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On the Convergence of an Interpolatory Product Rule for Evaluating Cauchy Principal Value Integrals*

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Abstract. The authors give convergence theorems for interpolatory product rules for evaluating Cauchy singular integrals and obtain asymptotic estimates of the remainder. Some results, previously established by other authors, are generalized and improved.

1. Introduction. Let $\Phi(wf; t)$ be the integral in the Cauchy principal value sense of the function f, defined by

$$(1.1)\qquad \begin{aligned}\Phi(wf;t) &= \int_{-1}^{1} \frac{f(x)}{x-t} w(x)\,dx \\ &= \lim_{\varepsilon\to 0^+} \int_{|x-t|\geqslant\varepsilon} \frac{f(x)}{x-t} w(x)\,dx, \qquad t \in A \subseteq (-1,1),\end{aligned}$$

where we assume that w is a nonnegative weight function on $I = [-1, 1]$ such that $0 < \int_{-1}^{1} w(x)\,dx < \infty$.

Let $C(I)$ be the set of continuous functions on I and $\omega(g; \delta)$ the modulus of continuity of the function $g \in C(I)$, defined by

$$\omega(g;\delta) = \max_{|x-y|\leqslant\delta} |g(x) - g(y)|, \qquad x,\ y \in I,\ \delta \geqslant 0.$$

If g is continuous on I and the modulus of continuity ω of g satisfies

$$\int_0^1 \delta^{-1}\omega(g;\delta)\,d\delta < \infty,$$

we say that the function g is of "Dini type" ($g \in DT(I)$). It is well known that a sufficient condition for the existence and the continuity of $\Phi(wf; t)$ is that w, $f \in DT(I)$. The requirement that $w \in DT(I)$ can be relaxed; indeed, w may have some singularities. Moreover, if $g \in DT(I)$ then the relation

$$\omega(\Phi g;\delta) = O(\omega(g;\delta)), \qquad \delta \to 0^+,$$

holds on any closed subset of A (see, e.g., [1]).

Let $\{p_n(w)\}$ be the sequence of the orthonormal polynomials on I associated with the weight function w. We denote the zeros of

$$p_n(x) = p_n(w;x) = \alpha_n x^n + \text{lower degree terms}, \qquad \alpha_n > 0,$$

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by $x_{n,i} = x_{n,i}(w)$, $i = 1, 2, \ldots, n$, ordered increasingly,

$$-1 < x_{n,1} < x_{n,2} < \cdots < x_{n,n} < 1.$$

Let $L_m(v; f; x)$ be the interpolating polynomial of f on the knots $x_{m,k} = x_{m,k}(v)$, $k = 1, 2, \ldots, m$, where v may or may not be equal to w. The polynomial $L_m(v; f; x)$ is defined by

$$L_m(v; f; x) = \sum_{k=1}^{m} l_{m,k}(v; x) f(x_{m,k}),$$

where

$$l_{m,k}(v; x) = \frac{p_m(v; x)}{(x - x_{m,k}) p_m'(v; x_{m,k})}, \qquad k = 1, 2, \ldots, m.$$

If we consider $L_m(v; f; x)$ instead of f in (1.1), we obtain the following interpolatory product rule for the numerical evaluation of $\Phi(wf; t)$:

$$\Phi_m^v(wf; t) = \sum_{k=1}^{m} A_{m,k}^w(v; t) f(x_{m,k}), \tag{1.2}$$

where

$$A_{m,k}^w(v; t) = \int_{-1}^{1} \frac{l_{m,k}(v; x)}{x - t} w(x)\, dx.$$

Various expressions for the coefficients $A_{m,k}^w(v; t)$ appear in [9], [12].

We note that the quadrature rule $\Phi_m^v(wf; t)$ has degree of exactness $m - 1$. (For some special values of t, the degree of exactness may be greater than $m - 1$.)

We denote the remainder term by $E_m^v(wf; t)$,

$$E_m^v(wf) = \Phi(wf) - \Phi_m^v(wf).$$

In this paper we prove convergence theorems for the quadrature rule $\Phi_m^v(wf; t)$. In Section 2 we study the convergence of the quadrature rule $\Phi_m^v(wf; t)$. Some results previously established by other authors [2]–[5], [7], [9], [12], [13] are generalized and improved upon. In addition, better estimates are given for the remainder. In Section 3 we give the proofs of the theorems stated in Section 2.

2. Convergence Theorems and Estimates of the Remainder. We start with some notations. The symbol "const" stands for a positive constant taking on different values on different occurrences. If A and B are two expressions depending on some variables, then we write: $A \sim B$ if and only if $|AB^{-1}| \leqslant \text{const}$ and $|A^{-1}B| \leqslant \text{const}$, uniformly for the variables under consideration.

In addition to the set $DT(I)$, defined in Section 1, we consider the following classes of functions:

$$LD(\lambda) = \{ f \in C(I) \mid \omega(f; \delta) \log^{\lambda} \delta^{-1} = o(1), \delta \to 0^+, \lambda > 0 \},$$

$$\mathrm{Lip}_M \lambda = \{ f \in C(I) \mid \omega(f; \delta) \leqslant M\delta^{\lambda}, \delta > 0, M > 0, \lambda \in (0, 1] \},$$

$$C^{(k)}(I) = \{ f \in C(I) \mid f^{(k)} \in C(I), k \geqslant 1 \}.$$

It is well known that

$$LD(1) \supset DT \supset LD(1 + \lambda) \supset \mathrm{Lip}_M \mu, \qquad \mu, \lambda > 0.$$

Throughout this paper, all functions are Riemann-integrable (possibly in the improper sense). Moreover, we set

$$\|f\|_E = \max_{x \in E \subseteq I} |f(x)|; \qquad \|f\| = \|f\|_I.$$

First, we assume that $w(x) = v(x)$, where

$$v(x) = \psi(x)(1-x)^{\alpha} \prod_{i=1}^{p} |t_i - x|^{\gamma_i} (1+x)^{\beta}, \tag{2.1}$$

with $-1 < t_1 < t_2 < \cdots < t_p < 1$; $\alpha, \beta, \gamma_i > -1$, $i = 1, 2, \ldots, p$; $p \geqslant 0$; $0 < \psi \in DT(I)$. We say that $v(x)$ is a "generalized smooth Jacobi" weight ($v \in$ GSJ).

In what follows, Δ is a closed set such that $\Delta \subset D := [-1,1] - \{\pm 1, t_1, \ldots, t_p\}$.

Concerning the convergence of (1.2), we will prove the following theorems.

THEOREM 2.1. *Let $f \in LD(1)$ and $w \in$ GSJ be such that the integral $\Phi(wf; t)$ exists for all values of $t \in D$. Then the sequence $\{\Phi_m^w(wf; t)\}_{m \in \mathbf{N}}$ converges to $\Phi(wf; t)$ for all $t \in D$.*

THEOREM 2.2. *If $w \in$ GSJ, then for any function $f \in DT(I)$ the sequence $\{\Phi_m^w(wf; t)\}_{m \in \mathbf{N}}$ converges uniformly to $\Phi(wf; t)$ on Δ.*

THEOREM 2.3. *If $w \in$ GSJ, the following relations hold,*

$$\|E_m^w(wf)\|_\Delta \leqslant o(\log^{-\lambda} m), \qquad f \in LD(1+\lambda),\ \lambda > 0, \tag{2.2}$$

$$\|E_m^w(wf)\|_\Delta \leqslant \text{const} \frac{\log m}{m^\lambda}, \qquad f \in \mathrm{Lip}_M \lambda,\ 0 < \lambda \leqslant 1, \tag{2.3}$$

$$\|E_m^w(wf)\|_\Delta \leqslant \text{const} \frac{\omega(f^{(k)}; m^{-1})}{m^k} \log m, \quad f \in C^{(k)}(I). \tag{2.4}$$

Remark 1. The results presented above hold in the special case in which $w = v = (1-x)^\alpha (1+x)^\beta$ is a Jacobi weight. This case is of practical interest and is studied in [4], [5], [9], [12].

The relation

$$\|E_m^w(wf)\|_\Delta \leqslant \frac{\text{const}}{m^{\lambda - 2\nu}}, \qquad \nu < \frac{\lambda}{2},\ f \in \mathrm{Lip}_M \lambda, \tag{2.5}$$

has been stated in [4], [5]; it is extended in [9] to read

$$\|E_m^w(wf)\|_\Delta \leqslant \frac{\text{const}}{m^{k+\lambda-2\nu}}, \qquad \nu < \frac{\lambda}{2},\ f^{(k)} \in \mathrm{Lip}_M \lambda. \tag{2.6}$$

Further, for any function $f \in \mathrm{Lip}_M \lambda$, the relations

$$\|E_m^w(wf)\|_\Delta \leqslant \text{const} \frac{\log m}{m^{\lambda - \gamma - 1/2}}, \quad \gamma = \max(\alpha, \beta) > -\tfrac{1}{2}, \tag{2.7}$$

$$\|E_m^w(wf)\|_\Delta \leqslant \text{const} \frac{\log^2 m}{m^\lambda}, \qquad \gamma = \max(\alpha, \beta) < -\tfrac{1}{2}, \tag{2.8}$$

$$\|E_m^w(wf)\|_\Delta \leqslant \text{const} \frac{\log m}{m^\lambda}, \qquad \alpha = \beta = -\tfrac{1}{2}, \tag{2.9}$$

are proved in [12]. The results of Theorem 2.3 thus generalize (2.9) and improve upon (2.5), (2.6), (2.7), (2.8). Moreover, the convergence Theorems 2.1 and 2.2 hold for spaces of functions which include $\mathrm{Lip}_M \lambda$.

We now consider the case in which the weight function w is not identical to v. We assume that $v \in \mathrm{GSJ}$ is defined by (2.1) and $w \in DT(\Delta)$ for all closed sets Δ such that $\Delta \subset D$.

Under these assumptions we have

THEOREM 2.4. *Assume that* $v \in \mathrm{GSJ}$ *and* $0 \leqslant w \in DT(\Delta)$. *If*

$$w \log^+ w \quad \text{is integrable on } I, \tag{2.10}$$

$$\frac{w}{\sqrt{v\sqrt{1-x^2}}} \quad \text{is integrable on } I, \tag{2.11}$$

then the results of Theorems 2.1, 2.2, 2.3 *hold for the quadrature rule* $\Phi_m^v(wf; t)$ *and for the corresponding remainder* $E_m^v(wf)$.

Remark 2. Particular cases of Theorem 2.4 are contained in [3], [12], [13]. Sheshko assumed

$$w = (1-x)^\alpha (1+x)^\beta, \qquad \alpha, \beta > -1; \qquad v = (1-x^2)^{-1/2}, \qquad f \in \mathrm{Lip}_M \lambda,$$

and

$$w = (1-x)^\alpha (1+x)^\beta \log\frac{1+x}{1-x}, \qquad \alpha, \beta > -1;$$
$$v = (1-x^2)^{-1/2}, \qquad f \in \mathrm{Lip}_M \lambda,$$

in [12] and in [13], respectively. In [3], Dagnino and Palamara Orsi studied the special case in which w is L^p-integrable on I, for some $p > 1$, and $w \in \mathrm{Lip}_M \lambda$ on an interval $[\xi, \eta] \subset (-1, 1)$ such that $t \in [\xi, \eta]$, and $v = (1-x^2)^{-1/2}$, $f^{(k)} \in \mathrm{Lip}_M \lambda$ for $k \geqslant 0$.

Relations of the same kind as (2.8) and (2.6) are proved in [12], [13] and in [3], respectively. Moreover, in [2], the authors have established the following result:

If $w \in DT(I)$, *for any matrix of knots* $M = \{x_{m,i}\}_{i=1,\dots,m}$, $m \in \mathbf{N}$, *the relation*

$$\|E_m(wf)\|_\Delta \leqslant \mathrm{const}(1 + \Lambda_m(M))\frac{\log m}{m^{k+\lambda}}, \qquad f^{(k)} \in \mathrm{Lip}_M \lambda, \tag{2.12}$$

holds, where $E_m(wf)$ *is the remainder term of the quadrature rule and* $\Lambda_m(M)$ *is the mth Lebesgue constant with respect to the matrix M.*

We observe that relation (2.12) gives results which may be improved under the assumptions of the present work.

Remark 3. Regarding the hypotheses (2.10) and (2.11) of Theorem 2.4, we note that (2.11) follows from (2.10) when $-1 < \max_i\{\alpha, \beta, \gamma_i\} \leqslant -\frac{1}{2}$. In the special case in which we assume that

$$\int_{-1}^{1} \left\{ w(x)(1-x)^{-\max[(2\alpha+1)/4,0]} \times \prod_{i=1}^{p} |x - t_i|^{-\max[\gamma_i/2,0]} (1+x)^{-\max[(2\beta+1)/4,0]} \right\}^q dx < \infty \tag{2.13}$$

holds for some $q > 1$, the hypotheses (2.10) and (2.11) are satisfied. In [14], [15], Sloan and Smith considered a condition of the same kind as (2.13).

Finally, the following corollary may be of practical interest.

COROLLARY 2.5. *Suppose that* $w = w^{\alpha,\beta} = (1-x)^{\alpha}(1+x)^{\beta}$ *and* $v = v^{\gamma,\delta} = (1-x)^{\gamma}(1+x)^{\delta}$, $\alpha, \beta, \gamma, \delta > -1$, *are Jacobi weight functions. If*

$$\gamma < 2\alpha + \tfrac{3}{2}, \qquad \delta < 2\beta + \tfrac{3}{2},$$

then the results of Theorems 2.1, 2.2, 2.3 *hold for the quadrature rule* $\Phi_m^{v^{\gamma,\delta}}(w^{\alpha,\beta}f; t)$, *and for the corresponding remainder* $E_m^{v^{\gamma,\delta}}(w^{\alpha,\beta}f)$.

3. Proofs of the Main Results. For the convenience of the reader, we collect some properties of generalized Jacobi polynomials $p_m(w; x)$, $w \in \mathrm{GSJ}$ (cf. [10], [11]), which will be used in the proofs of some preliminary lemmas.

The Christoffel function $\lambda_m(x)$ is defined by

$$\lambda_m(x) = \lambda_m(w; x) = \left[\sum_{k=0}^{m-1} p_k^2(w; x)\right]^{-1}.$$

The zeros of $p_m(w; x)$ are denoted by $x_{m,k} = x_{m,k}(w)$ and they are ordered so that

$$x_{m,1} < x_{m,2} < \cdots < x_{m,m}.$$

The numbers

$$\lambda_{m,k} = \lambda_{m,k}(w) = \lambda_m(w; x_{m,k}),$$

are the Christoffel constants. Set $x_{m,k}(w) = \cos\theta_{m,k}$ for $0 \leqslant k \leqslant m+1$, where $x_{m,0} = -1$, $x_{m,m+1} = 1$ and $0 \leqslant \theta_{m,k} \leqslant \pi$. Then,

$$\theta_{m,k} - \theta_{m,k+1} \sim m^{-1}, \tag{3.1}$$

uniformly for $0 \leqslant k \leqslant m$, $m \in \mathbf{N}$.

Define w_m by

$$w_m(x) = \left(\sqrt{1-x} + m^{-1}\right)^{2\alpha+1} \prod_{i=1}^{p} \left(|t_i - x| + m^{-1}\right)^{\gamma_i}$$
$$\times \left(\sqrt{1+x} + m^{-1}\right)^{2\beta+1}, \qquad -1 \leqslant x \leqslant 1.$$

Then,

(3.2) $\lambda_m(x) \sim m^{-1} w_m(x)$, uniformly for $-1 \leqslant x \leqslant 1$, $m \in \mathbf{N}$;

(3.3) $\lambda_{m,k} \sim m^{-1} w_m(x_{m,k})$, $m \in \mathbf{N}$;

(3.4) $p_{m-1}(x_{m,k}) \sim \sqrt{(1 - x_{m,k}^2)/w_m(x_{m,k})}$, $k = 1, 2, \ldots, m$, $m \in \mathbf{N}$;

(3.5) $|p_m(x)| \leqslant \mathrm{const}\left\{\left(w(x)\sqrt{1-x^2}\right)^{-1/2} + 1\right\}$, uniformly for $-1 \leqslant x \leqslant 1$, $m \in \mathbf{N}$.

Now let $x_{c(m)} = x_{m,c}$ be the knot closest to t, defined by $x_{m,k} \leqslant t \leqslant x_{m,k+1}$; $|t - x_{m,c}| = \min\{(t - x_{m,k}), (x_{m,k+1} - t)\}$, $k \geqslant 1$. Then it is known that for any $t \in I$,

$$|l_{m,c}(w; t)| \sim 1 \tag{3.6}$$

(see [10, proof of Theorem 33, p. 171]). Define

$$\sigma_m^*(t) = \sum_{\substack{i=1\\ i\neq c}}^{m} \frac{\lambda_{m,i}}{|x_{m,i} - t|}; \qquad \rho_m^*(t) = \sum_{\substack{i=1\\ i\neq c}}^{m} \frac{\lambda_{m,i}}{x_{m,i} - t};$$

$$A_m^*(t) = \Phi(w;t) - \rho_m^*(t); \qquad \delta_m^*(t) = \sum_{\substack{i=1\\ i\neq c}}^{m} \lambda_{m,i} \frac{|p_{m-1}(x_{m,i})|}{|x_{m,i} - t|}.$$

We first prove

LEMMA 3.1. *For any weight function $w \in \mathrm{GSJ}$, and for m sufficiently large, we have*

$$\sigma_m^*(t) \leqslant \text{const}[w(t)(1 + \log m) + \Gamma(t)], \qquad t \in D, \tag{3.7}$$

$$\rho_m^*(t) \leqslant \text{const}\left[1 + w(t)\log\frac{1-t}{1+t} + \Gamma(t)\right], \qquad t \in D, \tag{3.8}$$

$$\delta_m^*(t) \leqslant \text{const}\sqrt{\sigma_m^*(t)\log m}, \qquad t \in D, \tag{3.9}$$

where $\Gamma(t) = \int_{-1}^{1} |w(x) - w(t)||x - t|^{-1}\,dx$.

Proof. Without loss of generality we assume that $x_{m,c} = x_{m,k} \leqslant t < x_{m,k+1}$. Then

$$t - x_{m,k-1} \sim x_{m,k+1} - t \sim m^{-1}. \tag{3.10}$$

Further, for any $t \in D$, there exists a closed set Δ such that $\Delta \subset D$ and t is an interior point of Δ. Moreover, $x_{m,k-1}, x_{m,k}, x_{m,k+1}$ also are interior points of Δ for sufficiently large m. Then, by the generalized Markov-Stieltjes inequalities (see [6, Lemmas 3.2, 3.3]), we obtain

$$\int_{-1}^{x_{m,k-1}} \frac{w(x)}{t-x}\,dx \leqslant \sum_{i=1}^{k-1} \frac{\lambda_{m,i}}{t - x_{m,i}} \leqslant \frac{\lambda_{m,k-1}}{t - x_{m,k-1}} + \int_{-1}^{x_{m,k-1}} \frac{w(x)}{t-x}\,dx,$$

$$\int_{x_{m,k+1}}^{1} \frac{w(x)}{x-t}\,dx \leqslant \sum_{i=k+1}^{m} \frac{\lambda_{m,i}}{x_{m,i} - t} \leqslant \frac{\lambda_{m,k+1}}{x_{m,k+1} - t} + \int_{x_{m,k+1}}^{1} \frac{w(x)}{x-t}\,dx.$$

From these inequalities, and by (3.10) and (3.3), we get

$$\sigma_m^*(t) \leqslant \text{const} + \int_{-1}^{x_{m,k-1}} \frac{w(x)}{t-x}\,dx + \int_{x_{m,k+1}}^{1} \frac{w(x)}{x-t}\,dx,$$

$$\rho_m^*(t) \leqslant \text{const} + \int_{-1}^{x_{m,k-1}} \frac{w(x)}{x-t}\,dx + \int_{x_{m,k+1}}^{1} \frac{w(x)}{x-t}\,dx.$$

By an easy calculation (3.7) and (3.8) follow.

To prove (3.9), apply (3.3) and (3.4) to obtain

$$\lambda_{m,i}|p_{m-1}(x_{m,i})| \sim \sqrt{\lambda_{m,i}}\sqrt{\frac{1 - x_{m,i}^2}{m}}. \tag{3.11}$$

Therefore,

$$\delta_m^*(t) \sim \sum_{\substack{i=1\\ i\neq c}}^{m} \sqrt{\frac{\lambda_{m,i}}{|x_{m,i} - t|}}\sqrt{\frac{1 - x_{m,i}^2}{m|x_{m,i} - t|}} \leqslant \sqrt{\sigma_m^*(t)}\left[\sum_{\substack{i=1\\ i\neq c}}^{m} \frac{1}{m|x_{m,i} - t|}\right]^{1/2},$$

from which (3.9) follows. □

Before proceeding further, we observe that, by (3.8),

$$|A_m^*(t)| \leqslant \text{const}\left[1 + w(t)\log\frac{1-t}{1+t} + \Gamma(t)\right]. \tag{3.12}$$

Moreover,

$$\|\sigma_m^*\|_\Delta \leqslant \text{const}(\log m + 1), \tag{3.13}$$

$$\|\delta_m^*\|_\Delta \leqslant \text{const}(\log m + 1), \tag{3.14}$$

$$\|\rho_m^*\|_\Delta \leqslant \text{const}, \tag{3.15}$$

$$\|A_m^*\|_\Delta \leqslant \text{const}, \tag{3.16}$$

where Δ is a closed set such that $\Delta \subset D$.

The Lebesgue function $L_m(w; x)$ is defined by

$$L_m(w; x) = \sum_{i=1}^{m} |l_{m,i}(w; x)|.$$

It can be estimated by

Lemma 3.2. *If $w \in$ GSJ, then the inequality*

$$\|L_m(w)\|_\Delta \leqslant \text{const}(\log m + 1) \tag{3.17}$$

holds for sufficiently large m, where Δ is a closed set such that $\Delta \subset D$.

Proof. From [6, III, (6.3)],

$$l_{m,i}(w; x) = \frac{\alpha_{m-1}}{\alpha_m}\lambda_{m,i}\frac{p_{m-1}(x_{m,i})}{x - x_{m,i}}p_m(x), \qquad \alpha_{m-1}/\alpha_m \leqslant 1.$$

Applying (3.5) we see that

$$|l_{m,i}(w; x)| \leqslant \text{const}\,\lambda_{m,i}\frac{|p_{m-1}(x_{m,i})|}{|x - x_{m,i}|}, \qquad i \neq c,$$

uniformly for $x \in \Delta$. Hence we can write

$$\begin{aligned} L_m(w; x) &= \sum_{\substack{i=1 \\ i \neq c}}^{m} |l_{m,i}(w; x)| + |l_{m,c}(w; x)| \\ &\leqslant \text{const}\,\delta_m^*(x) + |l_{m,c}(w; x)|, \qquad x \in \Delta, \end{aligned}$$

so that by (3.6) and (3.14) the lemma follows. □

Now let

$$\Psi_m^v(w; t) = \sum_{k=1}^{m} |A_{m,k}^w(v; t)|, \tag{3.18}$$

where $A_{m,k}^w$ are the coefficients in (1.2). The results presented so far are sufficient to prove the following

Lemma 3.3. *If $w \in$ GSJ, the relation*

$$\Psi_m^w(w; t) \leqslant \text{const}(\log m + 1), \qquad t \in D, \tag{3.19}$$

holds for m sufficiently large.

Proof. For any $t \in D$, there exists a closed set Δ such that $\Delta \subset D$ and t is an interior point of Δ. Moreover, $x_{m,c}$ also is an interior point of Δ for m sufficiently large. Since

$$l_{m,k}(w;x) = \frac{\alpha_{m-1}}{\alpha_m}\lambda_{m,k}\frac{p_{m-1}(x_{m,k})}{x - x_{m,k}}p_m(x), \qquad \alpha_{m-1}/\alpha_m \leqslant 1,$$

and applying the Gaussian quadrature rule

$$\begin{aligned} \int_{-1}^{1} \frac{l_{m,k}(w;x)}{x-t}w(x)\,dx = {} & \sum_{j=1}^{m}\lambda_{m,j}\frac{l_{m,k}(w;x_{m,j}) - l_{m,k}(w;t)}{x_{m,j}-t} \\ & + \Phi(w;t)l_{m,k}(w;t) \end{aligned}$$

or its limiting case, we see that

(i) if $t \neq x_{m,c}$, then

$$\begin{aligned} A_{m,k}(w;t) &= A_m^*(t)l_{m,k}(w;t) + \frac{\lambda_{m,k}}{x_{m,k}-t} - \frac{l_{m,k}(w;t)}{x_{m,c}-t}\lambda_{m,c} \\ &= A_m^*(t)l_{m,k}(w;t) + \frac{\lambda_{m,k}}{x_{m,k}-t} + \frac{\alpha_{m-1}}{\alpha_m}\lambda_{m,k}\frac{p_{m-1}(x_{m,k})}{t - x_{m,k}}p_m'(\tau_c)\lambda_{m,c}, \\ & \qquad \text{for } k \neq c; \end{aligned}$$

$$A_{m,c}(w;t) = A_m^*(t)l_{m,c}(w;t) + \lambda_{m,c}l_{m,c}'(w;\tau_c') \quad \text{for } k = c;$$

where $\tau_c, \tau_c' \in (t, x_{m,c}) \subset \Delta$;

(ii) if $t = x_{m,c}$,

$$A_{m,k}(w;x_{m,c}) = \frac{\lambda_{m,k}}{x_{m,k}-x_{m,c}} + \frac{\alpha_{m-1}}{\alpha_m}\lambda_{m,k}\frac{p_{m-1}(x_{m,k})}{x_{m,c}-x_{m,k}}p_m'(x_{m,c})\lambda_{m,c} \qquad \text{for } k \neq c;$$

$$A_{m,c}(w;x_{m,c}) = A_m^*(x_{m,c}) + l_{m,c}'(w;x_{m,c})\lambda_{m,c} \quad \text{for } k = c.$$

We then obtain from (3.3), (3.5), (3.6), and applying Bernstein's inequality for the derivative of a polynomial (see [8, p. 92]),

$$\left|\lambda_{m,c}p_m'(\tau_c)\right| \leqslant \text{const}\|p_m\|_\Delta \leqslant \text{const},$$

$$\left|\lambda_{m,c}l_{m,c}'(w;\tau_c')\right| \leqslant \text{const}\|l_{m,c}(w)\|_{[t,x_{m,c}]} \leqslant \text{const}.$$

In the same way, we obtain

$$\left|\lambda_{m,c}p_m'(x_{m,c})\right| \leqslant \text{const}, \qquad \left|\lambda_{m,c}l_{m,c}'(w;x_{m,c})\right| \leqslant \text{const}.$$

Thus, in both cases (i) and (ii), we deduce from (3.6) and (3.16) that

$$A_{m,c}(w;t) \leqslant \text{const},$$

$$A_{m,k}(w;t) \leqslant \frac{\lambda_{m,k}}{|x_{m,k}-t|} + \text{const}\left[|l_{m,k}(w;t)| + \lambda_{m,k}\frac{|p_{m-1}(x_{m,k})|}{|t-x_{m,k}|}\right], \qquad k \neq c.$$

We can thus write

$$\begin{aligned} \Psi_m^w(w;t) &\leqslant \text{const}(1 + L_m(w;t) + \sigma_m^*(t) + \delta_m^*(t)) \\ &\leqslant \text{const}\left(\|L_m(w)\|_\Delta + \|\sigma_m^*\|_\Delta + \|\delta_m^*\|_\Delta\right), \end{aligned}$$

so that by (3.13), (3.14), (3.17) the lemma follows. $\square$

We now set

$$r_{m-1} = f - q_{m-1}, \tag{3.20}$$

where q_{m-1} is the best approximation polynomial of degree $m-1$ for the function f. It is well known that

$$\|r_{m-1}\| \leqslant \text{const}\, \omega(f; m^{-1}), \qquad f \in C(I),$$
$$\|r_{m-1}\| \leqslant \text{const}\, m^{-k} \omega(f^{(k)}; m^{-1}), \quad f \in C^{(k)}(I),\ k \geqslant 1$$

(see, for example, [14, p. 6]). With these notations, we have

LEMMA 3.4. *Let $w \geqslant 0$ be integrable on I, and let $w \in DT(\Delta)$, where Δ is a closed set such that $\Delta \subset A \subseteq (-1,1)$. If the integral $\Phi(wf; t)$ exists for $t \in A \subseteq (-1,1)$, then for any function $f \in LD(1)$, we have*

$$\begin{aligned} R_m(f; t) &:= \left|\Phi\left(w\left[r_{m-1} - r_{m-1}(t)\right]; t\right)\right| \\ &\leqslant \text{const}\, \omega(f; m^{-1})\{w(t)(\log m + 1) + \Gamma(t)\} + o_t(1) \end{aligned}$$

where the subscript t in the notation o_t indicates that the "o" condition need only hold pointwise in t and not necessarily uniformly. Moreover, under the same hypotheses for w, if $f \in DT(I)$, then

$$\|R_m f\| \leqslant \text{const}\, \omega(f; m^{-1})(\log m + 1) + o(1), \quad f \in DT(I),$$
$$\|R_m f\| \leqslant o(\log^{-\gamma} m), \qquad f \in LD(1+\gamma),\ \gamma > 0,$$
$$\|R_m f\| \leqslant \text{const}\|r_{m-1}\| \log m, \qquad f \in \text{Lip}_M \lambda,\ 0 < \lambda \leqslant 1.$$

The proof of this lemma follows immediately from a property proved in [2, Proposizione 2.1, p. 11].

Proofs of Theorems 2.1, 2.2, 2.3. Since rule (1.2) has degree of exactness $m-1$, we have

$$\begin{aligned} E_m^w(wf; t) &= E_m^w(w(r_{m-1} - r_{m-1}(t)); t) \\ &= \Phi(w(r_{m-1} - r_{m-1}(t)); t) - \Phi_m^w(w(r_{m-1} - r_{m-1}(t)); t). \end{aligned}$$

Hence,

$$|E_m^w(wf; t)| \leqslant 2\|r_{m-1}\| \Psi_m^w(w; t) + R_m(f; t). \tag{3.21}$$

Thus Theorems 2.1, 2.2, 2.3 follow from Lemmas 3.3 and 3.4.

In order to prove Theorem 2.4, we need some more preliminary results.

LEMMA 3.5. *Let $v \in$ GSJ and let $u \geqslant 0$ on I, $u \in DT(\Delta)$ where $\Delta \subset D$ is a closed set. If*

$$u \log^+ u \qquad \textit{is integrable on } I,$$
$$u\left(v\sqrt{1-x^2}\right)^{-1/2} \quad \textit{is integrable on } I,$$

then

$$\int_{-1}^{1} L_m(v; x) u(x)\, dx < \text{const}. \tag{3.22}$$

The proof of this lemma follows immediately from an important result of Nevai (see [11, Theorem 1, p. 680]).

We are now in a position to prove the following

LEMMA 3.6. *Let $v \in$ GSJ, and let $w \geqslant 0$ on I, $w \in DT(\Delta)$, where $\Delta \subset D$ is a closed set. If the functions $w \log^+ w$ and $w(v^{-1/2}(1-x^2)^{-1/4})$ are integrable on I, then*

$$\Psi_m^w(v;t) \leqslant C\{\Psi_m^v(v;t) + \log m + 1\} \tag{3.23}$$

holds for m sufficiently large, where C is a constant dependent on t, when $t \in D$, while C is independent on t when $t \in \Delta$.

Proof. Let t be a fixed point belonging to $D = \bigcup_{i=0}^{p}(t_i, t_{i+1})$, where $t_0 = -1$, $t_{p+1} = 1$. There is then an index $j \in \{0, 1, \dots, p\}$ such that $t_j < t < t_{j+1}$. Define $d(t) = \frac{1}{2}\min\{t - t_j, t_{j+1} - t\}$. Clearly, $d(t) > 0$ for any $t \in D$. Since

$$A_{m,k}^w(v;t) = \frac{w(t)}{v(t)} A_{m,k}^v(v;t) - \frac{w(t)}{v(t)} \int_{-1}^{1} l_{m,k}(v;x) \frac{v(x)-v(t)}{x-t}\,dx + \int_{-1}^{1} l_{m,k}(v;x) \frac{w(x)-w(t)}{x-t}\,dx,$$

we get

$$\begin{aligned}\Psi_m^w(v;t) \leqslant{}& \frac{w(t)}{v(t)}\Psi_m^v(v;t) + \frac{w(t)}{v(t)} L_m(v;\tau) \int_{|x-t|\leqslant d(t)} |x-t|^{-1}\omega(v;|x-t|)\,dx \\ &+ L_m(v;\tau') \int_{|x-t|\leqslant d(t)} |x-t|^{-1}\omega(w;|x-t|)\,dx \\ &+ \frac{1}{d(t)}\left[\frac{w(t)}{v(t)} \int_{|x-t|>d(t)} L_m(v;x)v(x)\,dx \right.\\ &\left. + \int_{|x-t|>d(t)} L_m(v;x)w(x)\,dx\right]\end{aligned}$$

where $\tau, \tau' \in [t - d(t), t + d(t)] =: T_t$.

We then obtain

$$\begin{aligned}\Psi_m^w(v;t) \leqslant{}& \frac{w(t)}{v(t)}\left[\Psi_m^v(v;t) + 2\|L_m(v)\|_{T_t} \int_0^{d(t)} \delta^{-1}\omega(v;\delta)\,d\delta\right] \\ &+ 2\|L_m(v)\|_{T_t} \int_0^{d(t)} \delta^{-1}\omega(w;\delta)\,d\delta \\ &+ \frac{1}{d(t)}\left[\frac{w(t)}{v(t)} \int_{-1}^{1} L_m(v;x)v(x)\,dx + \int_{-1}^{1} L_m(v;x)w(x)\,dx\right].\end{aligned}$$

Since v, $w \in DT(\Delta)$, the first two integrals on the right are bounded. Applying Lemma 3.5 for $u = v$, $u = w$, we deduce that the other integrals are also bounded. By Lemma 3.2, the first part of the lemma follows.

Now assume that $t \in \Delta = \bigcup_{i=0}^{p}[a_i, b_i]$, where $[a_i, b_i] \subset (t_i, t_{i+1})$, $i = 0, 1, \ldots, p$. Let

$$\mu = \frac{1}{2}\min_i \{a_i - t_i, t_{i+1} - b_i\} > 0.$$

Obviously, the number μ is independent of t and $[t - \mu, t + \mu] \subset D$ for all $t \in D$. Then, we can repeat the previous proof with μ instead of $d(t)$. Finally, since w and v are bounded on Δ, the proof of the lemma is completed. □

Proof of Theorem 2.4. Proceeding as in the proof of the previous theorems, we obtain

$$|E_m^v(wf; t)| \leqslant 2\|r_{m-1}\|\Psi_m^w(v; t) + R_m(f; t).$$

Thus, by Lemmas 3.6, 3.5, 3.3, the theorem follows.

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