm \frac{1}{2}$. Then there is a set \mathcal{E} in $(-1, 1)$ of linear Lebesgue measure zero with the following property: If $g \in \text{Lip}(1)$ in $[a, b]$, then

$$\left| E_n \left[|x - y|^{-\delta} g \right] \right| \geq c |g(y)| n^{-1+2\delta} (\log n)^\delta (\log \log n)^\delta$$

for infinitely many integers n and all $0 < \delta < 1$, whenever $y \notin \mathcal{E}$. Here c is a positive constant independent of g, n, y , and δ .

Thus, provided the set of zeros of g has linear Lebesgue measure zero, and if $\delta \geq \frac{1}{2}$, $E_n[|x - y|^{-\delta} g] \rightarrow 0$ as $n \rightarrow \infty$ for almost all $y \in [a, b]$.

(iii) Assume $(a, b) = (-1, 1)$ and that $\omega(x)$ is the Jacobi weight function $\omega^{(\alpha, \beta)}(x)$, where α and β satisfy (4.8). Then, if $y = \cos \pi \xi$, where ξ is an irrational number, and $g \in \text{Lip}(1)$ in $[-1, 1]$, then

$$\left| E_n \left[|x - y|^{-\delta} g \right] \right| \geq c |g(y)| n^{-1+2\delta}$$

for infinitely many n and all $0 < \delta < 1$. Here, c is a positive constant independent of g, n , and δ . Hence, for any $y \ni g(y) \neq 0$, if $\delta \geq \frac{1}{2}$, $E_n[|x - y|^{-\delta} g] \rightarrow 0$ as $n \rightarrow \infty$.

Proof. See the proof of Theorem 7.6 in [2].

In a similar fashion, one can use Theorem 5.2 to prove the following result for ignoring a logarithmic singularity:

THEOREM 7.3. Assume $\omega(x) \in IB(y)$ for each y interior to (a, b) . Then there is a set \mathcal{E} of linear Lebesgue measure zero (even further of Hausdorff dimension zero) with the following property: If $g \in \text{Lip}(1)$ in $[a, b]$, then

$$E_n \left[(-\log|x - y|)g \right] = O(n^{-1} \log n), \quad \text{whenever } y \notin \mathcal{E}.$$

8. Endpoint Singularities for More General Functions. In extending the results of Section 6 to more general functions, we shall assume throughout that $(a, b) = (-1, 1)$ and that $\omega(x)$ is a generalized Jacobi weight function given by (6.3). Our main result for endpoint singularities is as follows:

THEOREM 8.1. (i) Let $0 < \delta < \min\{1, 1 + \alpha\}$, and let l be the smallest integer $\geq 2(1 + \alpha - \delta)$. Let $g \in C^l[-1, 1]$ and assume there exists $\eta > 0$ such that $g^{(l)}(x) \in \text{Lip}(\delta; \eta)$ near 1. Then

$$E_n \left[(1 - x)^{-\delta} g \right] = O(n^{-2(1 + \alpha - \delta)}).$$

(ii) Let k be the smallest integer $\geq 2(1 + \alpha)$. Let $g \in C^k[-1, 1]$ and assume there exists $\eta > 1$ such that $g^{(k)}(x) \in \text{Lip}(0; \eta)$ near 1. Then

$$E_n[(\log(1 - x))g] = O(n^{-2(1+\alpha)} \log n).$$

Proof. See the proof of Theorem 8.2 in [2] with ν replaced by α .

Department of Applied Mathematics
The Weizmann Institute of Science
Rehovot 76100, Israel

1. G. FREUD, *Orthogonal Polynomials*, Pergamon Press, New York, 1966.
2. D. S. LUBINSKY & P. RABINOWITZ, "Rates of convergence of Gaussian quadrature for singular integrands," *Math. Comp.*, v. 43, 1984, pp. 219–242.
3. P. NEVAI, "Mean convergence of Lagrange interpolation. III," *Trans. Amer. Math. Soc.*, v. 282, 1984, pp. 669–698.
4. P. RABINOWITZ, "Ignoring the singularity in numerical integration," in *Topics in Numerical Analysis III* (J. J. H. Miller, ed.), Academic Press, London, 1977, pp. 361–368.
5. P. RABINOWITZ, "Numerical integration in the presence of an interior singularity," *J. Comput. Appl. Math.* (To appear.)
6. G. SZEGÖ, *Orthogonal Polynomials*, rev. ed., Amer. Math. Soc. Colloq. Publ., vol. 23, Amer. Math. Soc., Providence, R. I., 1959.