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# A TABLE OF ELLIPTIC INTEGRALS: ONE QUADRATIC FACTOR

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ABSTRACT. Integration in terms of real quantities is accomplished for 33 integrands that are rational except for the square root of a cubic or quartic polynomial with exactly one pair of conjugate complex zeros. Formulas are provided by which 45 more integrals of the same type can be expressed in terms of real quantities with the help of earlier papers. Neither limit of integration is assumed to be a singular point of the integrand. All the integrals are reduced to $R$-functions, for which Fortran programs are available. Most of the integrals are not listed in other tables.

## 1. INTRODUCTION

This paper treats integrands that are rational except for the square root of a cubic or quartic polynomial with exactly one pair of conjugate complex zeros. References [4, 5] dealt with elliptic integrals of the form

$$[p] = [p_1, \dots, p_5] = \int_y^x \prod_{i=1}^{5} (a_i + b_i t)^{p_i/2}\, dt, \tag{1.1}$$

where all quantities are real, $p_1, \dots, p_5$ are integers (omitted if 0), and the number of odd $p$'s is exactly three ("cubic cases") or four ("quartic cases"). Quartic cases were reduced by recurrence relations to the integrals

$$\begin{aligned} I_1 &= [-1, -1, -1, -1], & I_2 &= [1, -1, -1, -3], \\ I_3 &= [1, -1, -1, -1, -2], & I_3' &= [1, -1, -1, -1], \end{aligned} \tag{1.2}$$

and cubic cases were reduced to

$$I_{1c} = [-1, -1, -1], \quad I_{2c} = [1, -1, -1], \quad I_{3c} = [1, -1, -1, -2]. \tag{1.3}$$

All seven of these integrals have $p_2 = p_3$.

In §§2 and 3 we consider integrals in which $p_2 = p_3$ but $a_2 + b_2 t$ and $a_3 + b_3 t$ are conjugate complex:

$$[p_1, p_2, p_2, p_4, p_5] = \int_y^x (f + gt + ht^2)^{p_2/2} \prod_{i=1,4,5} (a_i + b_i t)^{p_i/2}\, dt, \tag{1.4}$$

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where all quantities are real, $x > y$, $f + gt + ht^2 > 0$ for all real $t$, $p_1$ and $p_2$ are odd integers, $p_4$ may be odd or even, and $p_5$ is even (zero if $p_4$ is even). Section 2 contains quartic cases ($p_4$ odd) and §3 contains cubic cases ($p_4$ even). Proofs are given in §§4 and 5. All integral formulas have been checked by numerical integration; some details of the checks are given in §6.

We assume that the integral is well defined, possibly as a Cauchy principal value, and, in particular, that $a_i + b_i t > 0$ for $y < t < x$ if $p_i$ is odd. The formulas of [4, 5] still hold but contain $a_2$, $b_2$, and their complex conjugates. The goal is to rewrite these formulas in terms of real quantities.

The integrals $I_1, I_2, \dots, I_{3c}$ are expressed in terms of four $R$-functions:

$$R_F(x, y, z) = \frac{1}{2}\int_0^\infty [(t+x)(t+y)(t+z)]^{-1/2}\,dt, \tag{1.5}$$

$$R_J(x, y, z, w) = \frac{3}{2}\int_0^\infty [(t+x)(t+y)(t+z)]^{-1/2}(t+w)^{-1}\,dt, \tag{1.6}$$

and two special cases,

$$R_C(x, y) = R_F(x, y, y) \quad \text{and} \quad R_D(x, y, z) = R_J(x, y, z, z). \tag{1.7}$$

When their last argument is negative, $R_C$ and $R_J$ are interpreted as Cauchy principal values. The functions $R_F$, $R_D$, and $R_J$ respectively replace Legendre's elliptic integrals of the first, second, and third kinds, while $R_C$ includes the inverse circular and inverse hyperbolic functions. Fortran codes for numerical computation of all four functions, including Cauchy principal values, are listed in the Supplements to [3, 4].

The main task is to express $I_1, I_2, \dots, I_{3c}$ in terms of real quantities by using Landen's transformation of $R_F$ and $R_J$. It is then simple to put formulas from [4, 5] with $p_2 = p_3$ in real form. However, cases like $[1, -1, -1, 1]$ require further work with recurrence relations because they are not listed in [4, 5], where the odd $p$'s are always in descending order.

## 2. Table of quartic cases

All quartic cases are reduced to the integrals $I_1$, $I_2$, $I_3$, and $I_3'$. We assume $x > y$, $a_1 + b_1 t > 0$ and $a_4 + b_4 t > 0$ for $y < t < x$, and $f + gt + ht^2 > 0$ for all real $t$. The first set of definitions will apply in §3 to cubic cases also:

$$X_i = (a_i + b_i x)^{1/2}, \quad Y_i = (a_i + b_i y)^{1/2}, \quad d_{ij} = a_i b_j - a_j b_i, \tag{2.1}$$

$$\alpha_i = 2fb_i - ga_i, \quad \beta_i = gb_i - 2ha_i, \quad \delta^2 = 4fh - g^2 > 0, \tag{2.2}$$

$$c_{ij}^2 = 2fb_i b_j - g(a_i b_j + a_j b_i) + 2ha_i a_j, \tag{2.3}$$

$$\xi = (f + gx + hx^2)^{1/2}, \qquad \eta = (f + gy + hy^2)^{1/2}, \tag{2.4}$$

$$A(p_1, p_2, p_2, p_4, p_5) = X_1^{p_1}\xi^{p_2}X_4^{p_4}X_5^{p_5} - Y_1^{p_1}\eta^{p_2}Y_4^{p_4}Y_5^{p_5}. \tag{2.5}$$

The following definitions are used only in quartic cases:

$$(2.6)\qquad \begin{aligned}(x-y)M &= (X_1Y_4+Y_1X_4)[(\xi+\eta)^2-h(x-y)^2]^{1/2}\\ &= (X_1Y_4+Y_1X_4)[2\xi\eta+2f+g(x+y)+2hxy]^{1/2},\end{aligned}$$

$$(2.7)\qquad L_\pm^2 = M^2+c_{14}^2\pm c_{11}c_{44},\qquad W_+^2 = M^2+d_{14}(c_{15}^2+c_{11}c_{55})/d_{15},$$

$$(2.8)\qquad (x-y)U = X_1X_4\eta+Y_1Y_4\xi,\qquad W^2 = U^2-c_{11}^2d_{45}/2d_{15},$$

$$(2.9)\qquad Q = X_5Y_5W/X_1Y_1,\qquad P^2 = Q^2+c_{55}^2d_{45}/2d_{15}.$$

When $a_5 = 1$ and $b_5 = 0$, the quantities $W^2$, $Q$, and $P^2$ reduce to

$$(2.10)\qquad W_1^2 = U^2-c_{11}^2b_4/2b_1,\quad Q_1 = W_1/X_1Y_1,\quad P_1^2 = Q_1^2+hb_4/b_1,$$

and $W_+^2-M^2$ reduces to

$$(2.11)\qquad \rho = d_{14}[\beta_1-(2h)^{1/2}c_{11}]/b_1.$$

When one limit of integration is infinite, $M$, $U$, and $Q$ become, for $x = +\infty$,

$$(2.12)\qquad \begin{aligned}M &= (b_1^{1/2}Y_4+b_4^{1/2}Y_1)(2h^{1/2}\eta+g+2hy)^{1/2},\\ U &= (b_1b_4)^{1/2}\eta+h^{1/2}Y_1Y_4,\qquad Q = (b_5/b_1)^{1/2}Y_5W/Y_1,\end{aligned}$$

or, for $y = -\infty$,

$$(2.13)\qquad \begin{aligned}M &= [(-b_1)^{1/2}X_4+(-b_4)^{1/2}X_1](2h^{1/2}\xi-g-2hx)^{1/2},\\ U &= (b_1b_4)^{1/2}\xi+h^{1/2}X_1X_4,\qquad Q = (b_5/b_1)^{1/2}X_5W/X_1.\end{aligned}$$

The four basic integrals (1.2) are

$$(2.14)\qquad I_1 = 4R_F(M^2, L_-^2, L_+^2),$$

$$(2.15)\qquad \begin{aligned}I_2 = (2c_{11}/3c_{44})[&4(c_{14}^2+c_{11}c_{44})R_D(M^2, L_-^2, L_+^2)\\ &-6R_F(M^2, L_-^2, L_+^2)+3/U]+2X_1Y_1/X_4Y_4U,\end{aligned}$$

$$(2.16)\qquad \begin{aligned}I_3 = (2c_{11}/3c_{55})[&4(d_{14}/d_{15})(c_{15}^2+c_{11}c_{55})R_J(M^2, L_-^2, L_+^2, W_+^2)\\ &-6R_F(M^2, L_-^2, L_+^2)+3R_C(U^2, W^2)]\\ &+2R_C(P^2, Q^2),\end{aligned}$$

$$(2.17)\qquad \begin{aligned}I_3' = (2c_{11}^2/9h)^{1/2}[&4\rho R_J(M^2, L_-^2, L_+^2, M^2+\rho)-6R_F(M^2, L_-^2, L_+^2)\\ &+3R_C(U^2, W_1^2)]\\ &+2R_C(P_1^2, Q_1^2).\end{aligned}$$

Integrals that converge when one limit of integration is infinite do not involve $I'_3$.

Of the 41 quartic cases listed in [4], 29 have $p_2 = p_3$. In these 29 cases the formulas given there can be used to evaluate

$$[p_1, p_2, p_2, p_4, p_5] = \int_y^x (f + gt + ht^2)^{p_2/2} \prod_{i=1,4,5} (a_i + b_i t)^{p_i/2}\, dt \tag{2.18}$$

by using (2.13) to (2.17) and usually doing a small amount of algebra to express coefficients involving $a_2$, $b_2$, $a_3$, $b_3$ in terms of $f$, $g$, and $h$. To aid in doing so, we list several identities:

$$\begin{aligned} &b_2 b_3 = h, \quad d_{2i} d_{3i} = h b_i^2 r_{2i} r_{3i} = c_{ii}^2/2, \quad d_{23}^2 = h^2 r_{23}^2 = -\delta^2, \\ &a_2 d_{3i} + a_3 d_{2i} = \alpha_i, \quad b_2 d_{3i} + b_3 d_{2i} = \beta_i, \quad d_{2i} d_{3j} + d_{2j} d_{3i} = c_{ij}^2, \\ &r_{2i} + r_{3i} = \beta_i / h b_i, \qquad r_{2i}^2 + r_{3i}^2 = (c_{ii}^2 / h b_i^2) - \delta^2/h^2, \\ &r_{2i}^{-1} + r_{3i}^{-1} = 2 b_i \beta_i / c_{ii}^2, \qquad r_{i2} - r_{3j} = r_{i2} + r_{j3} = a_i/b_i + a_j/b_j - g/h. \end{aligned} \tag{2.19}$$

We recall that $d_{ij}$ and $r_{ij} = d_{ij}/b_i b_j$ are antisymmetric in $i$ and $j$.

Instead of rewriting all 29 formulas in terms of $f$, $g$, and $h$, it should suffice to give two examples. A simple one is

$$\begin{aligned} [1, 1, 1, -3] = {} & [2(b_1 \beta_4 + h d_{14}) I'_3 + 3 b_1 c_{44}^2 I_2 - b_4 c_{11}^2 I_1]/4 b_1 b_4^2 \\ & + A(1, 1, 1, -1)/b_4, \end{aligned} \tag{2.20}$$

and one of the most complicated is

$$\begin{aligned} [1, 1, 1, -1, -4] = {} & [(\beta_5/b_5 - c_{45}^2/d_{45} + b_1 c_{55}^2/b_5 d_{15}) I_3 + 4 h I'_3/b_5 \\ & + c_{44}^2 I_2/d_{45} - c_{11}^2 I_1/d_{15} - 4A(1, 1, 1, -1, -2)]/4 b_5. \end{aligned} \tag{2.21}$$

The coefficient of $I_3$ has been slightly simplified by using the identities

$$2 h d_{ij} = b_i \beta_j - b_j \beta_i, \qquad d_{ij} \beta_k = b_i c_{jk}^2 - b_j c_{ik}^2. \tag{2.22}$$

Among integrals with $p_2 = p_3$, $\sum |p_i| \le 8$, and $\sum p_i \le 0$, there are 16 that are not listed in [4] because the odd $p$'s are not in descending order. Six are integrals of the second kind with $p_5 = 0$ and $\sum p_i < -2$:

$$\begin{aligned} [-3, -1, -1, -3] = {} & [(b_4^2 + b_1^2 c_{44}^2/c_{11}^2) I_2 - 2 b_1 b_4 I_1]/d_{14}^2 \\ & + 4 b_1^2 A(-1, 1, 1, -1)/d_{14} c_{11}^2, \end{aligned} \tag{2.23}$$

$$[-3, 1, 1, -3] = (c_{44}^2 I_2 - c_{14}^2 I_1)/d_{14}^2 + 2A(-1, 1, 1, -1)/d_{14}, \tag{2.24}$$

$$[-1, 1, 1, -5] = [-c_{14}^2 I_2 + c_{11}^2 I_1 - 2 d_{14} A(1, 1, 1, -3)]/3 d_{14}^2, \tag{2.25}$$

$$\begin{aligned} [1, -3, -3, -1] = {} & [\beta_4 I_2 - \beta_1 I_1 + 2(g + 2hx) X_1 / X_4 \xi \\ & - 2(g + 2hy) Y_1 / Y_4 \eta]/\delta^2, \end{aligned} \tag{2.26}$$

$$[1, -3, -3, 1] = [-c_{44}^2 I_2 + c_{14}^2 I_1 - 2(\alpha_4 + \beta_4 x)X_1/X_4\xi + 2(\alpha_4 + \beta_4 y)Y_1/Y_4\eta]/\delta^2, \tag{2.27}$$

$$\begin{aligned}[-1, -3, -3, -1] = (2/\delta^2 c_{11}^2)\{&(\delta^2 b_1 b_4 - hc_{14}^2)I_2 + hc_{11}^2 I_1 \\ &+ 2[\delta^2 b_1 - h(\alpha_1 + \beta_1 x)]X_1/X_4\xi \\ &- 2[\delta^2 b_1 - h(\alpha_1 + \beta_1 y)]Y_1/Y_4\eta\}.\end{aligned} \tag{2.28}$$

Four integrals of the third kind with $p_5 = 0$ or 2 and $\sum p_i \geq -2$ involve $I_3'$ but not $I_3$:

$$\begin{aligned}[1, -1, -1, 1] = {}&[(-b_1\beta_4 - b_4\beta_1)I_3' + b_1 c_{44}^2 I_2 - b_4 c_{11}^2 I_1]/4hb_1 \\ &+ b_4 A(1, 1, 1, -1)/h,\end{aligned} \tag{2.29}$$

$$[-1, 1, 1, -3] = (2hd_{14}I_3' + b_1 c_{44}^2 I_2 - b_4 c_{11}^2 I_1)/2b_1 b_4 d_{14}, \tag{2.30}$$

$$\begin{aligned}[-1, 1, 1, -1] = {}&[(b_1\beta_4 + b_4\beta_1)I_3' + b_1 c_{44}^2 I_2 + b_4 c_{11}^2 I_1]/4b_1^2 b_4 \\ &+ A(1, 1, 1, -1)/b_1,\end{aligned} \tag{2.31}$$

$$\begin{aligned}[-1, 1, 1, -3, 2] = {}&[(b_1 b_5 \beta_4 + b_4 b_5 \beta_1 - 4hb_1 d_{45})I_3' \\ &+ (b_5 - 2b_1 d_{45}/d_{14})b_1 c_{44}^2 I_2 \\ &+ (b_5 + 2b_1 d_{45}/d_{14})b_4 c_{11}^2 I_1]/4b_1^2 b_4^2 \\ &+ b_5 A(1, 1, 1, -1)/b_1 b_4.\end{aligned} \tag{2.32}$$

In the last integral, $a_5$ and $b_5$ are unrestricted.

Six more quartic cases with $p_5 = -2$ or $-4$ involve $I_3$, and the three with $\sum p_i \geq -2$ involve $I_3'$ also:

$$[1, -1, -1, 1, -2] = (d_{45}I_3 + b_4 I_3')/b_5, \tag{2.33}$$

$$\begin{aligned}[3, -1, -1, 1, -2] = {}&d_{15}d_{45}I_3/b_5^2 \\ &+ [(4hb_4 d_{15}/b_5 - b_1\beta_4 - b_4\beta_1)I_3' \\ &+ b_1 c_{44}^2 I_2 - b_4 c_{11}^2 I_1 \\ &+ 4b_1 b_4 A(1, 1, 1, -1)]/4hb_5,\end{aligned} \tag{2.34}$$

$$[-1, 1, 1, -1, -2] = (b_1 c_{55}^2 I_3 + 2hd_{15}I_3' - b_5 c_{11}^2 I_1)/2b_1 b_5 d_{15}, \tag{2.35}$$

$$[-1, 1, 1, -3, -2] = (d_{14}c_{55}^2 I_3 - d_{15}c_{44}^2 I_2 + d_{45}c_{11}^2 I_1)/2d_{14}d_{15}d_{45}, \tag{2.36}$$

$$\begin{aligned}[1, -1, -1, 1, -4] = {}&[(2d_{15}c_{45}^2 - d_{14}c_{55}^2)I_3 + d_{15}c_{44}^2 I_2 \\ &- d_{45}c_{11}^2 I_1]/2d_{15}c_{55}^2 \\ &- 2d_{45}A(1, 1, 1, -1, -2)/c_{55}^2,\end{aligned} \tag{2.37}$$

$$[-1, 1, 1, -1, -4] = [(\beta_5/b_5 - c_{45}^2/d_{45} - b_1 c_{55}^2/b_5 d_{15}) I_3 + c_{44}^2 I_2/d_{45} + c_{11}^2 I_1/d_{15} - 4A(1, 1, 1, -1, -2)]/4d_{15}. \tag{2.38}$$

## 3. Table of cubic cases

In addition to (2.1) to (2.5), the following definitions are used in cubic cases. They are obtained from (2.6) to (2.16), with omission of (2.10), by putting $a_4 = 1$ and $b_4 = 0$ and subsequently replacing $(a_5, b_5)$ by $(a_4, b_4)$:

$$\begin{aligned}(x-y)M &= (X_1 + Y_1)[(\xi+\eta)^2 - h(x-y)^2]^{1/2} \\ &= (X_1 + Y_1)[2\xi\eta + 2f + g(x+y) + 2hxy]^{1/2},\end{aligned} \tag{3.1}$$

$$L_\pm^2 = M^2 - \beta_1 \pm (2h)^{1/2} c_{11}, \qquad W_+^2 = M^2 - b_1(c_{14}^2 + c_{11}c_{44})/d_{14}, \tag{3.2}$$

$$(x-y)U = X_1\eta + Y_1\xi, \qquad W^2 = U^2 - c_{11}^2 b_4/2d_{14}, \tag{3.3}$$

$$Q = X_4 Y_4 W/X_1 Y_1, \qquad P^2 = Q^2 + c_{44}^2 b_4/2d_{14}, \tag{3.4}$$

$$\rho = (2h)^{1/2} c_{11} - \beta_1. \tag{3.5}$$

When one limit of integration is infinite, $M^2$, $U$, and $Q$ become, for $x = +\infty$,

$$\begin{aligned}M^2 = b_1(2h^{1/2}\eta + g + 2hy), \qquad U = h^{1/2} Y_1, \\ Q = (b_4/b_1)^{1/2} Y_4 W/Y_1,\end{aligned} \tag{3.6}$$

or, for $y = -\infty$,

$$\begin{aligned}M^2 = -b_1(2h^{1/2}\xi - g - 2hx), \qquad U = h^{1/2} X_1, \\ Q = (b_4/b_1)^{1/2} X_4 W/X_1.\end{aligned} \tag{3.7}$$

The three basic integrals (1.3) are

$$I_{1c} = 4R_F(M^2, L_-^2, L_+^2), \tag{3.8}$$

$$\begin{aligned}I_{2c} = (2c_{11}^2/9h)^{1/2}[4\rho R_D(M^2, L_-^2, L_+^2) \\ - 6R_F(M^2, L_-^2, L_+^2) + 3/U] + 2X_1Y_1/U,\end{aligned} \tag{3.9}$$

$$\begin{aligned}I_{3c} = (2c_{11}/3c_{44})[(-4b_1/d_{14})(c_{14}^2 + c_{11}c_{44}) R_J(M^2, L_-^2, L_+^2, W_+^2) \\ - 6R_F(M^2, L_-^2, L_+^2) + 3R_C(U^2, W^2)] \\ + 2R_C(P^2, Q^2).\end{aligned} \tag{3.10}$$

When one limit of integration is infinite, the integral (1.1) with $\prod b_i \neq 0$ converges only if $\sum p_i < -2$. To allow for the possibility of an infinite limit of integration, integrals with $\sum p_i < -2$ should not be reduced in terms of $I_{2c}$, which has $\sum p_i = -1$ by (1.3). In [5] an integral called $J_{2c}$ was used in such cases, but in the present context $J_{2c}$ is complex and is replaced by

$$N_{2c} = (8h/9c_{11}^2)^{1/2}[4\rho R_D(M^2, L_-^2, L_+^2) - 6R_F(M^2, L_-^2, L_+^2) + 3/U] + 2/X_1Y_1U. \tag{3.11}$$

The last term vanishes if one limit of integration is infinite.

Of the 40 cubic cases with $\sum |p_i| \le 7$ and $\sum p_i \le 3$ listed in [5], 20 have $p_2 = p_3$. For brevity, some of the formulas contain the quantities $J_{1c}$ and $K_{2c}$, which can be computed in the present context from

$$\begin{aligned} J_{1c} &= c_{11}^2 I_{1c}/2 - 2b_1A(1, 1, 1), \\ K_{2c} &= c_{11}^2 N_{2c}/2 - 2d_{14}A(-1, 1, 1, -2). \end{aligned} \tag{3.12}$$

In 18 of the 20 cases the formulas given in [5] can be used to evaluate

$$[p_1, p_2, p_2, p_4] = \int_y^x (f + gt + ht^2)^{p_2/2} \prod_{i=1,4} (a_i + b_it)^{p_i/2}\, dt \tag{3.13}$$

by expressing coefficients involving $a_2$, $b_2$, $a_3$, $b_3$ in terms of $f$, $g$, and $h$ with the help of identities (2.19). Since the remaining two cases, as well as $[-3, -3, -3]$, involve the complex quantity $J_{2c}$, they are listed here in terms of $N_{2c}$:

$$[1, -3, -3] = [-c_{11}^2 N_{2c} - \beta_1 I_{1c} - 2(\alpha_1 + \beta_1 x)/X_1\xi + 2(\alpha_1 + \beta_1 y)/Y_1\eta]/\delta^2, \tag{3.14}$$

$$[-1, -3, -3] = [\beta_1 N_{2c} + 2hI_{1c} + 2(g + 2hx)/X_1\xi - 2(g + 2hy)/Y_1\eta]/\delta^2, \tag{3.15}$$

$$\begin{aligned} [-3, -3, -3] = (2/\delta^2 c_{11}^2)[(2b_1^2\delta^2 - hc_{11}^2)N_{2c} - h\beta_1 I_{1c} \\ - 2h(\alpha_1 + \beta_1 x)/X_1\xi + 2h(\alpha_1 + \beta_1 y)/Y_1\eta] \\ + 4b_1A(-1, -1, -1)/c_{11}^2. \end{aligned} \tag{3.16}$$

Among integrals with $p_2 = p_3$, $\sum |p_i| \le 7$, and $\sum p_i \le 3$, there are 12 that are not listed in [5] because the odd $p$'s are not in descending order. Five are integrals of the second kind with $p_4 = 0$:

$$[-3, -1, -1] = N_{2c}, \tag{3.17}$$

$$[-1, 1, 1] = [\beta_1 I_{2c} + c_{11}^2 I_{1c} + 2b_1A(1, 1, 1)]/3b_1^2, \tag{3.18}$$

$$[-3, 1, 1] = (2hI_{2c} + \beta_1 I_{1c})/b_1^2 - 2A(-1, 1, 1)/b_1, \tag{3.19}$$

$$[-5, -1, -1] = (-2/3c_{11}^2)[2\beta_1 N_{2c} + hI_{1c} + 2b_1A(-3, 1, 1)], \tag{3.20}$$

$$[-5, 1, 1] = [\beta_1 N_{2c} + 2hI_{1c} - 2b_1A(-3, 1, 1)]/3b_1^2. \tag{3.21}$$

Three integrals of the second kind have $p_4 = 2$ with no restrictions on $a_4$ or $b_4$:

$$(3.22)\qquad [-3, -1, -1, 2] = (-d_{14}N_{2c} + b_4 I_{1c})/b_1,$$

$$(3.23)\qquad \begin{aligned}[-3, 1, 1, 2] = {} & [(b_4\beta_1 - 6hd_{14})I_{2c} + (b_4 c_{11}^2 - 3d_{14}\beta_1)I_{1c}]/3b_1^3 \\ & + 2[d_{14}A(-1, 1, 1) + b_4 A(1, 1, 1)/3]/b_1^2,\end{aligned}$$

$$(3.24)\qquad \begin{aligned}[-1, 1, 1, 2] = {} & [(2b_1^2 b_4\delta^2 - hb_4c_{11}^2 - 5hd_{14}\beta_1)I_{2c} \\ & - (b_4\beta_1 + 10hd_{14})J_{1c}]/15hb_1^3 \\ & + 2[b_4A(3, 1, 1)/5 - d_{14}A(1, 1, 1)]/b_1^2.\end{aligned}$$

The final four cases are integrals of the third kind with $p_4 = -2$ or $-4$:

$$(3.25)\qquad [-3, -1, -1, -2] = (b_4^2 I_{3c} - b_1 d_{14} N_{2c} - b_1 b_4 I_{1c})/d_{14}^2,$$

$$(3.26)\qquad \begin{aligned}[-3, 1, 1, -2] = {} & [b_1 c_{44}^2 I_{3c} - d_{14}c_{11}^2 N_{2c} \\ & - (b_4c_{11}^2 + 2d_{14}\beta_1)I_{1c}]/2b_1 d_{14}^2,\end{aligned}$$

$$(3.27)\qquad [-1, 1, 1 - 2] = (b_1c_{44}^2 I_{3c} + 2hd_{14}I_{2c} - b_4c_{11}^2 I_{1c})/2b_1b_4d_{14},$$

$$(3.28)\qquad [-1, 1, 1, -4] = [(2\beta_4 d_{14} - b_1c_{44}^2)I_{3c} + 2d_{14}K_{2c} + b_4c_{11}^2 I_{1c}]/4b_4d_{14}^2.$$

## 4. The basic integrals

To derive the expressions given in §§2 and 3 for the seven integrals $I_1$, $I_2, \dots, I_{3c}$ in terms of real quantities, it suffices to deal with $I_1$ and $I_3$ because the other five can be obtained from these. Although $I_1$ was treated in [2], we shall redo it here to have uniform notation and to prepare for $I_3$.

By [4, (2.13), (2.2), (2.3)] we have

$$(4.1)\qquad I_1 = [-1, -1, -1, -1] = 2R_F(U_{12}^2, U_{13}^2, U_{14}^2),$$

$$(4.2)\qquad \begin{gathered}(x-y)U_{ij} = X_iX_jY_kY_m + Y_iY_jX_kX_m, \\ X_i = (a_i + b_ix)^{1/2}, \qquad Y_i = (a_i + b_iy)^{1/2},\end{gathered}$$

where $i, j, k, m$ is any permutation of $1, 2, 3, 4$. Because $a_1 + b_1 t$ and $a_4 + b_4 t$ are assumed to be strictly positive on the open interval of integration, $X_1$, $Y_1$, $X_4$, and $Y_4$ are real and nonnegative. From $a_2 = \overline{a}_3$ and $b_2 = \overline{b}_3$, where an overbar denotes complex conjugation, it follows that $X_2 = \overline{X}_3$, $Y_2 = \overline{Y}_3$, $U_{12} = \overline{U}_{13}$, and $U_{14} \geq 0$. The real quantities $f$, $g$, and $h$ satisfy

$$(4.3)\qquad (a_2 + b_2t)(a_3 + b_3t) = f + gt + ht^2 > 0, \qquad -\infty < t < \infty.$$

Because only $f$, $g$, and $h$ are given, we may choose $b_2 = b_3 = h^{1/2}$ and $\mathrm{Im}(a_2) > 0$. If we assume $x$ and $y$ to be finite and take the principal branch of the square roots in (4.2), then $X_2$ and $Y_2$ lie in the open first quadrant of the complex plane, $X_3$ and $Y_3$ lie in the open fourth quadrant, and $X_2Y_3$ and $Y_2X_3$ have positive real part. Since we assume $x > y$ and since $X_1Y_4$ and $Y_1X_4$ cannot both vanish if $I_1$ is finite, we conclude that $\mathrm{Re}\, U_{12} = \mathrm{Re}\, U_{13} > 0$.

The variables of $R_F$ can be made real and nonnegative by Landen's transformation [6, (5.5)],

$$(4.4)\quad \begin{aligned} &R_F(U_{12}^2, U_{13}^2, U_{14}^2) = 2R_F(M^2, L_-^2, L_+^2), \qquad M = U_{12} + U_{13}, \\ &L_\pm = [(U_{12} + U_{14})(U_{13} + U_{14})]^{1/2} \pm [(U_{12} - U_{14})(U_{13} - U_{14})]^{1/2}, \\ &L_+L_- = 2MU_{14}, \qquad L_\pm^2 - M^2 = [(U_{14}^2 - U_{12}^2)^{1/2} \pm (U_{14}^2 - U_{13}^2)^{1/2}]^2. \end{aligned}$$

Recalling that $U_{14} \geq 0$, $U_{12} = \overline{U}_{13}$, and $\mathrm{Re}\, U_{12} > 0$, we see that $M > 0$, $L_+ > 0$, and $L_- \geq 0$, with equality if and only if $U_{14} = 0$ (the integral being then called complete). The last equation in (4.4) shows that $L_+^2 \geq M^2 \geq L_-^2$. Since

$$(4.5)\quad (x - y)M = (X_1Y_4 + Y_1X_4)(X_2Y_3 + Y_2X_3)$$

and

$$(X_2Y_3 + Y_2X_3)^2 = (X_2X_3 + Y_2Y_3)^2 - (X_2^2 - Y_2^2)(X_3^2 - Y_3^2),$$

we define

$$(4.6)\quad \xi = X_2X_3 = (f + gx + hx^2)^{1/2}, \qquad \eta = Y_2Y_3 = (f + gy + hy^2)^{1/2}$$

and obtain

$$(4.7)\quad (x - y)M = (X_1Y_4 + Y_1X_4)[(\xi + \eta)^2 - h(x - y)^2]^{1/2}.$$

Equation (4.2) implies

$$(4.8)\quad U_{ij}^2 - U_{ik}^2 = d_{im}d_{jk}, \qquad d_{ij} = a_ib_j - a_jb_i,$$

where $i, j, k, m$ is any permutation of $1, 2, 3, 4$. From (4.4) we see that

$$(4.9)\quad \begin{aligned} L_\pm^2 - M^2 &= [(d_{13}d_{42})^{1/2} \pm (d_{12}d_{43})^{1/2}]^2 \\ &= d_{12}d_{43} + d_{42}d_{13} \pm (2d_{12}d_{13})^{1/2}(2d_{42}d_{43})^{1/2} \\ &= c_{14}^2 \pm c_{11}c_{44}, \end{aligned}$$

where

$$(4.10)\quad \begin{aligned} c_{ij}^2 &= d_{i2}d_{j3} + d_{j2}d_{i3} \\ &= 2b_2b_3a_ia_j - (a_2b_3 + a_3b_2)(a_ib_j + a_jb_i) + 2a_2a_3b_ib_j \\ &= 2fb_ib_j - g(a_ib_j + a_jb_i) + 2ha_ia_j. \end{aligned}$$

We note that the arguments of $R_F$ in (4.4) differ by amounts that are independent of $x$ and $y$. The assumption that $f + gt + ht^2 > 0$ for all real $t$ implies

$f > 0$, $h > 0$, and $g^2 - 4fh < 0$. Hence, $c_{ii}^2$ is a positive definite quadratic form in the variables $a_i$ and $b_i$, and so we may take $c_{ii}$ to be real and positive.

The integral $I_1$ has now been expressed in terms of real quantities. Although the transformation (4.4) is unnecessary unless $f + gt + ht^2$ has complex zeros, it can still be used if $f + gt + ht^2$ has real zeros that do not interlace the zeros of $(a_1 + b_1 t)(a_4 + b_4 t)$. If they do interlace, the last member of the equation

$$(L_+^2 - L_-^2)^2 = 4c_{11}^2 c_{44}^2 = 16 d_{12} d_{13} d_{42} d_{43} \tag{4.11}$$

is negative, and $L_+$ and $L_-$ are then conjugate complex.

We turn next to $I_3 = [1, -1, -1, -1, -2]$, expressed by [4, (2.15), (2.5), (2.9)] as

$$\begin{gathered} I_3 = (2d_{12}d_{13}d_{14}/3d_{15}) R_J(U_{12}^2, U_{13}^2, U_{14}^2, W^2) + 2R_C(P^2, Q^2), \\ W^2 = U_{14}^2 - d_{12}d_{13}d_{45}/d_{15}, \qquad Q = X_5 Y_5 W / X_1 Y_1, \\ P^2 = Q^2 + d_{25}d_{35}d_{45}/d_{15}. \end{gathered} \tag{4.12}$$

Each $d$ or $U$ with a subscript 3 is the complex conjugate of the corresponding quantity with a subscript 2. By (4.10) we have

$$d_{12}d_{13} = c_{11}^2/2, \qquad d_{25}d_{35} = c_{55}^2/2. \tag{4.13}$$

Since $U_{14} \geq 0$, $W^2$ and the arguments of $R_C$ are real, and we need only express $R_J$ in terms of real quantities by Landen's transformation. In [6, (8.5)] we put $(x, y, z, w) = (U_{12}, U_{13}, U_{14}, W)$ and $(\alpha, z_\pm, w_\pm) = (M, L_\pm, W_\pm)/2$ to obtain

$$\begin{aligned} &\tfrac{1}{4}(W_+^2 - W_-^2) R_J(U_{12}^2, U_{13}^2, U_{14}^2, W^2) \\ &\quad = 4(W_+^2 - M^2) R_J(M^2, L_-^2, L_+^2, W_+^2) \\ &\qquad - 6R_F(M^2, L_-^2, L_+^2) + 3R_C(U_{14}^2, W^2), \end{aligned} \tag{4.14}$$

where $M$ and $L_\pm$ are given by (4.4) and where [6, (7.2)] implies

$$W_\pm^2 - M^2 = [(W^2 - U_{12}^2)^{1/2} \pm (W^2 - U_{13}^2)^{1/2}]^2. \tag{4.15}$$

From [4, (2.9)] we find

$$W^2 - U_{12}^2 = -d_{13}d_{14}d_{25}/d_{15}, \qquad W^2 - U_{13}^2 = -d_{12}d_{14}d_{35}/d_{15}, \tag{4.16}$$

from which it follows by (4.10) that

$$\begin{aligned} W_\pm^2 - M^2 &= (d_{14}/d_{15})(-d_{13}d_{25} - d_{12}d_{35}) \\ &\quad \pm 2(d_{14}^2/d_{15}^2)^{1/2}(d_{12}d_{13}d_{25}d_{35})^{1/2} \\ &= (d_{14}/d_{15})(c_{15}^2 \pm c_{11}c_{55}). \end{aligned} \tag{4.17}$$

We are free to choose $(d_{14}^2/d_{15}^2)^{1/2}$ to be $d_{14}/d_{15}$ regardless of the sign of the latter quantity because (4.14) still holds if $W_+$ and $W_-$ are interchanged (see the remark following [6, (8.5)]). Substitution in (4.14) and then in (4.12) leads to (2.15), wherein $U_{14}$ is abbreviated to $U$.

In $I_3$ we put $a_5 = a_4$ and $b_5 = b_4$ to get $I_2$, or $a_5 = 1$ and $b_5 = 0$ to get $I_3'$. To obtain $I_{1c}$, $I_{2c}$, and $I_{3c}$ from $I_1$, $I_2$, and $I_3$, we put $a_4 = 1$ and $b_4 = 0$ and subsequently replace the subscript 5 by 4. The quantity $N_{2c}$ is defined by

$$c_{11}^2 N_{2c}/2 = hI_{2c} - 2b_1 A(-1, 1, 1). \tag{4.18}$$

Both terms on the right side become infinite if one limit of integration is infinite. Substitution of (3.9) and use of the identity

$$hX_1^2 Y_1^2 - b_1 U X_1 Y_1 A(-1, 1, 1) = c_{11}^2/2 \tag{4.19}$$

lead to (3.11), in which all terms remain finite. We note from (3.17) that $N_{2c} = [-3, -1, -1]$. From [5, (2.59)] we have

$$K_{2c} = hI_{2c} - 2b_4 A(1, 1, 1, -2), \tag{4.20}$$

which implies (3.12) by way of (4.18) and [4, (4.8)]. In deriving (3.14) to (3.16), it is necessary to use

$$b_3 J_{2c} = hI_{2c} - 2b_3 A(1, 1, -1) = c_{11}^2 N_{2c}/2 - 2d_{13} A(-1, 1, -1), \tag{4.21}$$

where the first equality comes from [5, (2.17)] and the second from (4.18) and [4, (4.8)].

Defining

$$\delta^2 = 4fh - g^2 \tag{4.22}$$

and noting from (4.3) that

$$a_2 b_3 + a_3 b_2 = g, \quad (a_2 b_3)(a_3 b_2) = fh,$$

we find

$$2a_2 b_3 = g + i\delta, \quad 2a_3 b_2 = g - i\delta, \quad d_{23} = a_2 b_3 - a_3 b_2 = i\delta. \tag{4.23}$$

The last equation is used in (2.19).

## 5. Use of recurrence relations

The 16 quartic cases (2.23) to (2.38) are obtained by recurrence relations with occasional help from the integrals listed in [4]. Let $e_i$ denote an $n$-tuple with 1 in the $i$th place and 0's elsewhere (for example, $[p + 2e_1] = [p_1 + 2, p_2, \dots, p_n]$). The four relations used most frequently are reproduced from [4] for convenience:

$$(p_1 + \cdots + p_n + 2) b_i [p] = \sum_{j \neq i} p_j d_{ji} [p - 2e_j] + 2A(p + 2e_i), \tag{A$i$}$$

$$d_{ij}[p] = b_j[p + 2e_i] - b_i[p + 2e_j], \tag{B$ij$}$$

$$(\mathrm{C}ij) \qquad b_j[p] = b_i[p - 2e_i + 2e_j] + d_{ij}[p - 2e_i],$$

$$(\mathrm{D}ijk) \qquad d_{ij}[p] = d_{kj}[p + 2e_i - 2e_k] + d_{ik}[p + 2e_j - 2e_k].$$

A fifth relation [3, (5.6)], with $i = 1$ and $[p] = [-1, -1, -1, -3]$, is used to get (2.23). After obtaining $[-1, 1, -1, -3]$ and $[-1, -1, 1, -3]$ from (D142) and (D143), respectively, we can find (2.24) by choosing $[p] = [-1, 1, 1, -3]$ in (A4), and we can find (2.25) from (D143). After obtaining $[1, -1, -3, -1]$ from (D234) and interchanging subscripts 2 and 3 to get $[1, -3, -1, -1]$, we can find (2.26) from (B23) and also (2.27) from (C42). Similarly, after obtaining $[-1, -1, -3, -1]$ from (B13) and interchanging subscripts 2 and 3 to get $[-1, -3, -1, -1]$, we can find (2.28) from (B23). The ten quartic cases of the third kind, (2.29) to (2.38), follow in order from (C42), (C31), (C41), (C54), (C45), (C15), (B15), (B45), (C45), and (B15).

To reduce the first cubic case, (3.14), we first express $[1, -1, -3]$ in terms of $N_{2c}$ by using [5, (2.25)] and (4.21), interchange $(a_2, b_2)$ with $(a_3, b_3)$ to get $[1, -3, -1]$, and then obtain $[1, -3, -3]$ from (B23). Exactly the same procedure, starting with $[-1, -1, -3]$ from [5, (2.26)], yields (3.15). By choosing $[p] = [-1, -3, -3]$ and $i = 1$ in [3, (5.5)], we can then get (3.16). Equations (3.17), (3.18), and (3.19) follow from putting $a_4 = 1$ and $b_4 = 0$ in (2.23), (2.25), and (2.24), respectively. In the first case, and in later cases where necessary, $I_{2c}$ is expressed in terms of $N_{2c}$ by using (4.18). To get (3.20) we choose $[p] = [-3, -1, -1]$ and $i = 1$ in [3, (5.5)]. The same procedure, with $[p] = [-3, 1, 1]$, yields (3.21) with the help of the identity

$$\begin{aligned} b_i b_j A(p + 4e_k) + (b_i d_{jk} + b_j d_{ik}) A(p + 2e_k) + d_{ik} d_{jk} A(p) \\ = b_k^2 A(p + 2e_i + 2e_j). \end{aligned} \tag{5.1}$$

Equations (3.22), (3.23), and (3.24) all come from (C41), while (3.25) and (3.26) come from (B14). The final two formulas, (3.27) and (3.28), are obtained by putting $a_4 = 1$ and $b_4 = 0$ and subsequently replacing the subscript 5 by 4 in (2.36) and (2.38), respectively.

## 6. Numerical checks

The 18 quartic cases in §2 and the 15 cubic cases in §3 were checked numerically when $x = 2.0$, $y = 0.5$, $(a_1, b_1) = (0.3, 0.2)$, $(f, g, h) = (0.4, -0.2, 0.1)$, $(a_4, b_4) = (0.9, -0.3)$, and $(a_5, b_5) = (0.4, 0.5)$. In each case the integral on the left side, defined by (2.18) or (3.13), was integrated numerically by the SLATEC code QNG. On the right side the seven basic integrals $I_1$, $I_2, \dots, I_{3c}$ were calculated from (2.14) to (2.17) and (3.8) to (3.10) by using the codes for $R$-functions in the Supplements to [3, 4]. The $A$'s were calculated by a simple code, and the remaining calculations were done with a hand calculator. For each of the 33 cases the values obtained for the two sides agreed to better than one part in a million.

Some intermediate values for the quartic cases are:

$$
\begin{aligned}
&M^2 = 0.62249271, &&R_F(M^2, L_-^2, L_+^2) = 1.2543726,\\
&L_-^2 = 0.54993185, &&R_D(M^2, L_-^2, L_+^2) = 1.7960842,\\
&L_+^2 = 0.74305357, &&R_J(M^2, L_-^2, L_+^2, W_+^2) = -0.99822609,\\
&W_+^2 = -0.54216139, &&R_C(U^2, W^2) = 1.7237432,\\
&U^2 = 0.16410988, &&R_C(P^2, Q^2) = 0.98880184,\\
&W^2 = -0.13717583, &&R_J(M^2, L_-^2, L_+^2, M^2+\rho) = 1.5689637,\\
&M^2 + \rho = 0.92172730, &&R_C(U^2, W_1^2) = 2.2358652,\\
&W_1^2 = 0.21960988, &&R_C(P_1^2, Q_1^2) = 1.16864877,
\end{aligned}
$$

$$
\begin{aligned}
&A(1, 1, 1, -1) = 0.54975858, &&I_1 = 5.0174903,\\
&A(1, 1, 1, -1, -2) = 0.04955294, &&I_2 = 5.8882786,\\
&A(-1, 1, 1, -1) = 0.33929812, &&I_3 = 2.7228427,\\
&A(1, 1, 1, -3) = 2.6651950, &&I_3' = 2.7668674.
\end{aligned}
$$

Some intermediate values for the cubic cases are:

$$
\begin{aligned}
&M^2 = 1.1713435, &&R_F(M^2, L_-^2, L_+^2) = 0.89978529,\\
&L_-^2 = 1.1496883, &&R_D(M^2, L_-^2, L_+^2) = 0.67751039,\\
&L_+^2 = 1.3929988, &&R_J(M^2, L_-^2, L_+^2, W_+^2) = 0.71986645,\\
&W_+^2 = 1.2606479, &&R_C(U^2, W^2) = 1.7844272,\\
&U^2 = 0.34181141, &&R_C(P^2, Q^2) = 1.9470611,\\
&W^2 = 0.30070030, &&
\end{aligned}
$$

$$
\begin{aligned}
&A(-1, -1, -1) = -0.88367862, &&I_{1c} = 3.5991412,\\
&A(-1, 1, 1) = -0.14545887, &&I_{2c} = 1.9453098,\\
&A(-1, 1, 1, -2) = 1.3179127, &&I_{3c} = 4.0022901,\\
&A(1, 1, 1) = 0.16859514, &&N_{2c} = 6.8301223,\\
&A(-3, 1, 1) = -1.1735711, &&J_{1c} = 0.06573017,\\
&A(3, 1, 1) = 0.22618313, &&K_{1c} = 0.96438740.
\end{aligned}
$$

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