Convergence of a Shock-Capturing Streamline Diffusion Finite Element Method for a Scalar Conservation Law in Two Space Dimensions

By Anders Szepessy

Abstract. We prove a convergence result for a shock-capturing streamline diffusion finite element method applied to a time-dependent scalar nonlinear hyperbolic conservation law in two space dimensions. The proof is based on a uniqueness result for measure-valued solutions by DiPerna. We also prove an almost optimal error estimate for a linearized conservation law having a smooth exact solution.

1. Introduction. In this note we continue the analysis of shock-capturing streamline diffusion finite element methods (SC-metods for short below) for hyperbolic conservation laws initiated by Johnson and Szepessy [6], [7], where convergence for Burgers' equation in one dimension was proved using the theory of compensated compactness. We prove here strong convergence in L_1^{loc} for a SC-method with piecewise linear elements applied to the following scalar conservation law in two dimensions:

$$(1.1a) \qquad u_t + \sum_{i=1}^{2} \frac{\partial}{\partial x_i} f_i(u) = 0 \quad \text{in } R^2 \times R_+ \equiv R_+^3,$$

$$(1.1b) \qquad u(x,0) = u_0(x) \quad \text{for } x \in R^2,$$

where the $f_i \colon R \to R$ are given smooth functions and we assume that the initial data $u_0 \in L_\infty(R^2)$ have compact support. The convergence result is obtained using a uniqueness result by DiPerna [1] for measure-valued solutions by proving that the finite element solutions are uniformly bounded in L_∞, weakly consistent with all entropy inequalities and strongly consistent with the initial condition. We also show that the accuracy of the method when applied to a linearized conservation law is at least $\mathscr{O}(h^{3/2})$, where h is the mesh parameter.

The streamline diffusion method is a general finite element method for hyperbolic problems which may be viewed as a certain combination of the standard Galerkin method and a least squares method. In the shock-capturing streamline diffusion method an artificial viscosity is added, with viscosity depending on the residual of the finite element solution and a certain mesh-dependent parameter (here the residual means the result of the hyperbolic operator applied to the finite element solution). The shock-capturing streamline diffusion method combines $\mathscr{O}(h^{3/2})$ accuracy (in the case of piecewise linear elements) with good stability obtained through the least squares control of the residual and the shock-capturing artificial viscosity.

For more information on streamline diffusion finite element methods we refer to Hughes and Mallet [2], Johnson et al. [4] and the references therein. An extension of the convergence result of this paper to include also boundary conditions is given in [9].

The shock-capturing artificial viscosity coefficients in this paper differ from those in Hughes and Mallet [2] and Johnson et al. [7], where the artificial viscosity was normalized by dividing by the gradient of the approximate solution. We compensate by using here a smaller mesh-dependent parameter in the coefficient of the artificial viscosity. This makes no essential difference in the convergence proof for nonsmooth solutions, but makes it possible to easily prove that the error for a linearized version of (1.1) is of the desired order $\mathscr{O}(h^{k+1/2})$ in regions where the exact solution is smooth, when using elements of order k.

An outline of the paper is as follows. In Section 2 we give some background on Young measures and state the uniqueness result for measure-valued solutions satisfying entropy conditions. In Section 3 we introduce the SC-method and in the main Theorem 3.1 we prove that the finite element solutions U^h converge strongly in L_p^{loc}, $1 \leq p < \infty$, to the unique L_∞-solution u of (1.1) as the mesh parameter h tends to zero. The proof is divided into Lemmas 3.1–3.3, which are proved in Sections 4–6. In Lemma 3.1 we prove that $\|U^h\|_{L_\infty}$ is uniformly bounded in h by proving L_p-estimates and letting p tend to infinity as in [7] in the case of one space dimension. In Lemma 3.2 we prove that the Young measure associated with U^h is a weak solution and satisfies all entropy inequalities corresponding to convex entropies. Finally, in Lemma 3.3 we prove convergence towards initial data as h tends to zero by combining weak convergence and L_2-stability. In Section 7 we prove error estimates for the SC-method applied to a linearized version of (1.1) with smooth solution, thus demonstrating that the shock-capturing modification in the SC-method does not degrade the accuracy for smooth solutions. We shall denote by C a positive constant not necessarily the same at each occurrence and always independent of h.

2. Measure-Valued Solutions. In this section we give the necessary background material on Young measures and measure-valued solutions of conservation laws following [10], [11], [1]. Our convergence result is based on Theorem 2.2 below.

THEOREM 2.1. *Let u_j be a uniformly bounded sequence in $L_\infty(R_+^3)$, i.e., for some constant K,*

$$(2.1) \qquad \|u_j\|_{L_\infty(R_+^3)} \leq K, \qquad j = 1, 2, 3, \ldots .$$

Then there exists a subsequence (again denoted) u_j and a family of measurable probability measures $\nu_y \in \text{Prob}(R)$, $y \in R_+^3$, such that $\text{supp}\,\nu_y$ is contained in $\{x \in R : |x| \leq K\}$ and the L_∞ weak-star limit,

$$(2.2a) \qquad g(u_j(\cdot)) \rightharpoonup \bar{g}(\cdot),$$

exists for all continuous functions g, where

$$(2.2b) \qquad \bar{g}(y) = \int_R g(\lambda)\, d\nu_y \equiv \langle \nu_y, g(\lambda) \rangle$$

for almost all points $y \in R_+^3$. Here, ν_y being measurable means that $\langle \nu_y, g \rangle$ is measurable with respect to y for each continuous function g.

COROLLARY 2.1. *The sequence u_j in Theorem 2.1 converges strongly to u if and only if the associated Young measure ν_y reduces at almost all points y to a Dirac measure concentrated at $u(y)$, i.e.,*

$$(2.3) \qquad \nu_y = \delta_{u(y)}.$$

Since u_j is uniformly bounded in $L_\infty(R_+^3)$, strong convergence in L_p^{loc} for some p, $1 \leq p < \infty$, is equivalent to strong convergence in L_q^{loc} for all q, $1 \leq q < \infty$.

We can now define the measure-valued (mv) solution of (1.1) introduced by DiPerna [1].

Definition. A measurable map $\nu : y \to \nu_y \in \text{Prob}(R)$ from the domain R_+^3 is a mv solution of (1.1a) if

$$(2.4) \qquad \frac{\partial}{\partial t} \langle \nu_y, \lambda \rangle + \sum_{i=1}^{2} \frac{\partial}{\partial x_i} \langle \nu_y, f_i(\lambda) \rangle = 0 \quad \text{in } \mathscr{D}'(R_+^3)$$

i.e., in the sense of distributions on R_+^3,

$$\int_{R_+^3} \left(\phi_t \langle \nu_y, \lambda \rangle + \sum_{i=1}^{2} \langle \nu_y, f_i(\lambda) \rangle \phi_{x_i} \right) dx \, dt = 0$$

for all $\phi \in \mathscr{C}_0^1(R_+^3)$. Further, a mv solution ν of (1.1a) is admissible if

$$(2.5) \qquad \frac{\partial}{\partial t} \langle \nu_y, \eta(\lambda) \rangle + \sum_{i=1}^{2} \frac{\partial}{\partial x_i} \langle \nu_y, q_i(\lambda) \rangle \leq 0 \quad \text{in } \mathscr{D}'(R_+^3),$$

for all convex entropy pairs $(\eta, q) = (\eta, q_1, q_2)$.

We recall that (η, q) is a convex entropy pair if the entropy $\eta : R \to R$ is continuous and convex, the corresponding entropy flux $q = (q_1, q_2) : R \to R^2$ is continuous and all $\mathscr{C}^1$-solutions $u(x, t)$ of (1.1a) satisfy the additional conservation law

$$(2.6) \qquad \frac{\partial}{\partial t} \eta(u) + \sum_{i=1}^{2} \frac{\partial}{\partial x_i} q_i(u) = 0 \quad \text{in } \mathscr{D}'(R_+^3).$$

For smooth η, (2.6) holds if and only if

$$\eta' f_i' = q_i', \qquad i = 1, 2,$$

which means that every convex smooth function $\eta(u)$ forms an entropy pair (η, q) provided that the corresponding entropy flux $q = (q_1, q_2)$ is defined by

$$q_i(u) = \int_0^u \eta'(s) f_i'(s) \, ds, \qquad i = 1, 2.$$

We recall that the classical admissibility condition for a solution u to (1.1a) reads

$$(2.7) \qquad \frac{\partial}{\partial t} \eta(u) + \sum_{i=1}^{2} \frac{\partial}{\partial x_i} q_i(u) \leq 0 \quad \text{in } \mathscr{D}'(R_+^3),$$

for all convex entropy pairs. The basic existence and uniqueness result for the Kružkov L_∞-solution u to the scalar conservation law (1.1), see [8], reads as follows:

If $u_0 \in L_1(R^2) \cap L_\infty(R^2)$, then there exists a unique function $u \in L_\infty(R^3_+)$ which satisfies (2.7) for all entropies η_k of the form $\eta_k(\lambda) = |\lambda - k|$, $k \in R$, together with the initial condition

$$\lim_{t \to 0} \|u(\cdot, t) - u_0\|_{L_1(R^2)} = 0.$$

The following result [1, Theorem 4.2 and Remark 3] gives an extension of the Kružkov uniqueness result to measure-valued solutions.

THEOREM 2.2. *Suppose that $u_0 \in L_1(R^2) \cap L_\infty(R^2)$ and that ν is an admissible mv solution of (1.1a) generated by a uniformly bounded sequence u_j in $L_\infty(R^3_+)$, such that for some constant C*

$$(2.8) \qquad \int_{R^2} \langle \nu_y, |\lambda| \rangle \, dx \le C,$$

$$(2.9) \qquad \lim_{t \to 0^+} \frac{1}{t} \int_0^t \int_{R^2} \langle \nu_y, |\lambda - u_0(x)| \rangle \, dx \, dt = 0.$$

Then ν is the Dirac solution $\nu_y = \delta_{u(y)}$, where u is the unique L_∞-solution of (1.1).

According to Corollary 2.1 we then have $u_j \to u$ strongly in L_p^{loc}, $1 \le p < \infty$.

3. Formulation of the Method and the Main Theorem. In this section we formulate the SC-method and give a basic L_2-stability result and some interpolation estimates, which will be used below.

The SC-method is based on a space-time finite element discretization of R^3_+ defined as follows. Let $0 = t_0 < t_1 < t_2 < \cdots < t_{\tilde{N}} = T$ be a sequence of time levels with $t_{n+1} - t_n \sim h$, set $I_n = (t_n, t_{n+1})$ and introduce the "slabs" $S_n = R^2 \times I_n$ and the sets $R_n^2 = R^2 \times \{t_n\}$. Let $\tilde{T}_h^n$ be a quasi-uniform triangulation of R^2 into triangles $\tilde{K}$ with one right angle (cf. Remark 3.1), smallest angle uniformly bounded away from zero and diameter $h_K \sim h$. For each $\tilde{K} \in \tilde{T}_h^n$, let the prism $\tilde{K} \times I_n$ be divided into three tetrahedrons

$$T(\tilde{K}, 1) = feab, \quad T(\tilde{K}, 2) = decb, \quad T(\tilde{K}, 3) = abce,$$

according to Figure 3.1, and form the corresponding "triangulation" $T_h^n = \{T(\tilde{K}, i), i = 1, 2, 3, \forall \tilde{K} \in \tilde{T}_h^n\}$ of S_n into tetrahedrons.
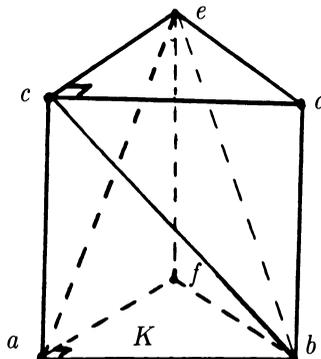

FIGURE 3.1

Remark 3.1. The requirement that the triangles $\tilde{K} \in \tilde{T}_h^n$ have one right angle is technically convenient when proving Lemma 4.2 and Proposition 5.3. We expect these results to hold also for a more general triangulation of $R^2 \times R_+$. In [9] we prove the corresponding results for a general triangulation of $R \times R_+$. $\quad\square$

Define

$$V_h^n = \left\{ v \in H^1(S_n) : v|_K \in P_1(K) \ \forall K \in T_h^n, \ v|_{R_+^3 \setminus \Omega_{\tilde{R}+2,M}} = 0 \right\},$$

$$\Omega_{s,s'} = \{(x,t) \in R_+^3 : |x| < s + ts'\}, \qquad s, s' > 0,$$

$$T_h = \bigcup_{n \geq 0} T_h^n,$$

where $P_1(K)$ denotes the set of linear functions on K, $\tilde{R} = \max\{|x|; x \in \operatorname{supp} u_0\}$ and M is a positive constant to be defined below. In other words, V_h^n consists of continuous piecewise linear functions on the slab S_n which are zero for large $|x|$.

We shall seek an approximate solution U^h in the space $V_h \equiv \prod_{n=0}^{N} V_h^n$, i.e., for $n = 0, 1, 2, \ldots, N$ we will have $U^h|_{S_n} \in V_h^n$. Note that the functions in V_h are continuous in x and possibly discontinuous in t at the discrete time levels t_n. The SC-method for (1.1) can now be formulated: Find $U^h \in V_h$ such that for $n = 0, 1, 2, \ldots$

$$(3.1) \quad \begin{aligned} &\int_{S_n} L(U^h) \left(v + \delta \left(v_t + \sum_{i=1}^{2} f_i'(U^h) v_{x_i} \right) \right) dx\, dt + \int_{R_n^2} (U_+^h - U_-^h) v_+ \, dx \\ &\quad + \int_{S_n} \varepsilon_1(U^h) \nabla U^h \cdot \nabla v \, dx\, dt \\ &\quad + \int_{S_n} \varepsilon_2(U^h) \nabla_x U^h \cdot \nabla_x v \, dx\, dt = 0 \quad \forall v \in V_h, \end{aligned}$$

where

$$L(U^h) = U_t^h + \sum_{i=1}^{2} (f_i(U^h))_{x_i},$$

$$\varepsilon_1(U^h) = \overline{\delta} |L(U^h)| (1 + |f_1'(U^h)| + |f_2'(U^h)|),$$

$$\varepsilon_2(U^h) = \overline{\overline{\delta}} |\tilde{U}|,$$

$$v_\pm(\cdot, t) = \lim_{s \to 0^\pm} v(\cdot, t + s),$$

$$\tilde{U}(x,t)|_K = \begin{cases} (U_+^h - U_-^h)(x)|_{K \cap R_n^2} & \text{if } \int_{\tilde{K} \cap R_n^2} dx > 0 \text{ for } K \in T_h^n, \\ 0 & \text{otherwise,} \end{cases}$$

$$U_-(\cdot, 0) = u_0(\cdot),$$

$$\nabla_x w \cdot \nabla_x v = \sum_{i=1}^{2} v_{x_i} w_{x_i},$$

$$\nabla w \cdot \nabla v = w_t v_t + \nabla_x w \cdot \nabla_x v,$$

and the positive parameters $\delta, \overline{\delta}, \overline{\overline{\delta}}$ satisfy

$$(3.2) \quad \begin{aligned} h/\delta \to 0 \text{ and } \delta/h^{1-\varepsilon} \to 0 &\quad \text{as } h \to 0, \ \forall \varepsilon > 0, \\ \overline{\delta} = Ch^{\alpha_1}, &\qquad \overline{\overline{\delta}} = Ch^{\alpha_2}, \end{aligned}$$

where $\frac{3}{2} < \alpha_1 < 2$, $\frac{1}{2} < \alpha_2 < 1$.

Existence of a solution to (3.1) follows from a variant of Brouwer's fixed point theorem as in [5]. From now on, $U = U^h$ will denote a solution of (3.1). Our main result is the following

THEOREM 3.1. *The solutions U^h of (3.1) converge strongly in $L_p^{\text{loc}}(R_+^3)$ for $1 \leq p < \infty$ to the unique L_∞-solution of (1.1) as h tends to zero.*

The proof is divided into three steps: Lemmas 3.1–3.3.

LEMMA 3.1. *There is a constant C such that the solutions U^h of (3.1) satisfy*

$$\|U^h\|_{L_\infty(R_+^3)} \leq C, \qquad 0 < h < 1.$$

LEMMA 3.2. *There is a subsequence of the solutions U^h of (3.1) that generates an admissible mv solution ν of (1.1).*

LEMMA 3.3. *There is a constant C such that the mv solution ν in Lemma 3.2 satisfies*

$$\int_{R^2} \langle \nu_y, |\lambda| \rangle \, dx \leq C$$

and the initial condition

$$\lim_{t \to 0^+} \frac{1}{t} \int_0^t \langle \nu_y, |\lambda - u_0(x)| \rangle \, dx \, dt = 0.$$

Theorem 3.1 now directly follows by combining Lemmas 3.1–3.3, Theorem 2.2 and Corollary 2.1.

Next, we give the following basic L_2-stability result obtained by taking $v = U^h$ in (3.1):

$$
\int_{R_{N+1}^2} (U_-^h)^2 \, dx + \sum_{n=0}^{N} \int_{R_n^2} (U_+^h - U_-^h)^2 \, dx + 2\delta \int_{S^N} (L(U^h))^2 \, dx \, dt
$$

(3.3)
$$
+ 2 \int_{S^N} \varepsilon_1(U^h) |\nabla U^h|^2 \, dx \, dt + 2 \int_{S^N} \varepsilon_2(U^h) |\nabla_x U^h|^2 \, dx \, dt
$$

$$
\leq \int_{R^2} (u_0)^2 \, dx,
$$

where $S^N = \bigcup_{n=0}^{N} S_n$ and integrals over S^N are interpreted as a sum of integrals over the S_n.

We shall need the following standard interpolation error estimate (3.4), "superapproximation" result (3.5) and inverse estimate (3.6), where $\pi w \in V_h$ is the usual piecewise linear interpolant of a function $w \in \Pi_{n \geq 0} \mathscr{C}(S_n)$. A proof of the superapproximation result is given in [7].

LEMMA 3.4. *There are constants C such that for $w \in W^{s,p}(\omega) \cap \mathscr{C}(S_n)$, $v \in V_h$, $n = 0, 1, 2, \ldots,$*

(3.4a) $\|w - \pi w\|_{W^{k,\infty}(\omega)} \leq Ch^{s-k} \|w\|_{\dot{W}^{s,\infty}(\omega)}, \qquad s = 1, 2, \ k = 0, 1, \ p = \infty,$

(3.4b) $\|w - \pi w\|_{H^k(\omega)} \leq Ch^{2-k} \|w\|_{\dot{H}^2(\omega)}, \qquad k = 0, 1, \ p = 2,$

(3.5a) $\begin{aligned} &\|vw - \pi(vw)\|_{W^{k,\infty}(\omega)} \\ &\qquad \leq Ch^{1-k} \|v\|_{L_\infty(\omega)} \|w\|_{\dot{W}^{1,\infty}(\omega)}, \qquad k = 0, 1, \ p = \infty, \end{aligned}$

(3.5b)
$$\|vw - \pi(vw)\|_{H^k(\omega)}$$
$$\leq Ch^{1-k}\|v\|_{L_\infty(\omega)}(\|w\|_{H^1(\omega)} + h\|w\|_{\dot{H}^2(\omega)}), \qquad k = 0, 1, \ p = 2,$$

(3.5c)
$$\|vw - \pi(vw)\|_{L_2(R_n^2)}$$
$$\leq Ch^{1/2}\|v\|_{L_\infty(S_n)}(\|w\|_{H^1(S_n)} + h\|w\|_{\dot{H}^2(S_n)}), \qquad p = 2,$$

(3.6a)
$$\|\nabla v\|_{L_p(\omega)} \leq Ch^{-1}\|v\|_{L_p(\omega)}, \qquad 1 \leq p \leq \infty,$$

(3.6b)
$$\|v\|_{L_\infty(S_n)} \leq Ch^{-3/2}\|v\|_{L_2(S_n)}, \qquad p = 2,$$

where $\omega = R_n^2, S_n, K \cap R_n^2$ *or* $K \cap S_n$ *for* $K \in T_h$, *and* $W^{s,p}(\omega)$ *is the usual Sobolev space (here dot denotes seminorm and* $W^{s,2} = H^2$*).*

4. Proof of Lemma 3.1 (The L_∞-Estimate). In order to prove the L_∞-estimate of Lemma 3.1, we shall need the following two preliminary lemmas.

LEMMA 4.1. *There are positive constants c and C independent of p such that for $p = 2m$, $m = 1, 2, 3, \ldots$, and $n = 0, 1, 2, \ldots$*

$$\frac{ch}{p^2} \sum_{K \in T_h^n} \int_{K \cap R_n^2} |U_+ - U_-| |\nabla_x U_+|^2 \|U_+\|_{L_\infty(K \cap R_n^2)}^{p-2} \, dx$$

$$\leq \int_{S_n} |\tilde{U}| \nabla_x U \cdot \nabla_x(\pi(U^{p-1})) \, dx \, dt,$$

$$ch \int_{R_n^2} |U_+ - U_-| \nabla_x U_+ \cdot \nabla_x v_+ \, dx \leq \int_{S_n} |\tilde{U}| |U_x U \cdot \nabla_x v| \, dx \, dt$$

$$\leq Ch \int_{R_n^2} |U_+ - U_-| |\nabla_x U_+ \cdot \nabla_x v_+| \, dx \quad \forall v \in V_h.$$

The proof of Lemma 4.1 is a simple combination of the proofs of Lemmas 4.1–4.2 in [7].

LEMMA 4.2. *There is a constant $c > 0$ independent of p such that for $p = 2m$, $m = 1, 2, 3, \ldots$, and $n = 0, 1, 2, \ldots$*

$$\int_{S_n} \varepsilon_1(U) \nabla U \cdot \nabla \pi(U^{p-1}) \, dx \, dt \geq \frac{c}{p^2} \sum_{K \in T_h^n} \int_K \varepsilon_1(U) |\nabla U|^2 \|U\|_{L_\infty(K)}^{p-2} \, dx \, dt.$$

Proof. Consider a tetrahedron $K \in T_h^n$. Since

$$\nabla U \cdot \nabla \pi(U^{p-1})|_K \quad \text{and} \quad |\nabla U|^2 \|U\|_{L_\infty(K)}^{p-2}$$

are constant on K, it is sufficient to prove that

$$\nabla U \cdot \nabla \pi(U^{p-1}) \geq \frac{c}{p^2} |\nabla U|^2 \|U\|_{L_\infty(K)}^{p-2}.$$

Define the function $f_p \colon S^3 \backslash \{\pm \frac{1}{2}(1, 1, 1, 1)\} \to R$ by

$f_p(y_1, y_2, y_3, y_4)$
$$= \frac{(y_1 - y_2)(y_1^{p-1} - y_2^{p-1}) + (y_1 - y_3)(y_1^{p-1} - y_3^{p-1}) + (y_3 - y_4)(y_3^{p-1} - y_4^{p-1})}{((y_1 - y_2)^2 + (y_1 - y_3)^2 + (y_3 - y_4)^2) \max(y_1^{p-2}, y_2^{p-2}, y_3^{p-2}, y_4^{p-2})}$$
$$= \frac{(y_1 - y_2)^2 \sum_{i=0}^{p-2} y_2^{p-2-i} y_1^i + (y_1 - y_3)^2 \sum_{i=0}^{p-2} y_3^{p-2-i} y_1^i + (y_3 - y_4)^2 \sum_{i=0}^{p-2} y_4^{p-2-i} y_3^i}{((y_1 - y_2)^2 + (y_1 - y_3)^2 + (y_3 - y_4)^2) \max(y_1^{p-2}, y_2^{p-2}, y_3^{p-2}, y_4^{p-2})}.$$

We note that f_p is continuous on S^3, and after an appropriate choice of coordinate directions we get by the definition of f_p

$$\nabla U \cdot \nabla \pi(U^{p-1}) \geq \inf_{y \in S^3} f_p(y) |\nabla U|^2 \|U\|_{L_\infty(K)}^{p-2}.$$

We have

$$f_p(y) \geq \frac{1}{4} \frac{(y_1 - y_2)^2(y_1^{p-2} + y_2^{p-2}) + (y_1 - y_3)^2(y_1^{p-2} + y_3^{p-2}) + (y_3 - y_4)^2(y_3^{p-2} + y_4^{p-2})}{((y_1 - y_2)^2 + (y_1 - y_3)^2 + (y_3 - y_4)^2) \max(y_1^{p-2}, y_2^{p-2}, y_3^{p-2}, y_4^{p-2})}.$$

Let us first assume that $|y_1|^{p-2} = \max_{1 \leq i \leq 4}(y_i^{p-2})$. If $(y_1 - y_2)^2 + (y_1 - y_3)^2 \geq 1/p^2$, then we clearly have

(4.1) $f_p(y) > c/p^2.$

In the case $(y_1 - y_2)^2 + (y_1 - y_3)^2 < 1/p^2$, we have

(4.2) $\left(\frac{y_i}{y_1}\right)^{p-2} = \left(1 - \frac{y_1 - y_i}{y_1}\right)^{p-2} > \exp\left(-\frac{2(p-2)}{p|y_1|}\right) > c, \qquad i = 2, 3,$

so that

$$f_p(y) \geq \frac{1}{4}\left(\frac{y_2}{y_1}\right)^{p-2} \frac{(y_1 - y_2)^2}{(y_1 - y_2)^2 + (y_1 - y_3)^2 + (y_3 - y_4)^2}$$

$$+ \frac{1}{4}\left(\frac{y_3}{y_1}\right)^{p-2} \frac{(y_1 - y_3)^2 + (y_3 - y_4)^2}{(y_1 - y_2)^2 + (y_1 - y_3)^2 + (y_3 - y_4)^2} > c,$$

which proves (4.1).

Next, we consider the case $|y_2|^{p-2} = \max_{1 \leq i \leq 4}(|y_i|^{p-2})$. If $|y_1 - y_2| \geq 1/p$, then (4.1) follows directly, while if $|y_1 - y_2| < 1/p$, then we have as in (4.2)

$$f_p(y) \geq \frac{1}{4} \frac{(y_1/y_2)^{p-2}((y_1 - y_2)^2 + (y_1 - y_3)^2) + (y_3/y_2)^{p-2}(y_3 - y_4)^2}{(y_1 - y_2)^2 + (y_1 - y_3)^2 + (y_3 - y_4)^2}$$

$$\geq \frac{1}{4} \frac{c((y_1 - y_2)^2 + (y_1 - y_3)^2) + (y_3/y_2)^{p-2}(y_3 - y_4)^2}{(y_1 - y_2)^2 + (y_1 - y_3)^2 + (y_3 - y_4)^2}.$$

If now $|y_1 - y_3| \geq 1/p$, then (4.1) holds, and finally, if $|y_1 - y_3| < 1/p$, then we have $|y_3 - y_2| < |y_1 - y_2| + |y_1 - y_3| < 2/p$, and as in (4.2) we obtain (4.1). By symmetry, this proves (4.1) for all $y \in S^3$. $\square$

Taking now $v = \pi(U^{p-1})$ in (3.1), where p is an even integer greater than 2, we get

$$0 = \int_{S_n} L(U)U^{p-1}\, dx\, dt + \int_{R_n^2} (U_+ - U_-)(U_+)^{p-1}\, dx$$

$$- \int_{S_n} L(U)(U^{p-1} - \pi(U^{p-1}))\, dx\, dt - \int_{R_n^2} (U_+ - U_-)(U_+^{p-1} - \pi(U_+^{p-1}))\, dx$$

$$+ \delta \int_{S_n} L(U)\left((U^{p-1})_t + \sum_i f_i'(U)(U^{p-1})_{x_i}\right) dx\, dt$$

$$- \delta \int_{S_n} L(U)\left((U^{p-1})_t - \pi(U^{p-1})_t + \sum_i f_i'(U)((U^{p-1})_{x_i} - \pi(U^{p-1})_{x_i})\right) dx\, dt$$

$$+ \int_{S_n} \varepsilon_1(U)\nabla U \cdot \nabla\pi(U^{p-1})\, dx\, dt$$

$$+ \int_{S_n} \varepsilon_2(U)\nabla_x U \cdot \nabla_x \pi(U^{p-1})\, dx\, dt \equiv \sum_{i=1}^{8} E_n^i.$$

Using the standard interpolation estimates (3.4) and the stability estimate (3.3), we have

$$|E_n^3| + |E_n^6| \le Cp^2 h(h+\delta) \sum_{K \in T_h^n} \int_K |L(U)| \left(1 + \sum_i |f_i'(U)|\right) |\nabla U|^2 \|U\|_{L_\infty(K)}^{p-3}$$

$$\le Cp^2 \frac{h\delta}{\overline{\overline{\delta}}} \sum_{K \in T_h^n} \int_{K \cap \{|U|>1\}} \varepsilon_1(U) |\nabla U|^2 \|U\|_{L_\infty(K)}^{p-2} \, dx \, dt$$

$$+ Cp^2 \frac{h\delta}{\overline{\overline{\delta}}} \int_{S_n} \varepsilon_1(U) |\nabla U|^2 \, dx \, dt = \mathrm{I}_n + \mathrm{II}_n,$$

where by Lemma 4.2

$$|\mathrm{I}_n| \le \frac{Cp^4 h\delta}{\overline{\overline{\delta}}} \int_{S_n} \varepsilon_1(U) \nabla U \cdot \nabla \pi(U^{p-1}) \, dx \, dt,$$

and by (3.3)

$$\sum_{n=0}^N |\mathrm{II}_n| \le Cp^2 h\delta/\overline{\delta}.$$

Further, using (3.3) and Lemma 4.1, we have

$$|E_n^4| \le Cp^2 h^2 \sum_{K \in T_h^n} \int_{K \cap R_n^2} |U_+ - U_-| |\nabla_x U|^2 \|U\|_{L_\infty(K \cap R_n^2)}^{p-3} \, dx$$

$$\le Cp^2 h^2 \sum_{K \in T_h^n} \int_{K \cap \{|U|>1\}} |U_+ - U_-| |\nabla_x U|^2 \|U\|_{L_\infty(K \cap R_n^2)}^{p-2} \, dx$$

$$+ Cp^2 h^2 \int_{R_n^2 \cap \{|U|<1\}} |U_+ - U_-| |\nabla_x U|^2 \, dx$$

$$\le Cp^4 h \int_{S_n} |\tilde{U}| \nabla_x U \cdot \nabla_x \pi(U^{p-1}) \, dx \, dt + Cp^2 h \int_{S_n} |\tilde{U}| |\nabla_x U|^2 \, dx \, dt.$$

With the aid of these estimates we get by summation over $n = 0, 1, 2, \ldots, N$ for $p^4 \le C \min(\overline{\delta}/(h\delta), \overline{\overline{\delta}}/h)$

$$\sum_{n=0}^N \int_{R_{n+1}^2} (U_-)^p \, dx - \int_{R_n^2} (U_+)^p \, dx - \int_{R_n^2} (U_- - U_+) p (U_+)^{p-1} \, dx$$

$$+ \delta p(p-1) \int_{S^N} (L(U))^2 U^{p-2} \, dx \, dt \le Cp^3 \left(\frac{h\delta}{\overline{\delta}} + \frac{h}{\overline{\overline{\delta}}}\right).$$

Using now the convexity of the function $U \to U^p$, we have

$$\|U_-\|_{L_p(R_{N+1}^2)}^p + \delta p(p-1) \int_{S^N} (L(U))^2 U^{p-2} \, dx \, dt$$

$$\le \|u_0\|_{L_p(R^2)}^p + Cp^3(h/\overline{\overline{\delta}} + h\delta/\overline{\delta}).$$

The next step is to obtain L_p-estimates for all $t \in (0, \infty)$. As in [7], we have for $t_n < t \leq t_{n+1}$

$$\|U(\cdot, t)\|_{L_p(R^2)}^p = \|U_-\|_{L_p(R_{n+1}^2)}^p - p \int_t^{t_{n+1}} \int_{R^2} \left(U_t U^{p-1} + \sum_i f_i' U^{p-1} U_{x_i} \right) dx\, dt$$

$$\leq \|U_-\|_{L_p(R_{n+1}^2)}^p + p \left(\int_{S_n} (L(U))^2 U^{p-2}\, dx\, dt \int_t^{t_{n+1}} \int_{R^2} U^p\, dx\, dt \right)^{1/2}$$

$$\leq \|U_-\|_{L_p(R_{n+1}^2)}^p + \delta p(p-1) \int_{S_n} (L(U))^2 U^{p-2}\, dx\, dt$$

$$+ \frac{p}{4\delta(p-1)} \int_t^{t_{n+1}} \|U(\cdot, s)\|_{L_p(R^2)}^p\, ds.$$

Thus, by Gronwall's inequality we obtain for $t_N \leq t \leq t_{N+1}$

$$(4.3) \qquad \|U(\cdot, t)\|_{L_p(R^2)}^p \leq e^{Ch/\delta} \|u_0\|_{L_p(R^2)}^p + Cp^3(h/\overline{\overline{\delta}} + h\delta/\overline{\delta}).$$

This proves by (3.2) the existence of positive constants c and α_0, independent of p and h, such that

$$(4.4) \qquad \sup_{t \geq 0} \|U(\cdot, t)\|_{L_p(R^2)} \leq C \quad \text{if } 4 \leq p \leq ch^{-\alpha_0}.$$

Further, there is a constant C, independent of q and h, such that all $v \in V_h$ satisfy the inverse estimate

$$(4.5) \qquad \|v_\pm\|_{L_\infty(R_n^2)} \leq (Cqh^{-1})^{2/q} \|v_\pm\|_{L_q(R_n^2)}, \qquad 1 \leq q < \infty,$$

which is proved analogously to Lemma 4.1. Finally, using (4.4)–(4.5), we have

$$\|U^h\|_{L_\infty(R_+^3)} = \|U\|_{L_\infty(R_+^3)} \leq C(ph^{-1})^{2/p} \sup_{t \geq 0} \|U(\cdot, t)\|_{L_p(R^2)}$$

$$\leq C \exp(c(1 + \alpha_0)h^{\alpha_0} \ln 1/h) \leq C,$$

for h sufficiently small, which completes the proof of Lemma 3.1.

5. Proof of Lemma 3.2 (The Entropy Condition).

To prove Lemma 3.2, we first note that by Lemma 3.1 the solutions U^h of (3.1) are uniformly bounded in the L_∞-norm, so that by Theorem 2.1 there exists a subsequence $\{U^h\}$ which converges in the weak-star topology in $L_\infty(R_+^3)$, and the limit can be represented by a family of probability measures ν_y such that for all continuous state variables g

$$(5.1) \qquad g(U^h(y)) \rightharpoonup \langle \nu_y, g(\lambda) \rangle.$$

Our next step is to prove that the Young measure ν is an admissible mv solution, i.e., that ν satisfies in the distribution sense

$$(5.2) \qquad \frac{\partial}{\partial t} \langle \nu_y, \lambda \rangle + \sum_{i=1}^{2} \frac{\partial}{\partial x_i} \langle \nu_y, f_i(\lambda) \rangle = 0,$$

and for all convex entropy pairs (η, q)

$$(5.3) \qquad \frac{\partial}{\partial t} \langle \nu_y, \eta(\lambda) \rangle + \sum_{i=1}^{2} \frac{\partial}{\partial x_i} \langle \nu_y, q_i(\lambda) \rangle \leq 0.$$

To prove (5.3), let (η, q_1, q_2) be a smooth convex entropy pair. Let $\Phi \in \mathscr{C}_0^\infty(R^2 \times R_+)$, $\Phi \geq 0$, and partition Φ as follows:

$$\Phi = \phi + \varphi,$$

where $\phi, \varphi \geq 0$, $\phi \in \mathscr{C}_0^\infty(\Omega_{\tilde{R}+2,M})$ and $\varphi \in \mathscr{C}_0^\infty(R^2 \times R_+ \backslash \Omega_{\tilde{R}+1,M})$. Taking $v = \pi(\eta'(U)\phi)$ in (3.1), we get

$$
\begin{aligned}
0 = &\int_{S_n} L(U)\eta'\phi \, dx \, dt + \int_{R_n^2} (U_+ - U_-)\eta'_+ \phi \, dx \\
&- \int_{S_n} L(U)(\eta'\phi - \pi(\eta'\phi)) \, dx \, dt - \int_{R_n^2} (U_+ - U_-)(\eta'\phi - \pi(\eta'\phi))_+ \, dx \\
&+ \delta \int_{S_n} L(U)\left((\eta'\phi)_t + \sum_i f_i'(\eta'\phi)_{x_i}\right) dx \, dt \\
&- \delta \int_{S_n} L(U)\left((\eta'\phi)_t - \pi(\eta'\phi)_t + \sum_i f_i'(\eta'\phi - \pi(\eta'\phi))_{x_i}\right) dx \, dt \\
&+ \int_{S_n} \varepsilon_1(U)\nabla U \cdot \nabla\pi(\eta'\phi) \, dx \, dt + \int_{S_n} \varepsilon_2(U)\nabla_x U \cdot \nabla_x \pi(\eta'\phi) \, dx \, dt \equiv \sum_{i=1}^8 E_n^i.
\end{aligned}
$$

Integrating by parts and summing over n, we have

$$
\begin{aligned}
(5.4) \quad &-\int_{S^N} \left(\eta(U)\phi_t + \sum_i q_i(U)\phi_{x_i}\right) dx \, dt \\
&+ \sum_{n=0}^N \left(\int_{R_{n+1}^2} \eta(U_-)\phi \, dx - \int_{R_n^2} (\eta(U_+) - (U_+ - U_-)\eta'(U_+))\phi \, dx\right) \\
&= -\sum_{n=0}^N \sum_{i=3}^8 E_n^i \equiv -\sum_{i=3}^8 R^i.
\end{aligned}
$$

Using now the convexity of η, we see that the sum of the integrals over R_n^2, R_{n+1}^2 is nonnegative.

PROPOSITION 5.1. *There holds*

$$\liminf_{h \to 0} \sum_{i=3}^8 R^i \geq 0.$$

PROPOSITION 5.2. *There are positive constants c, C such that*

$$\|U\|_{L_\infty(R^2 \times R_+ \backslash \Omega_{\tilde{R}+1,M})} \leq Ce^{-c/(h+\delta+\bar{\bar{\delta}}+\bar{\delta}/\sqrt{\delta})},$$

where $\tilde{R} = \{|x| : x \in \operatorname{supp} u_0\}$ and M is a positive constant to be defined below.

We postpone the proofs of these results to the end of this section. From (5.4) we get

$$-\int_{S^N}\left(\eta(U)\Phi_t + \sum_i q_i(U)\Phi_{x_i}\right) dx\, dt$$

(5.5)
$$\leq -\sum_{i=3}^{8} R^i - \int_{S^N}\left(\eta(U)\varphi_t + \sum_i q_i(U)\varphi_{x_i}\right) dx\, dt$$

$$= -\sum_{i=3}^{8} R^i - \int_{S^N}\left(\eta(0)\varphi_t + \sum_i q_i(0)\varphi_{x_i}\right) dx\, dt$$

$$+ \int_{S^N}\left((\eta(0)-\eta(U))\varphi_t + \sum_i(q_i(0)-q_i(U))\varphi_{x_i}\right) dx\, dt,$$

where the first integral on the right-hand side is zero and the second tends to zero as $h \to 0$ by Proposition 5.2. Letting $h \to 0$ in (5.5), using (5.1) and Proposition 5.1, we now obtain (5.3).

In order to treat all convex continuous entropies η, we observe that a standard regularization $\eta^\varepsilon = \eta * \omega_\varepsilon$ maintains the convexity, and η^ε, q_i^ε tend to η, q_i uniformly. Here, ω satisfies the following conditions:

$$\omega \in \mathscr{C}_0^\infty((-1,1)), \quad \omega \geq 0, \quad \int_R \omega\, dy = 1, \quad \omega_\varepsilon(y) = \varepsilon^{-1}\omega\left(\frac{y}{\varepsilon}\right).$$

By dominated convergence applied to (5.3) with $\eta^\varepsilon \to \eta$ we then obtain (5.3) in the general case. Next, by taking $\eta(\lambda) = \pm\lambda$ in (5.3) we get (5.2), thus proving that ν is an admissible mv solution.

It remains to prove the propositions. We shall estimate the R^i using the estimates (3.3)–(3.6) and the L_∞-estimate in Lemma 3.1. We have

$$|R^3| + |R^6| \leq Ch(h+\delta)\sum_{K \in T_h}\int_K |L(U)|\, \|\eta'\phi\|_{\dot{W}^{2,\infty}}\, dx\, dt$$

$$\leq Ch(h+\delta)\sum_{K \in T_h}\int_K |L(U)|(|\nabla U|^2 + |\nabla U| + 1)\|\phi\|_{W^{2,\infty}(K)}\, dx\, dt$$

$$\leq C\frac{h\delta}{\overline{\overline{\delta}}}\int_{S^N}\varepsilon_1(U)|\nabla U|^2 + Ch(h+\delta)\|L(U)\|_{L_2(S^N)}$$

$$\leq C\left(\frac{h\delta}{\overline{\overline{\delta}}} + h\sqrt{\delta}\right).$$

Using also Lemma 4.1, we have

$$|R^4| \leq \sum_{n=0}^{N}\sum_{K \in T_h^n} Ch^2\int_{K \cap R_n^2} |U_+ - U_-|(|\nabla_x U|^2 + |\nabla_x U| + 1)\|\phi\|_{W^{2,\infty}(K)}\, dx$$

$$\leq Ch\int_{S^N}|\tilde{U}|\,|\nabla_x U|^2\, dx\, dt + Ch^{3/2}\left(\sum_{n=0}^{N}\|U_+ - U_-\|_{L^2(R_n^2)}^2\right)^{1/2}$$

$$\leq C\left(\frac{h}{\overline{\overline{\delta}}} + h^{3/2}\right).$$

Further,

$$R^5 = \delta \int_{S^N} L(U)^2 \eta''(U)\phi\, dx\, dt + \delta \int_{S^N} L(U)\left(\phi_t + \sum_i f_i' \phi_{x_i}\right)\eta'(U)\, dx\, dt$$

$$\geq -C\delta\|L(U)\|_{L_2(S^N)} \geq -C\sqrt{\delta}.$$

To estimate R^7 and R^8, we shall use the following result.

PROPOSITION 5.3. *We have*

$$U_t(\pi(\eta'(U)))_t \geq 0, \quad \nabla_x U \cdot \nabla_x(\pi(\eta'(U))) \geq 0.$$

Proof. We see that $U_t(\pi(\eta'(U)))_t$ is constant on each tetrahedron $K \in T_h$. According to the construction of the tetrahedrons K, there is for each K an orthogonal coordinate transformation in the x_1, x_2-plane such that K always has one edge in each coordinate direction. Hence, let (x_1, x_2, t), $t_1' \leq t \leq t_2'$, be such an edge of K; then the sign of $U_t \pi(\eta'(U))_t$ is equal to the sign of $(U(x_1, x_2, t_1') - U(x_1, x_2, t_2'))(\eta'(U(x_1, x_2, t_1')) - \eta'(U(x_1, x_2, t_2')))$, which is nonnegative since η' is nondecreasing. An analogous argument in the x_i directions will then, after summation, complete the proof of Proposition 5.3, since $\nabla_x v \cdot \nabla_x w$ is invariant under orthogonal coordinate transformations. $\square$

Let $\mathscr{P}\varphi|_K \in P_0(K)\ \forall K \in T_h$ be defined by

$$(5.6) \qquad \mathscr{P}\varphi|_K = \int_K \varphi\, dx\, dt \Big/ \left(\int_K dx\, dt\right) \quad \forall \varphi \in L_2(R_+^3),$$

i.e., $\mathscr{P}\varphi$ is the L_2-projection of φ onto the set of piecewise constants. Then using (5.6), we have

$$R^7 = \sum_{\substack{K \in T_h \\ K \subset S^N}} \int_K \varepsilon_1(U)\nabla U \cdot \nabla(\pi(\eta'\phi) - \pi(\eta'\mathscr{P}\phi))\, dx\, dt$$

$$+ \sum_{\substack{K \in T_h \\ K \subset S^N}} \int_K \varepsilon_1(U)\nabla U \cdot \nabla\pi(\eta'\mathscr{P}\phi)\, dx\, dt \equiv \mathrm{I} + \mathrm{II},$$

$$|\mathrm{I}| \leq C \sum_{\substack{K \in T_h \\ K \subset S^N}} \int_K \varepsilon_1(U)\|\nabla(\eta'(\phi - \mathscr{P}\phi))\|_{L_\infty(K)}|\nabla U|\, dx\, dt$$

$$\leq C \sum_{\substack{K \in T_h \\ K \subset S^N}} \int_K \varepsilon_1(U)(h|\nabla U|^2 + |\nabla U|)\|\nabla\phi\|_{L_\infty(K)}\, dx\, dt$$

$$\leq Ch\int_{S^N} \varepsilon_1(U)|\nabla U|^2\, dx\, dt + C\left(\int_{S^N} \varepsilon_1(U)|\nabla U|^2\, dx\, dt\|\varepsilon_1(U)\|_{L_\infty(S^N)}\right)^{1/2}$$

$$\leq C(h + \sqrt{\overline{\delta}/h}),$$

where we have used that

$$\|(\mathrm{I} - \mathscr{P})\phi\|_{L_\infty(K)} \leq Ch\|\nabla\phi\|_{L_\infty(K)}$$

and the inverse estimate (3.6) to obtain

$$\|\varepsilon_1(U)\|_{L_\infty(R_+^3)} \leq C\overline{\delta}/h.$$

Next, by Proposition 5.3, and since $\mathscr{P}\phi$ is piecewise constant, we have

$$\mathrm{II} = \sum_{\substack{K \in T_n \\ K \subset S^N}} \int_K \varepsilon_1(U) \nabla U \cdot \nabla(\pi(\eta')) \mathscr{P}\phi \, dx \, dt \geq 0,$$

so that

$$R^7 \geq -C(h + \sqrt{\overline{\delta}}/h).$$

Finally,

$$R^8 = \sum_{\substack{K \in T_h \\ K \subset S^N}} \int_K \varepsilon_2(U) \nabla_x U \cdot \nabla_x(\pi(\eta'\phi) - \pi(\eta'\mathscr{P}\phi)) \, dx \, dt$$

$$+ \sum_{\substack{K \in T_n \\ K \subset S^N}} \int_K \varepsilon_2(U) \nabla_x U \cdot \nabla_x \pi(\eta'\mathscr{P}\phi) \, dx \, dt \equiv \mathrm{III} + \mathrm{IV}.$$

As above, we get by Lemma 4.1

$$|\mathrm{III}| \leq Ch\overline{\overline{\delta}} \sum_{n=0}^N \int_{R_n^2} |\tilde{U}| \nabla_x U \cdot \nabla_x(\pi(\eta'(\phi - \mathscr{P}\phi))) \, dx$$

$$\leq Ch\overline{\overline{\delta}} \sum_{\substack{K \in T_n \\ K \subset S^N}} \sum_{n=0}^N \int_K |\tilde{U}| \, |\nabla_x U| \, \|\nabla_x(\eta'(\phi - \mathscr{P}\phi))\|_{L_\infty(K \cap R_n^2)} \, dx$$

$$\leq C\overline{\overline{\delta}} \sum_{\substack{K \in T_n \\ K \subset S^N}} \int_K |\tilde{U}| \, |\nabla_x U| \, \|\nabla_x(\eta'(\phi - \mathscr{P}\phi))\|_{L_\infty(K)} \, dx \, dt$$

$$\leq C\overline{\overline{\delta}} \sum_{\substack{K \in T_n \\ K \subset S^N}} \int_K |\tilde{U}| (h|\nabla_x U|^2 + |\nabla_x U|) \|\nabla_x \phi\|_{L_\infty(K)} \, dx \, dt$$

$$\leq Ch \int_{S^N} \varepsilon_2(U)|\nabla_x U|^2 \, dx \, dt + C \left(\int_{S^N} \varepsilon_2(U)|\nabla_x U|^2 \, dx \, dt \|\varepsilon_2(U)\|_{L_\infty} \right)^{1/2}$$

$$\leq C(h + \sqrt{\overline{\overline{\delta}}}).$$

Finally, by Proposition 5.3,

$$\mathrm{IV} = \sum_{\substack{K \in T_h \\ K \subset S^N}} \int_K \varepsilon_2(U) \nabla_x U \cdot \nabla_x(\pi(\eta'(U))) \mathscr{P}\phi \, dx \, dt \geq 0.$$

Letting now $h \to 0$ in the above estimates, we obtain Proposition 5.1. □

Proof of Proposition 5.2. Let us introduce the cutoff function

$$\psi(x,t) = \begin{cases} 1 & \text{if } \beta(\tilde{R} + 1 - |x|) + t < 0, \\ \exp(-(\beta(\tilde{R} + 1 - |x|) + t)/\tau) & \text{otherwise,} \end{cases}$$

where

$$\frac{1}{\beta} = 8 \sup_{\substack{|w| \leq \|U^h\|_{L_\infty} \\ h > 0}} (|f_1'(w)|, |f_2'(w)|)$$

and

$$\tau = C(h + \delta + \overline{\overline{\delta}} + \overline{\delta}/h),$$

where C is a sufficiently large constant. We now take $M = 1/\beta$ in the definition of V_h^n. Further, we note that $\psi|_{R_+^3 \setminus \Omega_{\tilde{R}+1,M}} = 1$ and ψ is exponentially decreasing in $\Omega_{\tilde{R}+1,M}$. We then have for $U = U^h$ and $N = 1, 2, 3, \ldots$ the following local stability result:

$$(5.7) \qquad \begin{aligned} &\frac{1}{2} \left[\int_{R_{N+1}^2} U_-^2 \psi \, dx + \sum_{n=0}^{N} \int_{R_n^2} (U_+ - U_-)^2 \psi \, dx \right] \\ &\quad + \delta \int_{S^N} (L(U))^2 \psi \, dx \, dt + \int_{S^N} \varepsilon_1(U) |\nabla U|^2 \psi \, dx \, dt \\ &\quad + \int_{S^N} \varepsilon_2(U) |\nabla_x U|^2 \psi \, dx \, dt \le C \int u_0^2 \psi \, dx. \end{aligned}$$

A similar stability estimate is proven in Theorem 1.2 of [4]. Since $u_0 \in L_\infty(R^2)$ and $\operatorname{supp} u_0 \subset \{x : |x| \le \tilde{R}\}$, we obtain from (5.7)

$$\int_{R_N^2} U_\pm^2 \psi \, dx \le C\tau e^{-\beta/\tau},$$

so that by the inverse estimate (3.6b),

$$\|U\|_{L_\infty((R_+^3 \setminus \Omega_{\tilde{R}+1,1/\beta}) \cap \{t < T\})} \le C \frac{\sqrt{\tau}}{h} e^{-\beta/\tau},$$

which is the desired result. $\square$

6. Proof of Lemma 3.3 (The Initial Condition). Here we prove that the L_1-stability (2.8) and the initial condition (2.9) are satisfied. First we note by the definition of V_h that $\langle \nu_{(x,t)}, |\lambda| \rangle$, i.e., the L_∞ weak-star limit of $|U^h(x,t)|$, has support in $\Omega_{\tilde{R}+2,M}$ (i.e., compact support in x for fixed t). Next, by the following L_2-stability,

$$(6.1) \qquad \|U^h(\cdot, t)\|_{L_2(R^2)} \le \exp(Ch/\delta) \|u_0\|_{L_2(R^2)},$$

which is obtained from the stability estimate (3.3) by a Gronwall inequality as in (4.3), we get

$$(6.2) \qquad \|U^h(\cdot, t)\|_{L_1(R^2)} \le C \|u_0\|_{L_2(R^2)}, \qquad t < T.$$

Now using that $g(U^h(x,t)) \rightharpoonup \langle \nu_{(\overline{x},t)}, g(\lambda) \rangle$ in the L_∞ weak-star topology for every continuous state variable g, we obtain from (6.1) and (6.2)

$$(6.3) \qquad \int_{R^2} \langle \nu_{(x,t)}, \lambda^2 \rangle \, dx \le \int_{R^2} u_0^2 \, dx \quad \text{a.e. } t \in (0, T)$$

and

$$\int_{R^2} \langle \nu_{(x,t)}, |\lambda| \rangle \, dx \le C \quad \text{a.e. } t \in (0, T),$$

which proves (2.8).

To prove (2.9), we shall use a technique (see [1]) which involves the following weak convergence and the L_2-stability (6.3):

PROPOSITION 6.1. *For* $\phi \in \mathscr{C}_0^1(\{x \in R^2 : |x| < \tilde{R} + 2\})$ *we have*

$$\lim_{t \to 0} \int_{R^2} \langle \nu_{(x,t)}, \lambda \rangle \phi \, dx = \int_{R^2} u_0 \phi \, dx.$$

We postpone the proof of this result to the end of this section. Assuming first that $u_0 \in \mathscr{C}_0^1(R^2)$ and $\operatorname{supp} u_0 \subset \{x \in R^2 : |x| \leq \tilde{R}\}$, we have with the aid of Proposition 6.1 and (6.3),

(6.4)
$$
\limsup_{t \to 0} \int_{R^2} \langle \nu_{(x,t)}, (\lambda - u_0)^2 \rangle \, dx
$$
$$
= \limsup_{t \to 0} \int_{R^2} \langle \nu_{(x,t)}, \lambda^2 - u_0^2 - 2u_0(\lambda - u_0) \rangle \, dx
$$
$$
\leq -2 \limsup_{t \to 0} \int_{R^2} \langle \nu_{(x,t)}, \lambda - u_0 \rangle u_0 \, dx = 0.
$$

Further, by using that $\langle \nu_{(x,t)}, |\lambda - u_0| \rangle$ has its support in $\Omega_{\tilde{R}+2,M}$ and Jensen's inequality, we get

(6.5)
$$
\limsup_{t \to 0} \int_{R^2} \langle \nu_{(x,t)}, |\lambda - u_0(x)| \rangle \, dx
$$
$$
\leq C \limsup_{t \to 0} \left(\int_{R^2} \langle \nu_{(x,t)}, (\lambda - u_0)^2 \rangle \, dx \right)^{1/2} = 0,
$$

which proves the initial condition (2.9) for regular initial data. In the more general case $u_0 \in L_\infty(R^2)$ with $\operatorname{supp} u_0 \subset \{x \in R^2 : |x| \leq \tilde{R}\}$, we let the functions f_n satisfy $f_n \in \mathscr{C}_0^1(R^2)$, $\operatorname{supp} f_n \subseteq \{x \in R^2 : |x| \leq \tilde{R}\}$ and $\lim_{n \to \infty} \|f_n - u_0\|_{L_2(R^2)} = 0$, and use (6.3) and Jensen's inequality to obtain

$$
\lim_{t \to 0} \int_{R^2} \langle \nu_{(x,t)}, (\lambda - u_0)^2 \rangle \, dx = \lim_{n \to \infty} \lim_{t \to 0} \int_{R^2} \langle \nu_{x,t}, (\lambda - f_n)^2 \rangle.
$$

Now, as in (6.4) we have

$$
\lim_{n \to \infty} \lim_{t \to 0} \int_{R^2} \langle \nu_{(x,t)}, (\lambda - f_n)^2 \rangle \, dx = 0,
$$

so that

$$
\limsup_{t \to 0} \int_{R^2} \langle \nu_{(x,t)}, |\lambda - u_0| \rangle \, dx \leq \lim_{t \to 0} C \left(\int_{R^2} \langle \nu_{(x,t)}, (\lambda - u_0)^2 \rangle \, dx \right)^{1/2} = 0.
$$

We now turn to the proof of Proposition 6.1. Let $\phi \in \mathscr{C}_0^1(R^2)$, $\operatorname{supp} \phi \subset \{x \in R^2 : |x| \leq \tilde{R}\}$, $\psi \in \mathscr{C}_0^1([0,T))$, $\psi(0) = 1$ and take $v = \pi(\phi\psi)$ in (3.1). Letting then h tend to zero, we obtain as in (5.2)

(6.6)
$$
\int_{R_+} \int_{R^2} \langle \nu_{(x,t)}, \lambda \rangle \phi \, dx \, \psi_t \, dt
$$
$$
+ \int_{R_+} \int_{R^2} \sum_i (\langle \nu_{(x,t)}, f_i(\lambda) \rangle \phi_{x_i}) \, dx \, \psi \, dt + \int_{R^2} u_0 \phi \, dx = 0.
$$

Further, we define the functions $A, B \in L_\infty((0,T))$ by

$$
A(t) = \int_{R^2} \langle \nu_{(x,t)}, \lambda \rangle \phi(x) \, dx, \qquad B(t) = \int_{R^2} \sum_i \langle \nu_{(x,t)}, f_i(\lambda) \rangle \phi_{x_i}(x) \, dx.
$$

Since ν_y is a measure-valued solution, we have $A_t + B = 0$ in the sense of distributions on R_+. We note that $B \in L_1((0,T))$, which implies $A_t \in L_1(0,T)$. Hence, A has bounded variation and $\lim_{t \to 0} A(t)$ exists. Taking now

(6.7)
$$
\psi \equiv \psi_n = \begin{cases} (1 - nt)^2, & t \leq 1/n, \\ 0, & t > 1/n \end{cases}
$$

in (6.6), we get

$$\int_{R^2} u_0 \phi \, dx = -\lim_{n\to\infty} \int_{R^2} A(t)(\psi_n)_t \, dt = \lim_{t\to 0} A(t),$$

which proves the proposition.

Remark 6.1. An alternative method to prove the initial condition (2.9) is to establish the strong L_1 convergence

$$\lim_{t\to 0} \|U^h(\cdot, t) - u_0\|_{L_1(R^2)} = 0$$

by using standard error estimates with respect to a solution having smooth initial data. This is possible, since for short times such a solution remains smooth. Using this method, we get (2.9) choosing also $\delta = Ch$. Recall that to obtain (6.3) we assumed $h/\delta \to 0$ as $h \to 0$, cf. Remark 6.2.

Remark 6.2. The following argument, cf. [1], proves that Theorem 3.1 holds also for the case $\delta = h$. Analogously to (6.6) and (5.3), we obtain

(6.8)
$$\int_{R_+} \int_{R^2} \langle \nu_{(x,t)}, \lambda^2 \rangle \phi \, dx \, \psi_t \, dt$$
$$+ \int_{R_+} \int_{R^2} \sum_i (\langle \nu_{(x,t)}, \overline{q}_i(\lambda) \rangle \phi_{x_i}) \, dx \, \psi \, dt + \int_{R^2} (u_0)^2 \phi \, dx \geq 0,$$

where $\overline{q}_i$ are the entropy fluxes corresponding to the entropy λ^2 and ϕ, ψ are as in (6.6). Using now that ν satisfies (5.3) with $\eta = \lambda^2$ we note (cf. [1]) that the integral

$$\overline{A}(t) \equiv \int_{R^2} \langle \nu_{(x,t)}, \lambda^2 \rangle \phi(x) \, dx$$

has bounded variation as a function of t and hence the limit $\lim_{t\to 0} \overline{A}(t)$ exists. Taking now ψ as in (6.7) and $\phi(x) = 1$ for $x \in \Omega_{\tilde{R}+2,M}$, we have by (6.8)

$$\lim_{t\to 0} \int_{R^2} \langle \nu_{(x,t)}, \lambda^2 \rangle \, dx \leq \int_{R^2} (u_0)^2 \phi \, dx,$$

which combined with Proposition 6.1 proves (6.4) and (6.5) as above. We thus conclude that Lemma 3.3 holds also for $\delta = h$.

7. Error Estimates for the SC-Method. In this section we prove that the SC-method applied with piecewise polynomials of order k to the linear problem

(7.1)
$$L(u) \equiv u_t + \sum_{i=1}^{d} a_i u_{x_i} = 0 \quad \text{in } R^d \times R_+,$$
$$u(\cdot, 0) = u_0 \qquad \text{on } R^d,$$

with smooth solution has essentially the same order of convergence as the corresponding streamline diffusion finite element method (see [4]) obtained by choosing $\varepsilon_1 = \varepsilon_2 = 0$ in (3.1), i.e., $\mathcal{O}(h^{k+1/2})$. Introducing the bilinear form

$$B_M(v, w) = \int_{S^M} L(v)(w + \delta(L(w))) \, dx \, dt$$
$$+ \sum_{n=1}^{M} \int_{R_n^d} (v_+ - v_-) w_+ \, dx + \int_{R_0^d} v_+ w_+ \, dx,$$

the SC-method for (7.1) can be formulated, now with V_h consisting of piecewise polynomials of degree k: Find $U \in V_h$ such that for $M = 0, 1, 2, \ldots$

$$B_M(U, v) + \int_{S^M} \varepsilon_1(U) \nabla U \cdot \nabla v \, dx \, dt + \int_{S^M} \varepsilon_2(U) \nabla_x U \cdot \nabla_x v \, dx \, dt = \int_{R_0^d} u_0 v_+ \, dx,$$

with ε_1 and ε_2 as in Section 3. For the quantity $e = U - \pi u \in V_h$, we have

$$
\begin{aligned}
B_M(e, e) &+ \int_{S^M} \varepsilon_1(U) |\nabla e|^2 \, dx \, dt + \int_{S^M} \varepsilon_2(U) |\nabla_x e|^2 \, dx \, dt \\
&= B_M(u - \pi u, e) - \int_{S^M} \varepsilon_1(U) \nabla(\pi u) \cdot \nabla e \, dx \, dt \\
&\quad - \int_{S^M} \varepsilon_2(U) \nabla_x \pi u \cdot \nabla_x e \, dx \, dt \\
&\equiv \mathrm{I} + \mathrm{II} + \mathrm{III}.
\end{aligned}
$$

(7.2)

Now

$$
\begin{aligned}
B_M(e, e) &= \frac{1}{2} \left(\|e_-\|_{L_2(R_{M+1}^d)}^2 + \sum_{n=1}^{M} \|e_+ - e_-\|_{L_2(R_n^d)}^2 + \|e_+\|_{L_2(R_0^d)}^2 \right) \\
&\quad + \delta \int_{S^M} (L(e))^2 \, dx \, dt,
\end{aligned}
$$

(7.3)

and with the notation $\eta = u - \pi u$, we have

$$
\begin{aligned}
\mathrm{I} = B_M(\eta, e) &\leq \frac{1}{2} B(e, e) + \sum_{n=1}^{M+1} \|\eta\|_{L_2(R_n^d)}^2 \\
&\quad + \delta^{-1} \|\eta\|_{L_2(S^M)}^2 + \delta \|L(\eta)\|_{L_2(S^M)}^2.
\end{aligned}
$$

(7.4)

Further, by (3.6),

$$
\begin{aligned}
\mathrm{II} &\leq \frac{\delta}{8} \int_{S^M} (L(U))^2 \, dx \, dt + \frac{2(\bar{\delta})^2}{\delta} \int_{S^M} |\nabla(\pi u)|^2 |\nabla e|^2 \, dx \, dt \\
&\leq \frac{\delta}{4} \int_{S^M} (L(e))^2 \, dx \, dt + \frac{\delta}{4} \int (L(\pi u))^2 \, dx \, dt \\
&\quad + C \frac{(\bar{\delta})^2}{\delta h^2} \|\nabla \pi u\|_{L_\infty}^2 \int_{S^M} e^2 \, dx \, dt,
\end{aligned}
$$

(7.5)

and finally, by Lemma 4.1,

$$
\begin{aligned}
\mathrm{III} &\leq \frac{c}{h} \int_{S^M} |\tilde{U}|^2 \, dx \, dt + C h \bar{\bar{\delta}}^2 \int_{S^M} |\nabla_x \pi u|^2 |\nabla_x e|^2 \, dx \, dt \\
&\leq \frac{1}{8} \sum_{n=0}^{M} \|e_+ - e_-\|_{L_2(R_n^d)}^2 + C \|\nabla_x \pi u\|_{L_\infty}^2 \frac{\bar{\bar{\delta}}^2}{h} \int_{S^M} e^2 \, dx \, dt.
\end{aligned}
$$

(7.6)

Combining now (7.2)–(7.6) and the interpolation estimate (3.4), with V_h^n now consisting of piecewise polynomials of degree $k \geq 1$, we get for $u \in H^2(S^M) \cap W^{1,\infty}(S^M)$

$$B_M(e, e) \leq C h^{2k}(\delta + h^2/\delta) + C \int_{S^M} e^2 \, dx \, dt.$$

Thus by a Gronwall argument, as in (4.3), we have

$$B_M(e, e) \leq C h^{2k}(\delta + h^2/\delta).$$

Using now (3.4), this proves that for $M = 1, 2, 3, \ldots$

$$
\begin{aligned}
(7.7) \quad & \|(U - u)_-\|^2_{L_2(R^d_{M+1})} + \sum_{n=1}^{M} \|U_+ - U_-\|^2_{L_2(R^d_n)} \\
& + \|(U - u_0)_+\|^2_{L_2(R^d_0)} + \delta \int_{S^M} (L(U))^2 \, dx \, dt \le Ch^{2k}(\delta + h^2/\delta),
\end{aligned}
$$

which for $\delta = Ch$ gives the accuracy $\mathscr{O}(h^{k+1/2})$.

Remark 7.1. Following [4], we easily get local error estimates corresponding to (7.7) away from regions where the exact solution is nonsmooth.

Department of Mathematics
Chalmers University of Technology
  and University of Göteborg
S-412 96 Göteborg, Sweden
E-mail: CMSAS@SEGUC21.bitnet

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