

Discrete Solution of the Plateau Problem and Its Convergence

By Takuya Tsuchiya*

Abstract. In this paper we define a discrete solution of the Plateau problem and we prove theorems for the convergence of the discrete solution when the contour satisfies a certain condition. A numerical example is given.

1. Introduction. Let $D = \{(u, v) \in \mathbf{R}^2 \mid u^2 + v^2 < 1\}$ be the unit disk with boundary ∂D , and let Γ be a Jordan curve in n -dimensional Euclidean space \mathbf{R}^n , $n \geq 2$, that is, Γ is a topological image of ∂D . The *Plateau problem* is to find maps $x(u, v) = (x_1(u, v), \dots, x_n(u, v))$ from the closure \bar{D} of D into \mathbf{R}^n which have the following properties:

- (1) Each x_i is continuous on \bar{D} and harmonic in D .
- (2) The parameters u, v are *isothermal* in D , that is,

$$(1.1) \quad |x_u|^2 = |x_v|^2, \quad (x_u, x_v) = 0,$$

where $x_u = (\partial x_1 / \partial u, \dots, \partial x_n / \partial u)$ and $x_v = (\partial x_1 / \partial v, \dots, \partial x_n / \partial v)$.

(3) The restriction of x to ∂D is a homeomorphism between ∂D and Γ , in which three fixed points on ∂D correspond to three fixed points on Γ .

The notations $|\cdot|$ and (\cdot, \cdot) mean Euclidean norm and inner product of \mathbf{R}^n , respectively.

A solution of the Plateau problem is called a *minimal surface* spanned in Γ . In [9] and [10] numerical methods for approximating minimal surfaces were given. The purpose of this paper is to demonstrate that their methods are proper, that is, these numerical solutions converge to minimal surfaces in suitable function spaces.

It is known that the Plateau problem is equivalent to the following variational problem.

Let $C(\bar{D}; \mathbf{R}^n)$ be the space of continuous maps on \bar{D} , and let $H^1(D; \mathbf{R}^n)$ be the ordinary Sobolev space (the exact definitions will be given in Section 2). We define the class X_Γ of maps by

$$(1.2) \quad X_\Gamma = \{f \in C(\bar{D}; \mathbf{R}^n) \cap H^1(D; \mathbf{R}^n) \mid f(\partial D) = \Gamma, f|_{\partial D} \text{ is monotone}\},$$

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where *monotone* means that, for each $p \in \Gamma$, $(f|_{\partial D})^{-1}(p) \subset \partial D$ is connected. X_Γ may be empty [6, p. 58], but in this paper we assume that $X_\Gamma \neq \emptyset$. We choose six arbitrary distinct points $z_1, z_2, z_3 \in \partial D$ and $\zeta_1, \zeta_2, \zeta_3 \in \Gamma$, and we define the subset of X_Γ by

$$(1.3) \quad X_\Gamma^{tp} = \{f \in X_\Gamma \mid f(z_i) = \zeta_i, i = 1, 2, 3\},$$

where the superscript “*tp*” stands for “three-point condition”. For $f \in H^1(D; \mathbf{R}^n)$, the *energy* of f on D is defined by

$$(1.4) \quad E(f) = \frac{1}{2} \iint_D (|f_u|^2 + |f_v|^2) du dv.$$

E is called the *energy functional*. It is well known that $f \in X_\Gamma^{tp}$ is a minimal surface spanned in Γ if and only if f is a stationary point of the energy functional in X_Γ^{tp} . In particular, if f is a minimal point of E in X_Γ^{tp} , then f is a minimal surface spanned in Γ [4, pp. 107–118].

In Section 3 we define a (stable) discrete minimal surface by a variational method using the simplest finite element scheme. In Section 5 we prove the convergence of the discrete minimal surfaces.

For the existence of minimal surfaces, Douglas and Rado proved the following theorem ([4, pp. 101–105], [6, p. 71]):

THEOREM A. *Let $e_\Gamma = \inf\{E(f) : f \in X_\Gamma^{tp}\}$. If $X_\Gamma^{tp} \neq \emptyset$, then there exists a map $x \in X_\Gamma^{tp}$ such that $E(x) = e_\Gamma$.*

From the above facts, $x \in X_\Gamma^{tp}$ in Theorem A is a minimal surface spanned in Γ . Hence we can say that if $X_\Gamma^{tp} \neq \emptyset$, there exists at least one minimal surface in X_Γ^{tp} . An x as in Theorem A is called a *Douglas solution*. Note that if $n = 2$, Theorem A is the Riemann Mapping Theorem for domains bounded by Jordan curves. To prove Theorem A, the following important lemma was introduced:

LEMMA B. *Let M be a positive constant such that $e_\Gamma < M$, and let Y be the subset of X_Γ^{tp} defined by $Y = \{f \in X_\Gamma^{tp} \mid E(f) \leq M\}$. Then Y is equicontinuous on ∂D .*

In Section 4 we prove a lemma which is the discrete analogue of Lemma B. That lemma will play an important role in the proof of the convergence of discrete minimal surfaces.

2. Function Spaces. In this section we define the function spaces to be used. Let A be a compact subset of \mathbf{R}^2 . We define

$$(2.1) \quad C(A) = \{f : A \rightarrow \mathbf{R} \mid f \text{ is continuous on } A\},$$

with $\|f\|_{C(A)} = \max\{|f(u, v)| : (u, v) \in A\}$, and

$$(2.2) \quad C(A; \mathbf{R}^n) = \{f = (f_1, \dots, f_n) : A \rightarrow \mathbf{R}^n \mid f_i \in C(A), i = 1, \dots, n\},$$

with $\|f\|_{C(A; \mathbf{R}^n)} = \max\{\|f_i\|_{C(A)} : i = 1, \dots, n\}$.

For $1 \leq p \leq \infty$, we define

$$(2.3) \quad L^p(A; \mathbf{R}^n) = \{f = (f_1, \dots, f_n) : A \rightarrow \mathbf{R}^n \mid f_i \in L^p(A), i = 1, \dots, n\},$$

with $\|f\|_{L^p(A; \mathbf{R}^n)} = \max\{\|f_i\|_{L^p(A)} : i = 1, \dots, n\}$.

Let B be a bounded domain in \mathbf{R}^2 . We define

$$(2.4) \quad W^{1,p}(B) = \{f \in L^p(B) \mid f_u, f_v \in L^p(B)\},$$

with $\|f\|_{W^{1,p}(B)} = \|f\|_{L^p(B)} + \|f_u\|_{L^p(B)} + \|f_v\|_{L^p(B)}$, where f_u and f_v are distributional derivatives. We define

$$(2.5) \quad W^{1,p}(B; \mathbf{R}^n) = \{f = (f_1, \dots, f_n): B \rightarrow \mathbf{R}^n \mid f_i \in W^{1,p}(B), i = 1, \dots, n\},$$

with $\|f\|_{W^{1,p}(B; \mathbf{R}^n)} = \max\{\|f_i\|_{W^{1,p}(B)}: i = 1, \dots, n\}$. When $p = 2$, we denote $W^{1,p}(B)$ and $W^{1,p}(B; \mathbf{R}^n)$ by $H^1(B)$ and $H^1(B; \mathbf{R}^n)$, respectively. We define

$$(2.6) \quad HM = \{f = (f_1, \dots, f_n): D \rightarrow \mathbf{R}^n \mid f_i \text{ is harmonic in } D, i = 1, \dots, n\}.$$

An element of HM is called a *harmonic map*.

For simplicity, when there is no confusion, we omit “ \mathbf{R}^n ” from the notations.

3. Definition of the Discrete Minimal Surface. Let $\Omega \subset D$ be a triangulation of D , that is, Ω is a polygonal domain with the properties that (1) $\partial\Omega$, the boundary of Ω , inscribes ∂D , and (2) the set $\bar{\Omega}$ is the union $\bar{\Omega} = \cup K_i$ of finite triangles $\{K_i\}$ whose interiors are pairwise disjoint and such that all their edges are either an edge of another triangle or an edge of the polygon $\partial\Omega$.

With the triangulation Ω we associate the mesh size of Ω defined by

$$(3.1) \quad |\Omega| = \max_i \text{diam}(K_i).$$

We assume that there exists a positive constant ω which is independent of the triangulation Ω such that the following inequality holds for each triangle $K_i \subset \Omega$:

$$(H1) \quad \text{diam}(K_i)/\rho(K_i) \leq \omega,$$

where $\rho(K_i) = \sup\{\text{diam}(S); K_i \supset S: \text{ball}\}$.

Let S_Ω be the set of functions which are continuous on $\bar{\Omega}$ and linear on each triangle K_i . Let \mathcal{S}_Ω be the set of maps from $\bar{\Omega}$ into \mathbf{R}^n such that each component function belongs to S_Ω . Let $N_\Omega = \{b_i\}_{i=1}^{N+N'}$ be the set of nodal points of Ω where $b_i \in \Omega^\circ$, the interior of Ω , for $1 \leq i \leq N$, and $b_i \in \partial\Omega$ for $N + 1 \leq i \leq N + N'$. The basis $\{w_i\}_{i=1}^{N+N'} \subset S_\Omega$ of \mathcal{S}_Ω is defined by

$$(3.2) \quad w_i(b_j) = \delta_{ij}, \quad 1 \leq i, j \leq N + N'.$$

We assume that the following inequality holds:

$$(H2) \quad \iint_{\Omega} (w_{iu}w_{ju} + w_{iv}w_{jv}) \, du \, dv \leq 0 \quad \text{for } 1 \leq i \leq N, 1 \leq j \leq N + N', i \neq j.$$

We say that a triangulation Ω satisfying (H2) is of *nonnegative type*. We associate the admissible class of triangulations of D defined by

$$(3.3) \quad \Delta^p = \{\Omega \mid z_1, z_2, z_3 \in N_\Omega, \Omega \text{ satisfies (H1), (H2)}\}.$$

When Ω is given, we define

$$(3.4) \quad X_{\Gamma, \Omega} = \{f \in \mathcal{S}_\Omega \mid f(N_\Omega \cap \partial D) \subset \Gamma, f|_{\partial D} \text{ is } d\text{-monotone}\},$$

where *d-monotone* means that the order of nodal points on Γ is the same as the order of nodal points on ∂D . Let

$$(3.5) \quad X_{\Gamma, \Omega}^p = \{f \in X_{\Gamma, \Omega} \mid f(z_i) = \zeta_i, i = 1, 2, 3\},$$

and let $E_\Omega(f)$ be the energy functional on Ω defined by

$$(3.6) \quad E_\Omega(f) = \frac{1}{2} \iint_\Omega (|f_u|^2 + |f_v|^2) du dv.$$

We extend $f \in \mathbb{S}_\Omega$ to $D - \Omega$ as follows:

If $p \in \partial\Omega$ and $p \notin N_\Omega$, there exists an exterior normal half line L_p of $\partial\Omega$ on p . For arbitrary $q \in L_p \cap (D - \Omega)$, we define $f(q) = f(p)$. Then the following estimate is valid:

$$(3.7) \quad E_\Omega(f) \leq E(f) \leq (1 + C|\Omega|)E_\Omega(f) \quad \text{for any } f \in \mathbb{S}_\Omega,$$

where C is a constant which is independent of Ω and f .

Definition 1. Let $\Omega \in \Delta^p$.

(D1) $f \in \mathbb{S}_\Omega$ is a d -harmonic function on Ω if $E_\Omega(f) \leq E_\Omega(g)$ for any $g \in \mathbb{S}_\Omega$ with $g = f$ on $\partial\Omega$.

(D2) $f = (f_1, \dots, f_n) \in \mathbb{S}_\Omega$ is a d -harmonic map on Ω if f_i , $i = 1, \dots, n$, are d -harmonic functions. We denote by HM_Ω the set of d -harmonic maps on Ω .

(D3) Let $x \in HM \cap C(\bar{D}; \mathbf{R}^n)$. $f \in \mathbb{S}_\Omega$ is the FEM solution of x on Ω if $f \in HM_\Omega$ and $x(b) = f(b)$ for all $b \in N_\Omega \cap \partial D$.

(D4) $f \in X_{\Gamma, \Omega}^{t,p}$ is a stable d -minimal surface if there exists a positive constant δ such that

$$\|f - g\|_{C(\bar{\Omega})} < \delta \quad \text{implies} \quad E_\Omega(f) \leq E_\Omega(g) \quad \text{for } g \in X_{\Gamma, \Omega}^{t,p}.$$

(D5) $f \in X_{\Gamma, \Omega}^{t,p}$ is a d -Douglas solution if

$$E_\Omega(f) = \inf\{E_\Omega(g) : g \in X_{\Gamma, \Omega}^{t,p}\}. \quad \square$$

From these definitions it is obvious that a stable d -minimal surface is d -harmonic, and a d -Douglas solution is a stable d -minimal surface.

Remark 2. It is known that there exists an unstable minimal surface in $X_{\Gamma}^{t,p}$ if there exist two stable minimal surfaces in $X_{\Gamma}^{t,p}$ [4, pp. 236–243]. We can define the unstable d -minimal surface similarly as in Definition 1. But a proof of the convergence of unstable d -minimal surfaces has not yet been obtained. So we do not examine unstable d -minimal surfaces in this paper. \square

LEMMA 3. *Suppose that $\Omega \in \Delta^p$. Then a d -harmonic function f satisfies the discrete maximum principle,*

$$(3.8) \quad \min_{s \in \partial D} f(s) \leq f(z) \leq \max_{s \in \partial D} f(s) \quad \text{for any } z \in D.$$

Proof. We can prove this lemma by direct computation. See [1], [3]. \square

4. Relative Compactness. In this section we prove the relative compactness of bounded subsets of $X_{\Gamma, \Omega}^{t,p}$ when the Jordan curve satisfies a certain condition. This fact is the discrete analogue of Lemma B.

Let $\Gamma_0 \subset \mathbf{R}^n$ be a Jordan arc which connects two distinct points $a, b \in \mathbf{R}^n$, and let $c \neq a, b$ be a point on Γ_0 . Taking finite points $a = x_0, x_1, \dots, x_l = c, \dots, x_m = b$ monotonously on Γ_0 , and connecting x_{i-1} and x_i , $i = 1, \dots, m$, by segments, a continuous and piecewise linear arc is defined. We denote by $L_{\Gamma_0, c}$ the set of such arcs.

Let $\Gamma \subset \mathbf{R}^n$ be a Jordan curve and ζ_i ($i = 1, 2, 3$) be three distinct points on Γ . We define the three Jordan arcs Γ_i by $\Gamma_i = \overline{\zeta_{i-1}\zeta_i\zeta_{i+1}}$, $i = 1, 2, 3$, where $\zeta_0 = \zeta_3$ and $\zeta_4 = \zeta_1$.

We say that $\Gamma = \{\Gamma, \zeta_1, \zeta_2, \zeta_3\}$ has the property (A) if Γ satisfies the following condition:

(A) For an arbitrary positive number ε there exists a positive number σ such that for any $\eta \in L_{\Gamma, \zeta_i}$, $i = 1, 2, 3$, and any $p_1, p_2 \in \eta - \{\zeta_{i-1}, \zeta_{i+1}\}$ with $|p_1 - p_2| \geq \varepsilon$, $\eta - \{p_1, p_2\}$ has three connected components η_i , $i = 1, 2, 3$, with $\zeta_{i-1} \in \eta_1$ and $\zeta_{i+1} \in \eta_3$, and the distance between η_1 and η_3 is greater than σ , i.e.,

$$(4.1) \quad \text{dist}(\eta_1, \eta_3) \geq \sigma.$$

The condition (A) is very restrictive. This condition requires that Γ should be almost like a circle. A contour like in Figure 1 does not satisfy (A).

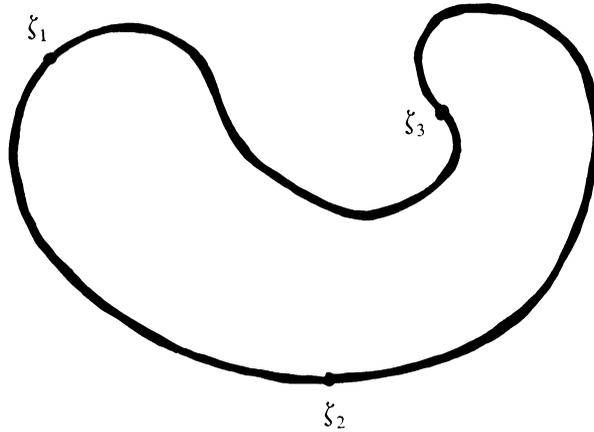


FIGURE 1

But if $n = 2$, the condition (A) does not restrict us. See Remark 10.

Let M be a positive constant with $e_\Gamma < M$. We define

$$(4.2) \quad Y' = \{f \in X_{\Gamma, \Omega}^{1,p} \mid \Omega \in \Delta^p, E(f) \leq M\}.$$

LEMMA 4. Suppose that $\{\Gamma, \zeta_1, \zeta_2, \zeta_3\}$ satisfies (A). Then Y' is equicontinuous on ∂D .

Proof. For arbitrary $z \in \mathbf{R}^2$ and arbitrary positive number r , we define

$$(4.3) \quad C_{r,z} = D \cap \{w \in \mathbf{R}^2: |w - z| = r\}.$$

Let $f \in Y'$, and let $l(C_{r,z})$ be the length of the image $f(C_{r,z})$. The following lemma is valid ([4, p. 102], [6, pp. 67–68]):

LEMMA 5. For arbitrary δ , $0 < \delta < 1$, there exists ρ , $\delta \leq \rho \leq \delta^{1/2}$, depending on f and z such that

$$(4.4) \quad l(C_{\rho,z})^2 \leq 2\pi\lambda(\delta),$$

where $\lambda(\delta) = 4M/\log(1/\delta)$.

The proof of Lemma 4 is by contradiction. Assume that Y' is not equicontinuous on ∂D . Then there exists a positive number ε_0 such that for arbitrary $\delta > 0$ there exist $\Omega \in \Delta'^p$, $f \in X_{\Gamma, \Omega}^{t,p}$, and $a_1, a_2 \in \partial D$ such that

$$(4.5) \quad |a_1 - a_2| < \delta \quad \text{and} \quad |f(a_1) - f(a_2)| \geq \varepsilon_0.$$

Let σ be the positive number associated with $\varepsilon = \varepsilon_0$ in (A). We choose δ such that (a) $\sigma > (2\pi\lambda(\delta))^{1/2}$, and (b) $\min\{|z_i - z_j|: i \neq j\} > 2\delta^{1/2}$.

For such δ , we take $\overline{\Omega} \in \Delta'^p$, $f \in X_{\Gamma, \Omega}^{t,p}$ and $a_1, a_2 \in \partial D$ as in (4.5). Let w be the center of the shorter arc $a_1 a_2$. By Lemma 5, there exists a number ρ with $\delta \leq \rho \leq \delta^{1/2}$ such that $l(C_{\rho, w}) \leq (2\pi\lambda(\delta))^{1/2}$. Let b_1, b_2 be the end points of $C_{\rho, w}$. ∂D is divided into two connected components by $C_{\rho, w}$. An arc A_1 contains w and, by (b), its complement A_2 contains at least two of the points z_i , $i = 1, 2, 3$. Then $f(b_j), f(a_j)$ ($j = 1, 2$) are on a Jordan arc which is an element of L_{Γ, ζ_i} for some i .

From (a),

$$|f(b_1) - f(b_2)| \leq l(C_{\rho, w}) \leq (2\pi\lambda(\delta))^{1/2} < \sigma,$$

and from (4.5)

$$|f(a_1) - f(a_2)| \geq \varepsilon_0.$$

Because of (A), this is the contradiction, and Lemma 4 is proved. \square

From Lemma 5 and the Ascoli-Arzelà theorem we immediately obtain the following

COROLLARY 6. *Let $\Delta'^p \supset \{\Omega_n\}_{n=1}^\infty$ such that $\lim_{n \rightarrow \infty} |\Omega_n| = 0$, and let $f_n \in X_{\Gamma, \Omega_n}^{t,p}$. Suppose that $\{\Gamma, \zeta_i\}$ satisfies (A) and $E(f_n)$ are uniformly bounded. Then there exists a subsequence $\{f_{n_i}\}$ such that $f_{n_i}|_{\partial D}$ converges uniformly to a continuous map $\varphi \in C(\partial D)$ on ∂D . Moreover, $\varphi(\partial D) = \Gamma$ and φ is monotone.*

5. Convergence of the Discrete Minimal Surfaces. In this section we prove the convergence of the discrete minimal surfaces.

LEMMA 7. *Let $\Delta'^p \supset \{\Omega_n\}_{n=1}^\infty$ such that $\lim_{n \rightarrow \infty} |\Omega_n| = 0$, and let $\{w_n \in \mathbb{S}_{\Omega_n}\}_{n=1}^\infty$ be a sequence of d -harmonic maps.*

Suppose that $w_n|_{\partial D}$ converges uniformly to a continuous map $\varphi \in C(\partial D)$. Let w be a harmonic map in D with $w = \varphi$ on ∂D . Then we have

$$(5.1) \quad E(w) \leq \liminf_{n \rightarrow \infty} E_{\Omega_n}(w_n).$$

Proof. Let $\{f_n\}_{n=1}^\infty$ be harmonic maps with $f_n = w_n$ on ∂D . Since $f_n|_{\partial D}$ converges uniformly to φ , and in view of the lower semicontinuity of the energy functional [4, p. 11], we have $E(w) \leq \liminf_{n \rightarrow \infty} E(f_n)$. Since $E(f_n) \leq E(w_n)$, and by (3.7), we obtain (5.1). \square

THEOREM 8. *Suppose that $\{\Gamma, \zeta_i\}$ satisfies (A). Let $\Delta'^p \supset \{\Omega_n\}_{n=1}^\infty$ be such that $\lim_{n \rightarrow \infty} |\Omega_n| = 0$, and let $\{x_n \in X_{\Gamma, \Omega_n}^{t,p}\}_{n=1}^\infty$ be a sequence of the d -Douglas solutions.*

Then there exists a subsequence $\{x_{n_i}\}$ which converges to one of the Douglas solutions $x \in X_{\Gamma}^{t,p}$ in the following sense:

$$(5.2) \quad \lim_{n_i \rightarrow \infty} \|x - x_{n_i}\|_{H^1(D)} = 0,$$

and if $x \in W^{1,p}(D)$, $p > 2$, then

$$(5.3) \quad \lim_{n_i \rightarrow \infty} \|x - x_{n_i}\|_{C(\bar{D})} = 0.$$

If the Douglas solution is unique, then x_n converges in the sense of (5.2) and (5.3).

Proof. Since \mathbb{S}_{Ω_n} are finite-dimensional vector spaces, and E_{Ω_n} is bounded from below with the property that $E_{\Omega_n}(f) \rightarrow \infty$ as $|f| \rightarrow \infty$ for $f \in X_{\Gamma, \Omega_n}^{1,p}$, it is obvious that d -Douglas solutions exist in each $X_{\Gamma, \Omega_n}^{1,p}$.

Let y be one of the Douglas solutions and $f_n \in X_{\Gamma, \Omega_n}^{1,p}$ the FEM solutions of y on Ω_n . It is well known that

$$(5.4) \quad \lim_{n \rightarrow \infty} \|y - f_n\|_{H^1(D)} = 0$$

[2, p.134]. Since $E_{\Omega_n}(x_n) \leq E_{\Omega_n}(f_n)$, $\{E_{\Omega_n}(x_n)\}$ is uniformly bounded. Thus, by Corollary 6, there exists a subsequence which converges uniformly to a continuous map $\varphi \in C(\partial D)$ on ∂D with the properties of Corollary 6. Let x be a harmonic map with $x = \varphi$ on ∂D . From Lemma 7 we see that

$$(5.5) \quad E(x) \leq \liminf_{n \rightarrow \infty} E_{\Omega_n}(x_n) \leq \lim_{n \rightarrow \infty} E(f_n) = E(y).$$

Hence x is one of the Douglas solutions.

Let $g_n \in X_{\Gamma, \Omega_n}^{1,p}$ be the FEM solutions of x on Ω_n . By Lemma 3 we have

$$(5.6) \quad \lim_{n_i \rightarrow \infty} \|x_{n_i} - g_{n_i}\|_{C(\bar{D})} = 0.$$

Hence, by (5.4), we obtain

$$(5.7) \quad \lim_{n_i \rightarrow \infty} \|x_{n_i} - x\|_{L^2(D)} = 0.$$

From (5.5), the limit of $E_{\Omega_n}(x_n)$ exists and is equal to $E(x)$. Hence, by standard Sobolev space theory, we obtain (5.2).

It is known [3] that, if $x \in W^{1,p}(D)$, $p > 2$, then

$$(5.8) \quad \lim_{n \rightarrow \infty} \|x - g_n\|_{C(\bar{D})} = 0.$$

Hence, with (5.6), we obtain (5.3).

When the Douglas solution is unique, the limit of convergent subsequences of $\{x_n\}$ is unique. Hence x_n converges in the sense of (5.2) and (5.3). \square

A sufficient condition for $x \in W^{1,p}(D)$ ($p > 2$) can be obtained in the following manner.

We say that the Jordan curve Γ is of $C^{m,\alpha}$ -class ($m \geq 1, 0 < \alpha < 1$) if there exists a homeomorphism $\gamma: \partial D \rightarrow \Gamma$ such that $\gamma \in C^{m,\alpha}(\partial D)$ and $\gamma'(s) > 0$ for all $s \in \partial D$. $C^{m,\alpha}$ denotes a Hölder space. The following theorem is known:

THEOREM C (NITSCHKE [7]). *Suppose that Γ is of $C^{m,\alpha}$ -class. Then arbitrary minimal surfaces spanned in Γ belong to $C^{m,\alpha}(\bar{D})$.*

Hence, if $\Gamma \in C^{1,\alpha}$, then arbitrary minimal surfaces spanned in Γ belong to $W^{1,p}(D)$, $p > 2$. This condition is rough but practical enough.

A harmonic map $x \in X_{\Gamma}^{lp}$ is said to be an *isolated stable minimal surface* if there exists a constant δ such that

$$0 < \|x - y\|_{C(\bar{D})} < \delta \quad \text{implies} \quad E(x) < E(y) \quad \text{for } y \in X_{\Gamma}^{lp}.$$

We have the following theorem:

THEOREM 9. *Suppose that $\{\Gamma, \zeta_i\}$ satisfies (A). Let $\Delta^{lp} \supset \{\Omega_n\}_{n=1}^{\infty}$ be such that $\lim_{n \rightarrow \infty} |\Omega_n| = 0$, and let $x \in X_{\Gamma}^{lp}$ be an isolated stable minimal surface. Then there exists a sequence $\{x_n \in X_{\Gamma, \Omega_n}^{lp}\}_{n=1}^{\infty}$ of stable d -minimal surfaces which converges to x in the sense of (5.2) and (5.3).*

Proof. Let $\{f_n \in X_{\Gamma, \Omega_n}^{lp}\}_{n=1}^{\infty}$ be the FEM solutions of x . We define δ -neighborhoods of f_n by

$$B_n(f_n, \delta) = \left\{ h_n \in X_{\Gamma, \Omega_n}^{lp} \cap HM_{\Omega_n} \mid \|f_n - h_n\|_{C(\bar{D})} \leq \delta \right\}.$$

Since $B_n(f_n, \delta)$ is a bounded closed set in finite-dimensional Euclidean space, there exists $x_n \in B_n(f_n, \delta)$ such that $E_{\Omega_n}(x_n)$ attains the minimum value in $B_n(f_n, \delta)$. By (5.4), $\{E(x_n)\}$ is uniformly bounded. Hence there exists a subsequence $\{x_{n_i}\}$ which converges uniformly to a continuous map φ on ∂D . Let $\psi \in X_{\Gamma}^{lp}$ be a harmonic map with $\psi = \varphi$ on ∂D . For arbitrary $\varepsilon > 0$ we take sufficiently large n_i such that $\|x|_{\partial D} - f_{n_i}|_{\partial D}\|_{C(\bar{D})} < \varepsilon/2$ and $\|\psi|_{\partial D} - x_i|_{\partial D}\|_{C(\bar{D})} < \varepsilon/2$. By (3.8), we obtain

$$\|\psi - x\|_{C(\bar{D})} = \|\psi|_{\partial D} - x|_{\partial D}\|_{C(\partial D)} < \varepsilon + \|f_{n_i}|_{\partial D} - x_{n_i}|_{\partial D}\|_{C(\partial D)} \leq \varepsilon + \delta.$$

Hence we show that $\|\psi - x\|_{C(\bar{D})} \leq \delta$ and $E(\psi) \geq E(x)$. From the proof of Theorem 8, the inverse inequality holds. Hence, by the assumption on x , we conclude that $x = \psi$. Again, from the proof of Theorem 8 we see that x_n converges to x in the sense of (5.2) and (5.3). It is clear that x_n are d -minimal surfaces. \square

Remark 10. The condition (A) is strong. But if $n = 2$ the condition (A) does not restrict us.

Let $D_0 \subset \mathbf{R}^2$ be an arbitrary simply connected polygonal domain. We can approximate the conformal map $x^{-1}: \bar{D}_0 \rightarrow \bar{D}$ instead of $x: \bar{D} \rightarrow \bar{D}_0$. Taking suitable ζ_i ($i = 1, 2, 3$) on ∂D , $\{\partial D, \zeta_i\}$ satisfies (A). Thus for x^{-1} we have the same conclusion as in Theorem 8. \square

6. A Numerical Example. In this section we give a numerical example. Let $n = 2$. We identify \mathbf{R}^2 with C by the identification $z = u + iv$ and $x = x_1 + ix_2$. Let $\Gamma = \{x \in C: |x| = 1\}$ and $\zeta_k = e^{2(k-1)\pi i/3}$, $k = 1, 2, 3$. It is easy to check that $\{\Gamma, \zeta_k\}$ satisfies (A). We take the corresponding points $z_1 = 1$, $z_2 = e^{4\pi i/3}$ and $z_3 = e^{5\pi i/3}$. The conformal map $x: \bar{D} \rightarrow \bar{D}$ is the unique Douglas solution. The exact solution is $x = e^{4\pi i/3}(z - \alpha)/(\bar{\alpha}z - 1)$, where $\alpha = (1 + \sqrt{3}i)/4$.

Here, we approximate this map on triangulations like in Figure 2, and we compare numerical solutions with the exact solution. This triangulation satisfies (H2).

We employ the generalized Newton method for the computations. For the details of the computations, see [9]. We give the results in Table 1.

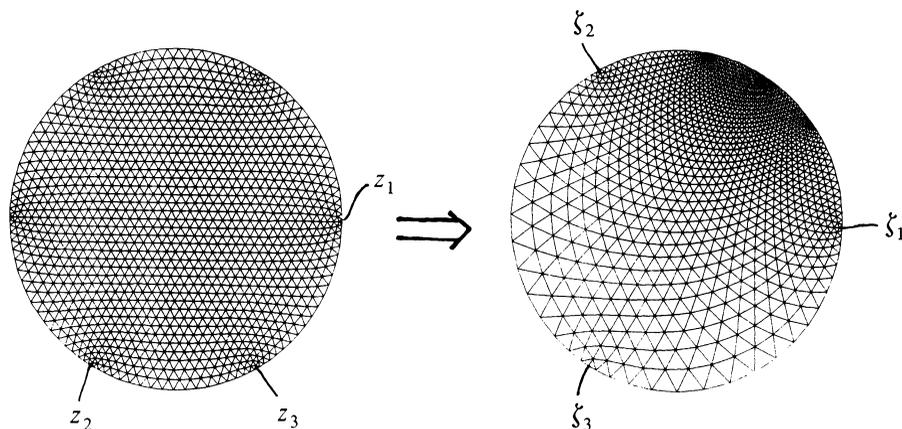


FIGURE 2

TABLE 1
Results of computations

mesh size	norm	energy
1.38918×10^{-1}	1.25348×10^{-2}	3.12988
9.35603×10^{-2}	8.77615×10^{-3}	3.13606
7.01455×10^{-2}	6.57107×10^{-3}	3.13805

In Table 1, "norm" means $\max\{|x(b_i) - x_h(b_i)|: b_i \in N_\Omega\}$ where x_h is the numerical solution.

The computations were carried out using the FACOM M-382 computer system at the Computer Center, Kyushu University.

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Department of Mathematics
Kyushu University 33
Fukuoka 812, Japan

1. P. G. CIARLET, "Discrete maximum principle for finite difference operators," *Aequationes Math.*, v. 4, 1970, pp. 338-352.
2. P. G. CIARLET, *The Finite Element Method for Elliptic Problems*, North-Holland, Amsterdam, 1978.
3. P. G. CIARLET & P. A. RAVIART, "Maximum principle and uniform convergence for the finite element method," *Comput. Methods Appl. Mech. Engrg.*, v. 2, 1973, pp. 17-31.
4. R. COURANT, *Dirichlet's Principle, Conformal Mapping, and Minimal Surfaces*, Interscience, New York, 1950.
5. R. COURANT & D. HILBERT, *Methods of Mathematical Physics*, Vol. II, Interscience, New York, 1962.
6. H. B. LAWSON, JR., *Lectures on Minimal Submanifolds*, Vol. I, Publish or Perish, Berkeley, 1980.
7. J. C. C. NITSCHKE, "The boundary behavior of minimal surfaces. Kellogg's theorem and branch points on the boundary," *Inventiones Math.*, v. 8, 1969, pp. 313-333.
8. R. OSSERMAN, *A Survey of Minimal Surfaces*, Van Nostrand Reinhold, New York, 1969.
9. T. TSUCHIYA, "On two methods for approximating minimal surfaces in parametric form," *Math. Comp.*, v. 46, 1986, pp. 517-529.
10. W. L. WILSON, JR., "On discrete Dirichlet and Plateau problems," *Numer. Math.*, v. 3, 1961, pp. 359-373.