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ON BASE AND TURYN SEQUENCES

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ABSTRACT. Base sequences of lengths $n+1$, $n+1$, n, n are constructed for all decompositions of $4n+2$ into four squares for $n = 19, \dots, 24$. The construction is achieved through an algorithm which is also presented. It is proved through an exhaustive search that Turyn sequences do not exist for $n = 18, \dots, 27$; since Turyn sequences cannot exist for $n = 28$ or 29, the first unsettled case is $n = 30$.

1. INTRODUCTION

Given the sequence $A = \{a_1, \dots, a_n\}$ of length n, the nonperiodic autocorrelation function $N_A(s)$ is defined as

$$N_A(s) = \sum_{i=1}^{n-s} a_i a_{i+s}, \qquad s = 0, 1, \dots, n-1. \tag{1}$$

If $A(z) = a_1 + a_2 z + \dots + a_n z^{n-1}$ is the polynomial associated with the sequence A, then

$$A(z)A(z^{-1}) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j z^{i-j} = N_A(0) + \sum_{s=1}^{n-1} N_A(s)(z^s + z^{-s}), \qquad z \neq 0. \tag{2}$$

If $A^* = \{a_n, \dots, a_1\}$ is the reversed sequence, then

$$A^*(z) = z^{n-1}A(z^{-1}). \tag{3}$$

Base and Turyn sequences are finite sequences, with zero autocorrelation function, useful in constructing orthogonal designs and Hadamard matrices [3], in communications engineering [5], etc.

Definition 1. The four sequences X, Y, Z, W of length n, with entries 0, 1, -1, are called T-sequences if

$$\text{(4)}\quad \begin{aligned} &\text{(i)} \quad |x_i| + |y_i| + |z_i| + |w_i| = 1, \qquad i = 1, \dots, n, \\ &\text{(ii)} \quad N_X(s) + N_Y(s) + N_Z(s) + N_W(s) = \begin{cases} 0, & s = 1, \dots, n-1, \\ n, & s = 0. \end{cases} \end{aligned}$$

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Yang [6] gives another name for T-sequences and calls them four-symbol δ-codes; he also calls the quadruple $Q = X + Y$, $R = X - Y$, $S = Z + W$, $T = Z - W$ a regular δ-code of length n, where X, Y, Z, W are T-sequences of length n.

If Williamson type matrices of size w exist, and T-sequences of length n exist, then Hadamard matrices of size $4nw$ can be constructed (Cooper and (Seberry) Wallis [2]).

If $X(z)$, $Y(z)$, $Z(z)$, $W(z)$ are the associated polynomials, then from Definition 1 and (2) we see that (ii) in (4) can be replaced by

$$(5)\quad X(z)X(z^{-1}) + Y(z)Y(z^{-1}) + Z(z)Z(z^{-1}) + W(z)W(z^{-1}) = n, \qquad z \neq 0.$$

In §2 we develop an algorithm and we construct base sequences of lengths $n+1$, $n+1$, n, n for $n = 19, \dots, 24$ for all decompositions of $4n+2$ into four squares. These are given in Table 1. In §3 we use an exhaustive search to prove that Turyn sequences of lengths $n+1$, $n+1$, n, n, do not exist for $n = 18, \dots, 27$.

2. Base sequences

Definition 2. The four sequences A, B, C, D of lengths $n+p$, $n+p$, n, n, with entries $+1$, -1, are called base sequences if

$$(6)\quad \begin{aligned} &N_A(s) + N_B(s) + N_C(s) + N_D(s) = \begin{cases} 0, & s = 1, \dots, n-1, \\ 4n + 2p, & s = 0, \end{cases} \\ &N_A(s) + N_B(s) = 0, \qquad s = n, \dots, n+p-1. \end{aligned}$$

Equivalently, (6) can be replaced by (see (2))

$$(7)\quad A(z)A(z^{-1}) + B(z)B(z^{-1}) + C(z)C(z^{-1}) + D(z)D(z^{-1}) = 4n + 2p, \qquad z \neq 0,$$

where $A(z)$, $B(z)$, $C(z)$, $D(z)$ are the associated polynomials. Base sequences of lengths $n+p$, $n+p$, n, n, whenever $p = 1$, are denoted by $\mathrm{BS}(2n+1)$.

From (7), and for $p = 1$, if we set $z = 1$, we obtain

$$(8)\quad a^2 + b^2 + c^2 + d^2 = 4n + 2,$$

where a, b, c, d are the sum of the elements of A, B, C, D, respectively. $\mathrm{BS}(2n+1)$ for all decompositions of $4n+2$ into four squares for $n = 1, \dots, 18$ are given in [1, 4]. Also, $\mathrm{BS}(2n+1)$ for $n = 19, 20, \dots, 23, 25, 26, 29$ and $n = 2^a 10^b 26^c$ (Golay numbers), a, b, c nonnegative integers, are given in Yang [7].

In Table 1 we give $\mathrm{BS}(2n+1)$ for all decompositions of $4n+2$ into four squares for $n = 19, \dots, 24$.

Theorem 1. *If* $A = \{a_1, \dots, a_{n+1}\}$, $B = \{b_1, \dots, b_{n+1}\}$, $C = \{c_1, \dots, c_n\}$, $D = \{d_1, \dots, d_n\}$ *are* BS$(2n+1)$, *then*

$$(9)\qquad \begin{aligned} &a_s + b_s + a_{n+2-s} + b_{n+2-s} \equiv \begin{cases} 2 \bmod 4, & s = 1, \\ 0 \bmod 4, & s = 2, \dots, [(n+1)/2], \end{cases} \\ &c_s + d_s + c_{n+1-s} + d_{n+1-s} \equiv 0 \bmod 4, \qquad s = 1, \dots, [n/2]. \end{aligned}$$

Proof. We have

$$N_A(s) + N_B(s) = \sum_{i=1}^{n+1-s} (a_i a_{i+s} + b_i b_{i+s}), \quad s = 1, \dots, n,$$

$$N_C(s) + N_D(s) = \sum_{i=1}^{n-s} (c_i c_{i+s} + d_i d_{i+s}), \quad s = 1, \dots, n-1.$$

Then

$$(10)\qquad \begin{aligned} N_A(s) + N_B(s) + N_C(s) + N_D(s) = &\sum_{i=1}^{n-s} (a_i a_{i+s} + b_i b_{i+s} + c_i c_{i+s} + d_i d_{i+s}) \\ &+ a_{n+1} a_{n+1-s} + b_{n+1} b_{n+1-s} = 0, \\ &\qquad s = 1, \dots, n-1, \end{aligned}$$

and since $xy \equiv (x + y - 1) \bmod 4$ whenever x, $y = \pm 1$, we have from (10)

$$\begin{aligned} F(s) := &\sum_{i=1}^{n-s} (a_i + a_{i+s} + b_i + b_{i+s} + c_i + c_{i+s} + d_i + d_{i+s}) \\ &+ a_{n+1} + a_{n+1-s} + b_{n+1} + b_{n+1-s} \equiv 2 \bmod 4, \qquad s = 1, \dots, n-1, \end{aligned}$$

or

$$(11)\qquad \begin{aligned} F(s) = &\sum_{i=1}^{n-s} (a_i + b_i + c_i + d_i) + \sum_{i=s+1}^{n} (a_i + b_i + c_i + d_i) \\ &+ a_{n+1} + b_{n+1} + a_{n+1-s} + b_{n+1-s} \\ \equiv &\ 2 \bmod 4, \qquad s = 1, \dots, n-1. \end{aligned}$$

Now

$$(12)\qquad \begin{aligned} F(s-1) - F(s) = &\ a_s + b_s + c_s + d_s + a_{n+2-s} + b_{n+2-s} + c_{n+1-s} + d_{n+1-s} \\ \equiv &\ 0 \bmod 4, \qquad s = 2, \dots, n-1, \end{aligned}$$

and for $s = n - 1$ we have from (11)

$$(13)\quad a_1 + b_1 + c_1 + d_1 + a_n + b_n + c_n + d_n + a_{n+1} + b_{n+1} + a_2 + b_2 \equiv 2 \bmod 4.$$

Also, from

$$N_A(n) + N_B(n) = a_1 a_{n+1} + b_1 b_{n+1} = 0$$

we have

$$(14)\qquad a_1 + b_1 + a_{n+1} + b_{n+1} \equiv 2 \bmod 4.$$

From (12)–(14) we obtain

$$(a_s + b_s + a_{n+2-s} + b_{n+2-s}) + (c_s + d_s + c_{n+1-s} + d_{n+1-s}) \equiv 0 \bmod 4, \qquad s = 2, \ldots, n. \tag{15}$$

If we set $n + 2 - s$ instead of s in (15), then

$$(a_{n+2-s} + b_{n+2-s} + a_s + b_s) + (c_{n+2-s} + d_{n+2-s} + c_{s-1} + d_{s-1}) \equiv 0 \bmod 4, \qquad s = 2, \ldots, n. \tag{16}$$

From (15) and (16) we have

$$c_{s-1} + c_s + d_{s-1} + d_s + c_{n+1-s} + c_{n+2-s} + d_{n+1-s} + d_{n+2-s} \equiv 0 \bmod 4, \qquad s = 2, \ldots, n. \tag{17}$$

For n odd, (17) becomes

$$c_s + c_{n+1-s} + d_s + d_{n+1-s} \equiv 0 \bmod 4, \quad s = 2, \ldots, n, \tag{18}$$

because with $s = (n+1)/2$, (17) gives

$$c_{(n-1)/2} + c_{(n+3)/2} + d_{(n-1)/2} + d_{(n+3)/2} \equiv 0 \bmod 4;$$

then set $s = (n+1)/2 - 1$, etc.

From (15) and (18) we have

$$a_s + b_s + a_{n+2-s} + b_{n+2-s} \equiv 0 \bmod 4, \qquad s = 2, \ldots, n. \tag{19}$$

If we set $n + 1 - s$ instead of s in (15), then we have similarly

$$a_s + a_{s+1} + b_s + b_{s+1} + a_{n+1-s} + b_{n+1-s} + a_{n+2-s} + b_{n+2-s} \equiv 0 \bmod 4, \qquad s = 2, \ldots, n-1. \tag{20}$$

For n even, using the same argument as before, (20) becomes

$$a_s + b_s + a_{n+2-s} + b_{n+2-s} \equiv 0 \bmod 4, \qquad s = 2, \ldots, n-1, \tag{21}$$

and from (15)

$$c_s + d_s + c_{n+1-s} + d_{n+1-s} \equiv 0 \bmod 4, \qquad s = 2, \ldots, n-1. \tag{22}$$

Therefore, (14), (18), (19), (21), and (22) give the required result. □

In (19) and (21) it is enough to take $s = 2, \ldots, [(n+1)/2]$, and in (18) and (22), $s = 1, \ldots, [n/2]$.

Before describing the algorithm, we need the following:

Given the sequence $E = \{e_1, \ldots, e_n\}$, we define the m subsequences, for some $m = 2, 3, \ldots, n$:

$$E_1 = \{e_1, e_{1+m}, \ldots, e_{1+s_1 \cdot m}\} \quad \text{with } s_1 = \left[\frac{n-1}{m}\right],$$

$$E_2 = \{e_2, e_{2+m}, \ldots, e_{2+s_2 \cdot m}\} \quad \text{with } s_2 = \left[\frac{n-2}{m}\right],$$

$$\vdots$$

$$E_m = \{e_m, e_{2m}, \ldots, e_{m+s_m \cdot m}\} \quad \text{with } s_m = \left[\frac{n-m}{m}\right],$$

or

$$E_i = \{e_i, e_{i+m}, \ldots, e_{i+s_i \cdot m}\} \quad \text{with } s_i = \left[\frac{n-i}{m}\right], \qquad i = 1, \ldots, m,$$

with associated polynomials

$$E_i(z) = \sum_{j=0}^{s_i} z^j \cdot e_{i+j\cdot m}, \qquad i = 1, \ldots, m.$$

Then

$$E(z) = E_1(z^m) + zE_2(z^m) + \cdots + z^{m-1}E_m(z^m)$$

or

$$E(z) = \sum_{i=1}^{m} z^{i-1} E_i(z^m). \tag{23}$$

Theorem 2. *If* $A = \{a_1, \ldots, a_{n+1}\}$, $B = \{b_1, \ldots, b_{n+1}\}$, $C = \{c_1, \ldots, c_n\}$, $D = \{d_1, \ldots, d_n\}$ *are* $(1, -1)$*-sequences of lengths* $n+1$, $n+1$, n, n, *then they are* BS$(2n+1)$ *if and only if for some* $m = 2, \ldots, n+1$

$$\begin{aligned}
&\sum_{i=1}^{m}(A_i(z^m)A_i(z^{-m}) + B_i(z^m)B_i(z^{-m}) \\
&\qquad + C_i(z^m)C_i(z^{-m}) + D_i(z^m)D_i(z^{-m})) = 4n+2, \\
&\sum_{i=1}^{m-s}(A_i(z^m)A_{i+s}(z^{-m}) + B_i(z^m)B_{i+s}(z^{-m}) \\
&\qquad + C_i(z^m)C_{i+s}(z^{-m}) + D_i(z^m)D_{i+s}(z^{-m})) \\
&\qquad + z^m\sum_{i=1}^{s}(A_{i+m-s}(z^m)A_i(z^{-m}) + B_{i+m-s}(z^m)B_i(z^{-m}) \\
&\qquad\qquad + C_{i+m-s}(z^m)C_i(z^{-m}) + D_{i+m-s}(z^m)D_i(z^{-m})) = 0, \\
&\qquad\qquad\qquad s = 1, \ldots, [m/2].
\end{aligned} \tag{24}$$

Proof. Writing $A(z)$, $B(z)$, $C(z)$, $D(z)$ as in (23), and equating all coefficients of z^t in (7), where $t \equiv s \bmod m$, we find the above relations (24) for $s = 1, \ldots, m-1$. By setting z^{-1} instead of z in (24) we see that it is enough to take $s = 1, \ldots, [m/2]$. □

For $m = 2$ we have

$$\begin{aligned}
A(z) &= A_1(z^2) + zA_2(z^2), \\
B(z) &= B_1(z^2) + zB_2(z^2), \\
C(z) &= C_1(z^2) + zC_2(z^2), \\
D(z) &= D_1(z^2) + zD_2(z^2).
\end{aligned} \tag{25}$$

From (24) and (25) we conclude that there are four isomorphic transformations for the BS$(2n+1)$, A, B, C, D, i.e.,

(26)
- (i) interchange A and B and/or C and D,
- (ii) reverse one or more sequences,
- (iii) negate one or more sequences,
- (iv) negate alternate elements in all sequences.

Corollary 1. *If* A, B, C, D *are* BS$(2n+1)$ *and given* $m \in \{2, 3, \dots, n+1\}$,

$$\text{(i)}\quad k_{im} = \sum_{j \equiv i \bmod m} a_j, \qquad r_{im} = \sum_{j \equiv i \bmod m} b_j, \qquad p_{im} = \sum_{j \equiv i \bmod m} c_j, \qquad q_{im} = \sum_{j \equiv i \bmod m} d_j, \tag{27}$$

$$\text{(ii)}\quad K_m = \{k_{1m}, \dots, k_{mm}\}, \qquad R_m = \{r_{1m}, \dots, r_{mm}\}, \qquad P_m = \{p_{1m}, \dots, p_{mm}\}, \qquad Q_m = \{q_{1m}, \dots, q_{mm}\}, \tag{28}$$

$$\text{(iii)}\quad N_K(s) = \sum_{i=1}^{m-s} k_{im}k_{i+s,m}, \qquad N_R(s) = \sum_{i=1}^{m-s} r_{im}r_{i+s,m}, \qquad N_P(s) = \sum_{i=1}^{m-s} p_{im}p_{i+s,m}, \qquad N_Q(s) = \sum_{i=1}^{m-s} q_{im}q_{i+s,m}, \tag{29}$$

then for the given $m \in \{2, 3, \dots, n+1\}$,

$$\begin{aligned} &N_K(0) + N_R(0) + N_P(0) + N_Q(0) \\ &\quad = k_{1m}^2 + \cdots + k_{mm}^2 + r_{1m}^2 + \cdots + r_{mm}^2 \\ &\qquad + p_{1m}^2 + \cdots + p_{mm}^2 + q_{1m}^2 + \cdots + q_{mm}^2 \\ &\quad = 4n + 2, \end{aligned} \tag{30}$$

$$\begin{aligned} &N_K(s) + N_R(s) + N_P(s) + N_Q(s) + N_K(m-s) + N_R(m-s) \\ &\quad + N_P(m-s) + N_Q(m-s) = 0, \qquad s = 1, \dots, [m/2]. \end{aligned} \tag{31}$$

Proof. If we set $z^m = 1$, then $A_i(z^m)$, $B_i(z^m)$, $C_i(z^m)$, $D_i(z^m)$ give

$$A_i(1) = k_{im}, \quad B_i(1) = r_{im}, \quad C_i(1) = p_{im}, \quad D_i(1) = q_{im},$$

and (24) becomes

$$\sum_{i=1}^{m} (k_{im}^2 + r_{im}^2 + p_{im}^2 + q_{im}^2) = 4n + 2,$$

$$\begin{aligned} &\sum_{i=1}^{m-s} (k_{im}k_{i+s,m} + r_{im}r_{i+s,m} + p_{im}p_{i+s,m} + q_{im}q_{i+s,m}) \\ &\quad + \sum_{i=1}^{s} (k_{im}k_{i+m-s,m} + r_{im}r_{i+m-s,m} + p_{im}p_{i+m-s,m} + q_{im}q_{i+m-s,m}) = 0, \end{aligned}$$

that is,

$$N_K(0) + N_R(0) + N_P(0) + N_Q(0) = 4n + 2,$$
$$N_K(s) + N_R(s) + N_P(s) + N_Q(s) + N_K(m-s) + N_R(m-s)$$
$$+ N_P(m-s) + N_Q(m-s) = 0, \qquad s = 1, \dots, [m/2]. \quad \square$$

Note that

$$(32) \qquad \begin{array}{ll} k_{jm} = k_{j,2m} + k_{j+m,2m}, & r_{jm} = r_{j,2m} + r_{j+m,2m}, \\ p_{jm} = p_{j,2m} + p_{j+m,2m}, & q_{jm} = q_{j,2m} + q_{j+m,2m}, \end{array} \qquad j = 1, \dots, m.$$

From (9) we can find further restrictions for k_{im}, r_{im}, p_{im}, q_{im}, as noted in the following theorem.

Theorem 3. *If* A, B, C, D *are* BS$(2n+1)$, *then for a given* $m \in \{2, 3, \dots, n+1\}$,

$$(33) \qquad \begin{aligned} & k_{1m} + r_{1m} + k_{n+1,m} + r_{n+1,m} \equiv \begin{cases} 2 \bmod 4 & \text{if } n \not\equiv 0 \bmod m, \\ 0 \bmod 4 & \text{if } n \equiv 0 \bmod m, \end{cases} \\ & k_{jm} + r_{jm} + k_{n+2-j,m} + r_{n+2-j,m} \equiv 0 \bmod 4, \qquad j = 2, \dots, m, \\ & p_{jm} + q_{jm} + p_{n+1-j,m} + q_{n+1-j,m} \equiv 0 \bmod 4, \qquad j = 1, \dots, m, \end{aligned}$$

where k_{im}, r_{im}, p_{im}, q_{im} *are defined in* (27).

Proof. Summing all relations in (9) with $s \equiv j \bmod m$, we obtain the above result. $\square$

The Algorithm. Since it is difficult to find directly the values of $a_1, \dots, a_{n+1}$, $b_1, \dots, b_{n+1}$, $c_1, \dots, c_n$, $d_1, \dots, d_n$, we find the values of

$$k_{1m}, \dots, k_{mm}, r_{1m}, \dots, r_{mm}, p_{1m}, \dots, p_{mm}, q_{1m}, \dots, q_{mm}$$

as defined in (27). Our algorithm relies on Corollary 1. To avoid calculating isomorphic BS$(2n+1)$, on the basis of the four properties given in (26), we can always take

$$(34) \qquad k_{11} \ge r_{11} \ge 0, \qquad p_{11} \ge q_{11} \ge 0,$$

where k_{11}, r_{11}, p_{11}, q_{11} is the sum of the elements of A, B, C, D, respectively. If n is odd, then $k_{12} \ge k_{22} \ge 0$, $r_{12} \ge r_{22}$. If n is even, then $p_{12} \ge p_{22} \ge 0$, $q_{12} \ge q_{22}$.

Step 1. Find all k_{11}, r_{11}, p_{11}, q_{11} satisfying

$$k_{11} \ge r_{11} \ge 0, \quad p_{11} \ge q_{11} \ge 0, \quad k_{11}^2 + r_{11}^2 + p_{11}^2 + q_{11}^2 = 4n + 2,$$

and for

(i) n even, take k_{11}, r_{11} odd,
p_{11}, q_{11} even,
(ii) n odd, take k_{11}, r_{11} even,
p_{11}, q_{11} odd.

Step 2. For every quadruple k_{11}, r_{11}, p_{11}, q_{11} and given $m \in \{2, 3, \dots, n+1\}$, find $k_{1m}, \dots, k_{mm}, r_{1m}, \dots, r_{mm}, p_{1m}, \dots, p_{mm}, q_{1m}, \dots, q_{mm}$ satisfying

(i) $k_{11} = k_{1m} + \cdots + k_{mm}$, $r_{11} = r_{1m} + \cdots + r_{mm}$, $p_{11} = p_{1m} + \cdots + p_{mm}$, $q_{11} = q_{1m} + \cdots + q_{mm}$,

(ii) k_{jm}, r_{jm} are both odd (even) if $[(n+1-j)/m] + 1$ is odd (even) and

$$|k_{jm}| \leq \left[\frac{n+1-j}{m}\right] + 1, \quad |r_{jm}| \leq \left[\frac{n+1-j}{m}\right] + 1, \qquad j = 1, \dots, m,$$

p_{jm}, q_{jm} are both odd (even) if $[(n-j)/m] + 1$ is odd (even) and

$$|p_{jm}| \leq \left[\frac{n-j}{m}\right] + 1, \quad |q_{jm}| \leq \left[\frac{n-j}{m}\right] + 1, \qquad j = 1, \dots, m,$$

(iii)

$$k_{1m} + r_{1m} + k_{n+1,m} + r_{n+1,m} \equiv \begin{cases} 2 \bmod 4 & \text{if } n \not\equiv 0 \bmod m, \\ 0 \bmod 4 & \text{if } n \equiv 0 \bmod m, \end{cases}$$

$$k_{jm} + r_{jm} + k_{n+2-j,m} + r_{n+2-j,m} \equiv 0 \bmod 4, \qquad j = 2, \dots, m,$$

$$p_{jm} + q_{jm} + p_{n+1-j,m} + q_{n+1-j,m} \equiv 0 \bmod 4, \qquad j = 1, \dots, m,$$

(iv) $k_{1m}^2 + \cdots + k_{mm}^2 + r_{1m}^2 + \cdots + r_{mm}^2 + p_{1m}^2 + \cdots + p_{mm}^2 + q_{1m}^2 + \cdots + q_{mm}^2 = 4n+2$,

(v) $N_K(s) + N_K(m-s) + N_R(s) + N_R(m-s) + N_P(s) + N_P(m-s) + N_Q(s) + N_Q(m-s) = 0$, $s = 1, \dots, [m/2]$, where

$$N_K(s) = \sum_{j=1}^{m-s} k_{jm} k_{j+s,m}, \qquad N_R(s) = \sum_{j=1}^{m-s} r_{jm} r_{j+s,m},$$

$$N_P(s) = \sum_{j=1}^{m-s} p_{jm} p_{j+s,m}, \qquad N_Q(s) = \sum_{j=1}^{m-s} q_{jm} q_{j+s,m}.$$

Step 3. (i) For every $k_{1m}, \dots, k_{mm}, r_{1m}, \dots, r_{mm}, p_{1m}, \dots, p_{mm}, q_{1m}, \dots, q_{mm}$ found in Step 2, find $k_{1,2m}, \dots, k_{2m,2m}, r_{1,2m}, \dots, r_{2m,2m}, p_{1,2m}, \dots, p_{2m,2m}, q_{1,2m}, \dots, q_{2m,2m}$ satisfying

$$\begin{aligned} k_{jm} &= k_{j,2m} + k_{j+m,2m}, & r_{jm} &= r_{j,2m} + r_{j+m,2m}, \\ p_{jm} &= p_{j,2m} + p_{j+m,2m}, & q_{jm} &= q_{j,2m} + q_{j+m,2m}, \end{aligned} \qquad j = 1, \dots, m.$$

(ii) Go to Step 2(ii)–(v), setting $2m$ instead of m.

Step 4. Stop when $m \geq n+1$ and examine if

$$N_K(s) + N_R(s) + N_P(s) + N_Q(s) = 0, \qquad s = 1, \dots, m-1,$$

because for $m \geq n+1$,

$$k_{jm} = 0, 1, -1, \qquad r_{jm} = 0, 1, -1,$$
$$p_{jm} = 0, 1, -1, \qquad q_{jm} = 0, 1, -1.$$

TABLE 1

Base sequences $\mathrm{BS}(2n+1)\ A, B, C, D$ of lengths $n+1$, $n+1$, n, n or Suitable sequences $\mathrm{SS}(2n+1)\ (A+B)/2,\ (A-B)/2,\ (C+D)/2,\ (C-D)/2$

Length	Sum of Squares for Suitable Sequences	Sequence
$n = 19$	$2n+1 = 39 = 6^2+1^2+1^2+1^2$	{− + − + − − + + + − − − + + + + + − − −} {+ − − + − − + − − + + + − + + + + − + −} {+ − + + + + + − + + + − − + + − + − −} {+ + + + + + − − + − + − + + + − + + −}
$n = 19$	$2n+1 = 39 = 5^2+3^2+2^2+1^2$	{+ + + + − − + − + + − + + − + − + − − −} {+ − + − + − − − + + − + + + + + − − + +} {+ + + + − − + + + − + − − − − + − + +} {+ + − − + + − − + − + + + + + − + + +}
$n = 20$	$2n+1 = 41 = 6^2+2^2+1^2+0^2$	see [1], or {+ + − + − − + − − + + + − − + + + − + − −} {− − + + − + + + + − − − + + + − + − − + −} {+ − + − + + + + + + − + + − − − − − + +} {+ + + + + − + + + − + + + − + − + − − +}
$n = 20$	$2n+1 = 41 = 5^2+4^2+0^2+0^2$	see [7], or {− − + + + − − + + + − + + − + + − + + − +} {+ − + + + + + − − + + + − + − − − + + − +} {− + + − + − + + + + + + + − − − + − + −} {+ − − − + − + + + + + + + − − − + + − +}
$n = 20$	$2n+1 = 41 = 4^2+4^2+3^2+0^2$	{− − + − + + − + − − + − + + + + + − − + +} {+ − − + + − + + + − − − − + − − + + + + +} {+ − − − + − + − − − − + − + + + − + + +} {+ − + + + − − + + + + − + − + + + − + +}
$n = 21$	$2n+1 = 43 = 5^2+4^2+1^2+1^2$	{− + + + − − + + + + − − + + − − + − − − + −} {+ + + + + + − − + − − − − + + + − + − − + −} {− − − + + + − − + − + + + + + − + − − + −} {+ − + + + − + − + − + + + + − + + − + + +}
$n = 21$	$2n+1 = 43 = 5^2+3^2+3^2+0^2$	{+ + + + + − − + − − + − + − − − − + + − − +} {+ + − + + + − + + − + − + + − − + + + + − −} {− + − − + + + + + + − + + − + − + + + − −} {+ − − + + + + + + − − − + − + − + − + + +}
$n = 21$	$2n+1 = 43 = 4^2+3^2+3^2+3^2$	see [7], or {− − + − + − − − + + + + + + − − − − + − + +} {− − + + − + − + + + + + + + + − + + − − + −} {− − − + + + − − + + − + + − + − + − + + −} {+ − + + + − − + + − + − + + + + + − − + +}
$n = 22$	$2n+1 = 45 = 6^2+3^2+0^2+0^2$	{+ − + − − + − − − − + − + + + + + − − + + + +} {− + + − − + + + + − + + + + − − − − − + + − +} {+ + − + − − + − + + + − + + + − + − + + + −} {− + − − + + + − + + + − + + + − − + − + + +}
$n = 22$	$2n+1 = 45 = 6^2+2^2+2^2+1^2$	{− − + − + + + − − − − + − + + − + + + − − + +} {+ − − + + + − − − + − + − − + − − + + + + + +} {+ − − + + − + − + − + − − + − − + + + + + +} {+ + + + − − + + + + + − + + + − + − + − − +}
$n = 22$	$2n+1 = 45 = 5^2+4^2+2^2+0^2$	{− + − − + − − + + + + + + + − − − + + + + − +} {+ − + − + − + − − + − − − + + + + + + + − + +} {+ − + − + + − − − − + + + − − + + − + + + −} {+ − + + − − + + + − + + + + + − − + − + + −}
$n = 22$	$2n+1 = 45 = 4^2+4^2+3^2+2^2$	see [7], or {+ − − − + + − + + + + − + − + − + + − − − − +} {− + − − − + − + + − + + + + + − + + + − − + +} {+ + + − − + − + − − + − − + + − + − − − + +} {+ − + + + + + + + − + − − − + + + + + − − +}
$n = 23$	$2n+1 = 47 = 6^2+3^2+1^2+1^2$	{+ + + − − − + − − − + − + + − + + − + − − + − +} {+ + − + − − − + + + + − + + + − − + + − + − − −} {+ + + − + − + + + + − − − − − + + − − + + − +} {+ − + − − − + + − + + + + − + + + − + + + + +}
$n = 23$	$2n+1 = 47 = 5^2+3^2+3^2+2^2$	see [1], or {− + + + − + − + + + + − − − − − − + + − + + + −} {− + + − + + − + − − − + + + + + − + + + − + + +} {+ + − − + − + + − + − + + + − + − − + − − − +} {− − − + + + − − + + − + + + + − + + + + − + −}

TABLE 1 (*continued*)

Length	Sum of Squares for Suitable Sequences	Sequence
$n = 24$	$2n+1 = 49 = 7^2 + 0^2 + 0^2 + 0^2$	{− − − − + + + − + + − + + + + + + + + − + − + − +} {+ − − + + + + + + + + − − − − + + − + − + + + − +} {− + − + − + + − + − − + + − − + + − + + + − − −} {+ + − + − − + − + + − − − − + + + − − + + − − +}
$n = 24$	$2n+1 = 49 = 6^2 + 3^2 + 2^2 + 0^2$	{− + + + − − − − − + + − + + + + + + − + − + − − +} {+ − + − + − − − − + + + − − + + + + − + + − − + +} {+ + + + − + + − + + + + − − + − − − + − + − + −} {+ + − + + − − + + + + + − − + − + + − + + + + −}
$n = 24$	$2n+1 = 49 = 5^2 + 4^2 + 2^2 + 2^2$	{− + − + − + + + + + + + − − − − + + − − + − − + +} {+ + + + + + − + − + − + + − + − − + + − − − + + +} {− − + + + − − − + + − + + − + + − − + − + − + +} {+ + + + + − − + + − + + + + − + + − + − + − − −}
$n = 24$	$2n+1 = 49 = 4^2 + 4^2 + 4^2 + 1^2$	{+ + + + + − + + − − + − − − + − + + − + − + + − −} {+ − + − + + + − − + + + − + + + + − − − − − + + +} {+ − + + + − + + − + + − − − − + − + + − − − − +} {+ + − − + − + − + − − + + + + − + + + − + + + +}

Remark. C. H. Yang and J. Yang (private communication) have constructed the equivalent of the sequences shown in Table 1 by constructing base, normal, or near normal sequences.

3. TURYN SEQUENCES

Our algorithm requires too much computer time for the construction of $\mathrm{BS}(2n+1)$ for $n \geq 25$. It is desirable to construct $\mathrm{BS}(4n+3)$ from $\mathrm{BS}(2n+1)$.

Theorem 4. *If* A, B, C, D *are* $\mathrm{BS}(2n+1)$, *and their associated polynomials satisfy*

$$A(z)C(z^{-1}) + zC(z)A(z^{-1}) = 0, \qquad z \neq 0, \tag{35}$$

then the sequences

$$X = \{1, A|C\}, \quad Y = \{-1, A|C\}, \quad Z = \{B|D\}, \quad W = \{B| - D\} \tag{36}$$

are $\mathrm{BS}(4n+3)$, *where*

$$A|C = \{a_1, c_1, a_2, c_2, \ldots, a_n, c_n, a_{n+1}\},$$
$$B|D = \{b_1, d_1, b_2, d_2, \ldots, b_n, d_n, b_{n+1}\}.$$

Proof. The polynomials associated with X, Y, Z, W are

$$X(z) = 1 + z(A(z^2) + zC(z^2)), \qquad Y(z) = -1 + z(A(z^2) + zC(z^2)),$$
$$Z(z) = B(z^2) + zD(z^2), \qquad W(z) = B(z^2) - zD(z^2).$$

Then

$$\begin{aligned} X(z)X(z^{-1}) &+ Y(z)Y(z^{-1}) + Z(z)Z(z^{-1}) + W(z)W(z^{-1}) \\ &= 2 + 2(A(z^2)A(z^{-2}) + B(z^2)B(z^{-2}) + C(z^2)C(z^{-2}) + D(z^2)D(z^{-2})) \\ &\quad + 2z^{-1}(z^2C(z^2)A(z^{-2}) + A(z^2)C(z^{-2})) = 8n + 6, \qquad z \neq 0. \quad \square \end{aligned}$$

If we define the cross correlations

$$(37)\qquad N_{AC}(s)=\sum_{i=1}^{n-s}a_ic_{i+s},\quad N_{CA}(s)=\sum_{i=1}^{n+1-s}c_ia_{i+s},\qquad s=0,1,\dots,n,$$

then

$$C(z)A(z^{-1})=\sum_{s=0}^{n}N_{AC}(s)z^s+\sum_{s=1}^{n}N_{CA}(s)z^{-s}$$

and

$$zC(z)A(z^{-1})+A(z)C(z^{-1})=\sum_{s=1}^{n}(N_{AC}(s-1)+N_{CA}(s))(z^s+z^{-s+1}).$$

Therefore, $zC(z)A(z^{-1})+A(z)C(z^{-1})=0$ is equivalent to

$$(38)\qquad N_{AC}(s-1)+N_{CA}(s)=0,\qquad s=1,\dots,n.$$

Definition. If the sequences A, B, C, D are $\mathrm{BS}(2n+1)$ and satisfy

$$N_{AC}(s-1)+N_{CA}(s)=0,\qquad s=1,\dots,n,$$

then they are called *Turyn sequences* (abbreviated as $\mathrm{TS}(2n+1)$).

Theorem 5 (see also [3, pp. 139–142]). *If* A, B, C, D *are* $\mathrm{TS}(2n+1)$, *then*

(i) *For n odd,*

$$A=\{A_1,-A_1^*\},\qquad B=\{b_1,B_1,-B_1^*,b_1\},$$
$$C=\{C_1,c_{(n+1)/2},C_1^*\},\qquad D=\{D_1,d_{(n+1)/2},D_1^*\},$$

where

$$A_1=\{a_1,\dots,a_{(n+1)/2}\},\qquad B_1=\{b_2,\dots,b_{(n+1)/2}\},$$
$$C_1=\{c_1,\dots,c_{(n-1)/2}\},\qquad D_1=\{d_1,\dots,d_{(n-1)/2}\}.$$

(ii) *For n even,*

$$A=\{A_1,a_{n/2+1},A_1^*\},\qquad B=\{b_1,B_1,b_{n/2+1},B_1^*,-b_1\},$$
$$C=\{C_1,-C_1^*\},\qquad D=\{D_1,-D_1^*\},$$

where

$$A_1=\{a_1,\dots,a_{n/2}\},\qquad B_1=\{b_2,\dots,b_{n/2}\},$$
$$C_1=\{c_1,\dots,c_{n/2}\},\qquad D_1=\{d_1,\dots,d_{n/2}\}.$$

Proof. From (38) and from $x\cdot y\equiv(x+y-1)\bmod 4$ when x, $y=\pm1$, we have

$$F(s):=\sum_{i=1}^{n+1-s}a_i+\sum_{i=s+1}^{n+1}a_i+\sum_{i=1}^{n+1-s}c_i+\sum_{i=s}^{n}c_i\equiv 2(n+1-s)\bmod 4,\qquad s=1,\dots,n;$$

for $s = n + 1$, (37) gives

$$a_1 + a_{n+1} + c_1 + c_n \equiv 2 \bmod 4.$$

Then

$$F(s-1) - F(s) = a_{n+2-s} + a_s + c_{n+2-s} + c_{s-1} \equiv 2 \bmod 4, \quad s = 2, \dots, n+1. \tag{39}$$

Proceeding as in Theorem 1, we obtain from (39) and (9)

(i) For n odd:

$$\begin{aligned} &a_s + a_{n+2-s} \equiv 0 \bmod 4, && s = 1, \dots, n+1, \\ &c_s + c_{n+1-s} \equiv 2 \bmod 4, && s = 1, \dots, n, \\ &b_s + b_{n+2-s} \equiv \begin{cases} 2 \bmod 4, & s = 1, n+1, \\ 0 \bmod 4, & s = 2, \dots, n. \end{cases} \end{aligned}$$

(ii) For n even:

$$\begin{aligned} &a_s + a_{n+2-s} \equiv 2 \bmod 4, && s = 1, \dots, n+1, \\ &b_s + b_{n+2-s} \equiv \begin{cases} 0 \bmod 4, & s = 1, n+1, \\ 2 \bmod 4, & s = 2, \dots, n, \end{cases} \\ &c_s + c_{n+1-s} \equiv 0 \bmod 4, && s = 1, \dots, n, \\ &d_s + d_{n+1-s} \equiv 0 \bmod 4, && s = 1, \dots, n, \end{aligned}$$

which proves Theorem 5. □

Our algorithm in §2 can now be modified to give $\mathrm{TS}(2n+1)$. From Theorem 5 we have:

(i) For n odd:

$$\begin{aligned} &k_{jm} = -k_{n+2-j,m}, && j = 1, \dots, m, \\ &r_{1m} + r_{n+1,m} = 2, \\ &r_{jm} = -r_{n+2-j,m}, && j = 2, \dots, m, \\ &p_{jm} = p_{n+1-j,m}, && j = 1, \dots, m, \\ &q_{jm} = q_{n+1-j,m}, && j = 1, \dots, m. \end{aligned}$$

(ii) For n even:

$$\begin{aligned} &k_{jm} = k_{n+2-j,m}, && j = 1, \dots, m, \\ &r_{jm} = r_{n+2-j,m}, && j = 2, \dots, m, \\ &r_{1m} - r_{n+1,m} = 2 && \text{for } n \not\equiv 0 \bmod 4, \\ &p_{jm} = -p_{n+1-j,m}, && j = 1, \dots, m, \\ &q_{jm} = -q_{n+1-j,m}, && j = 1, \dots, m. \end{aligned}$$

It is known that $\mathrm{TS}(2n+1)$ exists for $n \leq 7$ and $n = 12, 14$; they cannot exist for $n = 10, 11, 16, 17$, and for $n = 8, 9, 13, 15$, $\mathrm{TM}(2n+1)$ might exist but an exhaustive machine search showed that they do not exist (see [1; 3, pp. 142–143]).

In this paper, applying our algorithm, we have done an exhaustive search, and we showed that $\mathrm{TS}(2n+1)$ does not exist for $n = 18, 19, \ldots, 27$; of course, for $n = 28, 29$, $\mathrm{TS}(2n+1)$ cannot exist.

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Bibliography

1. G. Cohen, D. Rubie, J. Seberry, C. Koukouvinos, S. Kounias, and M. Yamada, *A survey of base sequences, disjoint complementary sequences and* $OD(4t; t, t, t, t)$, J. Combin. Math. and Combin. Computing **5** (1989), 60–104.
2. J. Cooper and J. Wallis, *A construction for Hadamard arrays*, Bull. Australian Math. Soc. **7** (1972), 269–277.
3. A. V. Geramita and J. Seberry, *Orthogonal designs: Quadratic forms and Hadamard matrices*, Marcel Dekker, New York-Basel, 1979.
4. C. Koukouvinos, S. Kounias, and J. Seberry, *Further results on base sequences, disjoint complementary sequences,* $OD(4t; t, t, t, t)$ *and the excess of Hadamard matrices*, Ars Combin. (to appear).
5. R. J. Turyn, *Hadamard matrices, Baumert-Hall units, four-symbol sequences, pulse compressions and surface wave encodings*, J. Combin. Theory Ser. A **16** (1974), 313–333.
6. C. H. Yang, *A composition theorem for* δ*-codes*, Proc. Amer. Math. Soc. **89** (1983), 375–378.
7. ——, *On composition of four-symbol* δ*-codes and Hadamard matrices*, Proc. Amer. Math. Soc. **107** (1989), 763–776.

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