MATHEMATICS OF COMPUTATION
VOLUME 55, NUMBER 192
OCTOBER 1990, PAGES 683-692

BOOLEAN METHODS FOR DOUBLE INTEGRATION

FRANZ-J. DELVOS

ABSTRACT. This paper is concerned with numerical integration of continuous functions over the unit square U^2. The concept of the rth-order blending rectangle rule is introduced by carrying over the idea from Boolean interpolation. Error bounds are developed, and it is shown that rth-order blending rectangle rules are comparable with number-theoretic cubature rules. Moreover, rth-order blending midpoint rules are defined and compared with the rth-order blending rectangle rules.

1. BIVARIATE RECTANGLE RULES

The problem we consider is the numerical evaluation of integrals of the form

$$\mathfrak{I}(f) = \int_0^1 \int_0^1 f(x, y)\, dx\, dy, \tag{1.1}$$

where f is a continuous function on the unit square $U^2 = [0, 1]^2$. Moreover, we assume that f satisfies the periodicity conditions

$$f(x, 0) = f(x, 1), \quad f(0, y) = f(1, y) \qquad (0 \le x, y \le 1). \tag{1.2}$$

The inner product of $f, g \in L^2(U^2)$ is

$$(f, g) = \int_0^1 \int_0^1 f(x, y)\overline{g(x, y)}\, dx\, dy.$$

We introduce the notations

$$e_k(x) = \exp(i2\pi kx) \qquad (k \in \mathbb{Z}),$$
$$e_{k,l}(x, y) = e_k(x) \cdot e_l(y) \qquad (k, l \in \mathbb{Z}),$$

where $x, y \in U$. The functions $e_{k,l}$ $(k, l \in \mathbb{Z})$ form an orthonormal basis of the Hilbert space $L^2(U^2)$. We denote by $A(U^2)$ the Wiener algebra of those functions $f \in L^2(U^2)$ with the property that the Fourier series of f is absolutely convergent:

$$\sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} |(f, e_{k,l})| < \infty. \tag{1.3}$$

Received April 3, 1989; revised November 30, 1989.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 65D30, 65D32.
Key words and phrases. Multiple integration, blending methods.

Let $\mathscr{C}(U^2)$ denote the subspace of those functions $f \in L^2(U^2)$ which are continuous on U^2. Moreover, $\mathscr{C}_0(U^2)$ denotes the subspace of those functions $f \in \mathscr{C}(U^2)$ which satisfy the periodicity conditions (1.2). It follows from relation (1.3) that

$$A(U^2) \subseteq \mathscr{C}_0(U^2)$$

and, for $f \in A(U^2)$,

$$f(x, y) = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} (f, e_{k,l}) \cdot e_{k,l}(x, y) \qquad (x, y \in U). \tag{1.4}$$

Let m and n be positive integers. The most obvious cubature formula is the *bivariate rectangle rule*:

$$\mathfrak{I}_{m,n}(f) = \frac{1}{m \cdot n} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} f\left(\frac{j}{m}, \frac{k}{n}\right).$$

The bivariate rectangle rule is not an efficient cubature formula in view of the large number of function evaluations. On the other hand, $\mathfrak{I}_{m,n}(f)$ is a basic tool in constructing a more sophisticated cubature formula, the rth-order blending rectangle rule. For this reason we will briefly derive a convenient remainder formula for $\mathfrak{I}_{m,n}(f)$.

Proposition 1. *If $f \in A(U^2)$, then*

$$\mathfrak{I}_{m,n}(f) = \sum_{u=-\infty}^{\infty} \sum_{v=-\infty}^{\infty} (f, e_{um,vn}). \tag{1.5}$$

Proof. In view of (1.4), we have

$$\begin{aligned}
\mathfrak{I}_{m,n}(f) &= \frac{1}{m \cdot n} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} f\left(\frac{j}{m}, \frac{k}{n}\right) \\
&= \sum_{r=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} (f, e_{r,s}) \frac{1}{m \cdot n} \sum_{j=0}^{m-1} e_r\left(\frac{j}{m}\right) \sum_{k=0}^{n-1} e_s\left(\frac{k}{n}\right) \\
&= \sum_{u=-\infty}^{\infty} \sum_{v=-\infty}^{\infty} (f, e_{um,vn}). \qquad \square
\end{aligned}$$

It is useful to define the series

$$R_{m,\infty}(f) = \sum_{u \neq 0} (f, e_{um,0}), \qquad R_{\infty,n}(f) = \sum_{v \neq 0} (f, e_{0,vn}),$$
$$R_{m,n}(f) = \sum_{u \neq 0} \sum_{v \neq 0} (f, e_{um,vn}).$$

Proposition 2. *If $f \in A(U^2)$, then the error in the bivariate rectangle rule is*

$$\mathfrak{I}_{m,n}(f) - \mathfrak{I}(f) = R_{m,\infty}(f) + R_{\infty,n}(f) + R_{m,n}(f). \tag{1.6}$$

Proof. It follows from relation (1.5) that

$$\mathfrak{I}_{m,n}(f) = (f, e_{0,0}) + R_{m,\infty}(f) + R_{\infty,n}(f) + R_{m,n}(f).$$

Since $\mathfrak{I}(f) = (f, e_{0,0})$, Proposition 2 is proved. □

Following Korobov, we define, for each $a \geq 1$, the linear space

$$E^a(U^2) = \{f \in L^2(U^2) : (f, e_{m,n}) = \mathcal{O}((\overline{m} \cdot \overline{n})^{-a}) \ (m, n \to \infty)\},$$

where $\overline{m} = \max\{1, |m|\}$ $(m \in \mathbb{Z})$. It is easily seen that

$$E^a(U^2) \subseteq A(U^2) \qquad (a > 1). \tag{1.7}$$

We denote by $\mathscr{C}^{p,p}(U^2)$ the linear subspace of $\mathscr{C}(U^2)$ of those functions f whose partial derivatives satisfy

$$D^{k,l} f \in \mathscr{C}(U^2) \qquad (0 \leq k, l \leq p).$$

Similarly, $\mathscr{C}_0^{p,p}(U^2)$ is the linear subspace of $\mathscr{C}_0(U^2)$ of functions f with

$$D^{k,l} f \in \mathscr{C}_0(U^2) \qquad (0 \leq k, l \leq p).$$

It was shown in Baszenski and Delvos [1] that

$$\mathscr{C}_0^{q-1,q-1}(U^2) \cap \mathscr{C}^{q+1,q+1}(U^2) \subseteq E^{q+1}(U^2) \qquad (q \in \mathbb{N}). \tag{1.8}$$

Proposition 3. *If $f \in E^a(U^2)$ with $a > 1$, then the error in the bivariate rectangle rule satisfies*

$$\mathfrak{I}_{m,n}(f) - \mathfrak{I}(f) = \mathcal{O}(m^{-a} + n^{-a}) \qquad (m, n \to \infty). \tag{1.9}$$

Proof. Since $f \in E^a(U^2)$, we have

$$\begin{aligned} R_{m,\infty}(f) &= \mathcal{O}(m^{-a}), \qquad R_{\infty,n}(f) = \mathcal{O}(n^{-a}), \\ R_{m,n}(f) &= \mathcal{O}(m^{-a} \cdot n^{-a}) \qquad (m, n \to \infty), \end{aligned} \tag{1.10}$$

from which (1.9) follows by virtue of Proposition 2. □

Proposition 4. *If $f \in \mathscr{C}_0^{q-1,q-1}(U^2) \cap \mathscr{C}^{q+1,q+1}(U^2)$ with $q \in \mathbb{N}$, then the error in the bivariate rectangle rule satisfies*

$$\mathfrak{I}_{m,n}(f) - \mathfrak{I}(f) = \mathcal{O}(m^{-q-1} + n^{-q-1}) \qquad (m, n \to \infty). \tag{1.11}$$

Proof. Using (1.8), an application of Proposition 3 yields (1.11) □

2. *r*TH-ORDER BLENDING RECTANGLE RULES

We introduce the rth-order sum of bivariate rectangle rules

$$\mathbf{S}_r^2(f) = \sum_{m=1}^{r} \mathfrak{I}_{2^m, 2^{r+1-m}}(f) \qquad (r \in \mathbb{Z}_+). \tag{2.1}$$

Then the *rth-order blending rectangle rule* $\mathfrak{I}_r^2(f)$ is

$$\mathfrak{I}_r^2(f) = \mathbf{S}_r^2(f) - \mathbf{S}_{r-1}^2(f), \tag{2.2}$$

where $r \in \mathbb{N}$ and $r > 1$. The construction of the rth-order blending rectangle rule resembles the explicit formula of the interpolation projector of rth-order blending (Delvos and Posdorf [3] and Delvos [2]). The *cubature points* of $\mathfrak{I}_r^2(f)$ are mainly determined by the points occurring in $\mathbf{S}_r^2(f)$:

$$\bigcup_{m=1}^{r} \{(j \cdot 2^{-m}, k \cdot 2^{-r-1+m}) : 0 \le j < 2^m,\ 0 \le k < 2^{r+1-m}\}. \tag{2.3}$$

Their number is given by

$$n_r = (r+1) \cdot 2^r. \tag{2.4}$$

Next we will determine a remainder formula for the rth-order blending rectangle rule.

Proposition 5. *If $f \in A(U^2)$, then the error in the rth-order blending rectangle rule is*

$$\begin{aligned} \mathfrak{I}_r^2(f) - \mathfrak{I}(f) &= R_{2^r,\infty}(f) + R_{\infty,2^r}(f) \\ &\quad + \sum_{m=1}^{r} R_{2^m,2^{r+1-m}}(f) - \sum_{m=1}^{r-1} R_{2^m,2^{r-m}}(f). \end{aligned} \tag{2.5}$$

Proof. Using (1.6), we get

$$\begin{aligned} \mathfrak{I}_r^2(f) - \mathfrak{I}(f) &= \sum_{m=1}^{r} (\mathfrak{I}_{2^m,2^{r+1-m}}(f) - \mathfrak{I}(f)) - \sum_{m=1}^{r-1} (\mathfrak{I}_{2^m,2^{r-m}}(f) - \mathfrak{I}(f)) \\ &= \sum_{m=1}^{r} (R_{2^m,2^{r+1-m}}(f) + R_{2^m,\infty}(f) + R_{\infty,2^m}(f)) \\ &\quad - \sum_{m=1}^{r-1} (R_{2^m,2^{r-m}}(f) + R_{2^m,\infty}(f) + R_{\infty,2^m}(f)) \\ &= R_{2^r,\infty}(f) + R_{\infty,2^r}(f) \\ &\quad + \sum_{m=1}^{r} R_{2^m,2^{r+1-m}}(f) - \sum_{m=1}^{r-1} R_{2^m,2^{r-m}}(f). \quad \square \end{aligned}$$

Proposition 6. *If $f \in E^a(U^2)$ with $a > 1$, then the error in the rth-order blending rectangle rule is*

$$\mathfrak{I}_r^2(f) - \mathfrak{I}(f) = \mathcal{O}((r+1) \cdot (2^r)^{-a}) \qquad (r \to \infty). \tag{2.6}$$

Proof. From (1.10) we have

$$\begin{aligned} R_{2^r,\infty}(f) = \mathcal{O}((2^r)^{-a}), \quad R_{\infty,2^r}(f) &= \mathcal{O}((2^r)^{-a}) \qquad (r \to \infty), \\ R_{2^m,2^{r+1-m}}(f) &= \mathcal{O}((2^{r+1})^{-a}) \qquad (1 \le m \le r,\ r \to \infty), \\ R_{2^m,2^{r-m}}(f) &= \mathcal{O}((2^r)^{-a}) \qquad (1 \le m < r,\ r \to \infty). \end{aligned}$$

Now (2.6) follows from the remainder formula (2.5). $\square$

Remark 1. Recall that the number of cubature points of the rth-order blending rectangle rule $\mathfrak{I}_r^2(f)$ is bounded by

$$n_r = (r+1)2^r .$$

It is easily seen that the error relation (2.6) of the rth-order blending rectangle rule obtains the form

$$\mathfrak{I}_r^2(f) - \mathfrak{I}(f) = \mathscr{O}(\log(n_r)^{a+1} \cdot (n_r)^{-a}) \qquad (r \to \infty),$$

where $f \in E^a(U^2)$ with $a > 1$. Thus, the rth-order blending rectangle rule is comparable with the bivariate number-theoretic "good-lattice" rules (see Sloan [5]). The attractive feature of the rth-order blending rectangle rule is its easy computation based on relations (2.1) and (2.2).

Proposition 7. *If* $f \in \mathscr{C}_0^{q-1,q-1}(U^2) \cap \mathscr{C}^{q+1,q+1}(U^2)$ *with* $q \in \mathbb{N}$, *then the error in the* r*th-order blending rectangle rule satisfies*

$$\mathfrak{I}_r^2(f) - \mathfrak{I}(f) = \mathscr{O}((r+1) \cdot (2^r)^{-q-1}) \qquad (r \to \infty). \tag{2.7}$$

Proof. Use of (1.8) and an application of Proposition 6 yields (2.7). □

3. Bivariate midpoint rules

Let m and n be positive integers. A simple cubature formula closely related to the bivariate rectangle rule is the *bivariate midpoint rule*:

$$\mathfrak{M}_{m,n}(f) = \frac{1}{m \cdot n} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} f\left(\frac{2j+1}{2m}, \frac{2k+1}{2n}\right).$$

Again, the bivariate midpoint rule is not an efficient cubature formula in view of the large number of function evaluations. However, $\mathfrak{M}_{m,n}(f)$ is a basic tool in constructing the more sophisticated cubature formula of the rth-order blending midpoint rule. For this reason we will briefly derive a convenient remainder formula for $\mathfrak{M}_{m,n}(f)$.

Proposition 8. *If* $f \in A(U^2)$, *then*

$$\mathfrak{M}_{m,n}(f) = \sum_{u=-\infty}^{\infty} \sum_{v=-\infty}^{\infty} (f, e_{um,vn}) \cdot (-1)^{u+v}. \tag{3.1}$$

Proof. By (1.4), we have

$$\begin{aligned}
\mathfrak{M}_{m,n}(f) &= \frac{1}{m \cdot n} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} f\left(\frac{2j+1}{2m}, \frac{2k+1}{2n}\right) \\
&= \sum_{r=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} (f, e_{r,s}) \frac{1}{m \cdot n} \sum_{j=0}^{m-1} e_r\left(\frac{2j+1}{2m}\right) \sum_{k=0}^{n-1} e_s\left(\frac{2k+1}{2n}\right) \\
&= \sum_{u=-\infty}^{\infty} \sum_{v=-\infty}^{\infty} (f, e_{um,vn}) \cdot (-1)^{u+v}. \quad \square
\end{aligned}$$

We define the series

$$Q_{m,\infty}(f) = \sum_{u\neq 0}(f, e_{um,0}) \cdot (-1)^u,$$
$$Q_{\infty,n}(f) = \sum_{v\neq 0}(f, e_{0,vn}) \cdot (-1)^v,$$
$$Q_{m,n}(f) = \sum_{u\neq 0}\sum_{v\neq 0}(f, e_{um,vn}) \cdot (-1)^{u+v}.$$

Proposition 9. *If $f \in A(U^2)$, then the error in the bivariate midpoint rule is*

$$\mathfrak{M}_{m,n}(f) - \mathfrak{I}(f) = Q_{m,\infty}(f) + Q_{\infty,n}(f) + Q_{m,n}(f). \tag{3.2}$$

Proof. From (3.1) we get

$$\mathfrak{M}_{m,n}(f) = (f, e_{0,0}) + Q_{m,\infty}(f) + Q_{\infty,n}(f) + Q_{m,n}(f).$$

Since $\mathfrak{I}(f) = (f, e_{0,0})$, Proposition 9 follows. □

Proposition 10. *If $f \in E^a(U^2)$ with $a > 1$, then the error in the bivariate midpoint rule satisfies*

$$\mathfrak{M}_{m,n}(f) - \mathfrak{I}(f) = \mathscr{O}(m^{-a} + n^{-a}) \qquad (m, n \to \infty). \tag{3.3}$$

Proof. Since $f \in E^a(U^2)$, we have

$$\begin{aligned} Q_{m,\infty}(f) = \mathscr{O}(m^{-a}), \qquad & Q_{\infty,n}(f) = \mathscr{O}(n^{-a}), \\ Q_{m,n}(f) = \mathscr{O}(m^{-a} \cdot n^{-a}) \qquad & (m, n \to \infty), \end{aligned} \tag{3.4}$$

and (3.3) follows from Proposition 9. □

Proposition 11. *If $f \in \mathscr{C}_0^{q-1,q-1}(U^2) \cap \mathscr{C}^{q+1,q+1}(U^2)$ with $q \in \mathbb{N}$, then the error in the bivariate midpoint rule satisfies*

$$\mathfrak{M}_{m,n}(f) - \mathfrak{I}(f) = \mathscr{O}(m^{-q-1} + n^{-q-1}) \qquad (m, n \to \infty). \tag{3.5}$$

Proof. The proof of Proposition 11 is similar to that of Proposition 4. □

4. rTH-ORDER BLENDING MIDPOINT RULES

We introduce the rth-order sum of bivariate midpoint rules

$$\mathbf{T}_r^2(f) = \sum_{m=0}^{r-1} \mathfrak{M}_{2^m,2^{r-1-m}}(f) \qquad (r \in \mathbb{N}). \tag{4.1}$$

Then the *rth-order blending midpoint rule* $\mathfrak{M}_r^2(f)$ is

$$\mathfrak{M}_r^2(f) = \mathbf{T}_r^2(f) - \mathbf{T}_{r-1}^2(f), \tag{4.2}$$

where $r \in \mathbb{N}$ and $r > 1$. The construction of the rth-order blending midpoint rule is analogous to the construction of the rth-order blending rectangular rule. While the latter may be interpreted as an interpolatory cubature formula based on Boolean periodic spline interpolation, no such interpolatory characterization holds for the rth-order blending midpoint rule.

The *cubature points* of $\mathfrak{M}_r^2(f)$ are mainly determined by the points occurring in $\mathbf{T}_r^2(f)$:

$$\bigcup_{m=0}^{r-1}\{((2j+1)\cdot 2^{-m-1},\ (2k+1)\cdot 2^{-r+m}) : 0 \le j < 2^m,\ 0 \le k < 2^{r-1-m}\}. \tag{4.3}$$

Their number is given by

$$m_r = r \cdot 2^{r-1}. \tag{4.4}$$

Next we will determine a remainder formula for the rth-order blending midpoint rule.

Proposition 12. *If $f \in A(U^2)$, then the error in the rth-order blending midpoint rule is*

$$\begin{aligned}\mathfrak{M}_r^2(f) - \mathfrak{I}(f) &= Q_{2^{r-1},\infty}(f) + Q_{\infty,2^{r-1}}(f)\\ &\quad + \sum_{m=0}^{r-1} Q_{2^m,2^{r-1-m}}(f) - \sum_{m=0}^{r-2} Q_{2^m,2^{r-2-m}}(f).\end{aligned} \tag{4.5}$$

Proof. In view of relations (3.2), (3.4), (4.1), and (4.3), the proof of (4.5) is similar to that of (2.5). □

Proposition 13. *If $f \in E^a(U^2)$ with $a > 1$, then the error in the rth-order blending midpoint rule is*

$$\mathfrak{M}_r^2(f) - \mathfrak{I}(f) = \mathcal{O}(r \cdot (2^{r-1})^{-a}) \qquad (r \to \infty). \tag{4.6}$$

Proof. In view of relations (3.4) and (4.5), the proof of (4.6) is similar to that of (2.6). □

Remark 2. Recall that the number of cubature points of the rth-order blending midpoint rule $\mathfrak{M}_r^2(f)$ is mainly determined by $m_r = r \cdot 2^{r-1}$. It is easily seen that the error relation (4.6) of the rth-order blending midpoint rule obtains the form

$$\mathfrak{M}_r^2(f) - \mathfrak{I}(f) = \mathcal{O}(\log(m_r)^{a+1} \cdot (m_r)^{-a}) \qquad (r \to \infty),$$

where $f \in E^a(U^2)$ with $a > 1$. Thus, the rth-order blending midpoint rule is comparable with the bivariate number-theoretic "good lattice" rules (see Sloan [5]). Again, the attractive feature of the rth-order blending midpoint rule is its easy computation based on relations (4.1) and (4.2).

Proposition 14. *If $f \in \mathscr{C}_0^{q-1,q-1}(U^2) \cap \mathscr{C}^{q+1,q+1}(U^2)$ with $q \in \mathbb{N}$, then the error in the rth-order blending midpoint rule satisfies*

$$\mathfrak{M}_r^2(f) - \mathfrak{I}(f) = \mathcal{O}(r \cdot (2^{r-1})^{-q-1}) \qquad (r \to \infty). \tag{4.7}$$

Proof. The proof of Proposition 14 is similar to that of Proposition 7. □

5. A NUMERICAL EXAMPLE

We consider the double integral

$$\mathfrak{I}(f) = \int_0^1 \int_0^1 f(x, y)\, dx\, dy$$

with the function

$$f(x, y) = \frac{x+y}{1+x\cdot y} \qquad (x, y \in U).$$

The function f is an element of the Korobov space $E^1(U^2)$. Following Hua and Wang [4, p. 122] we introduce the function

$$g(x, y) = \tfrac{1}{4}(f(x, y) + f(x, 1-y) + f(1-x, y) + f(1-x, 1-y)).$$

It is easily seen that

$$\mathfrak{I}(g) = \mathfrak{I}(f) = 2\cdot(\log(4) - 1)$$

and

$$g \in \mathscr{C}_0^{0,0}(U^2) \cap \mathscr{C}^{2,2}(U^2).$$

It follows from relation (1.8) that Propositions 4 and 7 are applicable to g with $q = 1$. The errors and the number of cubature points for the blending rectangle rule and the ordinary rectangle rule are shown in Table 1.

TABLE 1
Errors in blending and ordinary rectangle rules

r	$(r+1)\cdot 2^r$	$\mathfrak{I}_r^2(g) - \mathfrak{I}(g)$	2^{2r}	$\mathfrak{I}_{2^r, 2^r}(g) - \mathfrak{I}(g)$
1	4	0.01009	4	0.01009
2	12	0.00365	16	0.00282
3	32	0.00120	64	0.00072
4	80	0.00037	256	0.00018
5	192	0.00011	1024	0.00005
6	448	0.00003	4096	0.00001

Similarly, it follows from relation (1.8) that Propositions 11 and 14 are applicable to g with $q = 1$. Table 2 shows the errors and the number of cubature points for the blending midpoint rule and the ordinary midpoint rule. In Figure 1 we exhibit the distribution of cubature points in rth-order sum of midpoint rules.

Remark 3. It follows from (2.3) and (4.3) that the cubature points of $\mathbf{T}_r^2(f)$ form a subset of the cubature points of $\mathbf{S}_r^2(f)$ which are not contained in the set of cubature points of $\mathbf{S}_{r-1}^2(f)$.

Remark 4. The Boolean methods for double integration can be extended to arbitrary dimensions by using the method of d-variate Boolean interpolation developed in [2]. This is the topic of a forthcoming paper.

TABLE 2
Errors in blending and ordinary midpoint rules

r	$r \cdot 2^{r-1}$	$\mathfrak{M}_r^2(g) - \mathfrak{I}(g)$	2^{2r-2}	$\mathfrak{M}_{2^{r-1},2^{r-1}}(g) - \mathfrak{I}(g)$
1	1	−0.02741	1	−0.02741
2	4	−0.00317	4	−0.00611
3	12	0.00028	16	−0.00148
4	32	0.00035	64	−0.00037
5	80	0.00016	256	−0.00009
6	192	0.00006	1024	−0.00002

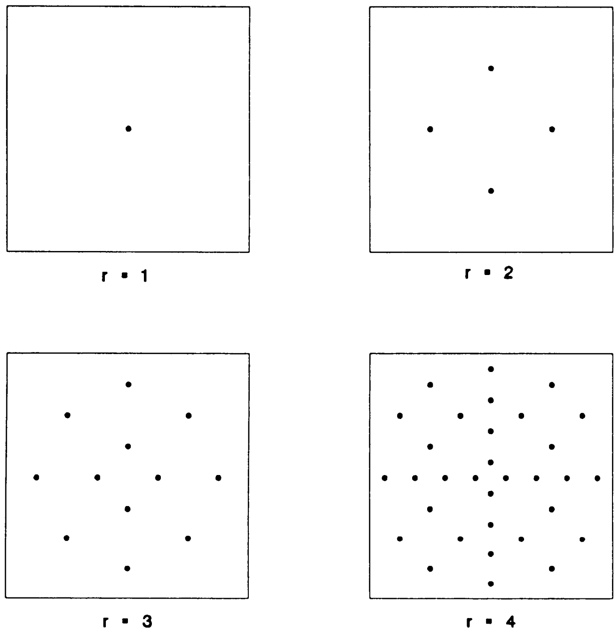

FIGURE 1
Points of rth-order sum of midpoint rules

BIBLIOGRAPHY

1. G. Baszenski and F.-J. Delvos, *Boolean methods in Fourier approximation*, Topics in Multivariate Approximation (C. K. Chui, L. L. Schumaker, and F. Utreras, eds.), Academic Press, 1987, pp. 1–11.

2. F.-J. Delvos, *d-variate Boolean interpolation*, J. Approx. Theory **34** (1982), 99–114.

3. F.-J. Delvos and H. Posdorf, *N-th order blending*, Constructive Theory of Functions of Several Variables (W. Schempp and K. Zeller, eds.), Lecture Notes in Math., vol. 571, Springer-Verlag, 1977, pp. 53–64.

4. Hua Loo Keng and Wang Yuan, *Applications of number theory to numerical analysis*, Springer-Verlag, 1981.

5. I. H. Sloan, *Lattice methods for multiple integration*, J. Comput. Appl. Math. **12–13** (1985), 131–143.

LEHRSTUHL FÜR MATHEMATIK I, UNIVERSITÄT GH SIEGEN, HÖLDERLINSTRASSE 3, D-5900 SIEGEN, WEST GERMANY