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# CONVERGENCE OF A SECOND-ORDER SCHEME FOR SEMILINEAR HYPERBOLIC EQUATIONS IN $2+1$ DIMENSIONS

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ABSTRACT. A second-order energy-preserving scheme is studied for the solution of the semilinear Cauchy Problem $u_{tt} - u_{xx} - u_{yy} + u^3 = 0$ $(t > 0; x, y \in \mathbb{R})$. Smooth data functions of compact support are prescribed at $t = 0$. On any time interval $[0, T]$, second-order convergence (up to logarithmic corrections) to the exact solution is established in both the energy and uniform norms.

## 1. INTRODUCTION

We study the numerical approximation to a smooth solution of the semilinear Cauchy Problem

$$u_{tt} - u_{xx} - u_{yy} + u^3 = 0 \qquad (t > 0,\ x, y \in \mathbb{R}), \tag{1}$$

$$u(0, x, y) = \phi(x, y), \qquad u_t(0, x, y) = \psi(x, y). \tag{2}$$

The data $\phi$, $\psi$ are to be smooth and of compact support. The cubic nonlinearity is chosen for convenience; the method easily extends to any odd power $u^{2N+1}$, $N = 1, 2, \ldots$.

An energy-preserving scheme for equations as above has been known for some years (cf. [8]). We have proved convergence for the analogous one-dimensional situation (unpublished); the same result is obtained in [9]. In both cases, an $L^\infty$-bound on the discrete solution follows easily from the Sobolev inequality and the standard energy estimate. However, in space dimension two, one would need $L^2$-estimates on *second derivatives* to conclude such an $L^\infty$-bound. Yet, in analogy to the continuous case, it is difficult to obtain $H^2$-estimates directly: an $L^\infty$-bound seems to be needed first. The novel aspect of the present work is that we use a recently obtained *representation* for the discrete solution in order to establish the convergence to the exact solution in the uniform norm. The rate of convergence is second-order, modulo logarithmic factors. As pointed out above, $H^2$-convergence then follows as well.

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To describe the scheme for (1), choose a space stepsize $\Delta x = \Delta y \equiv h > 0$, and let $\Delta t > 0$ be the time step. As usual, we denote

$$x_k = kh, \quad y_j = jh, \quad t^n = n\Delta t \qquad (k, j \in \mathbb{Z}, \ n \in \mathbb{N}), \tag{3}$$

$$u_{kj}^n \cong u(t^n, x_k, y_j), \tag{4}$$

where $u$ is the exact solution to (1), (2). We choose the largest possible time step $\Delta t$ as allowed by the classical CFL condition:

$$\Delta t = h/\sqrt{2}. \tag{5}$$

We let $L_h$ be the usual approximation to the wave operator $\partial_t^2 - \Delta$ via centered second-order differences:

$$(L_h u)_{kj}^n \equiv \frac{u_{kj}^{n+1} - 2u_{kj}^n + u_{kj}^{n-1}}{\Delta t^2} - \left[\frac{u_{k+1,j}^n - 2u_{kj}^n + u_{k-1,j}^n}{h^2}\right] - \left[\frac{u_{k,j+1}^n - 2u_{kj}^n + u_{k,j-1}^n}{h^2}\right]. \tag{6}$$

Lastly, we write

$$G(u) = \int_0^u s^3\, ds = \frac{1}{4}u^4. \tag{7}$$

Then the scheme we analyze is

$$(L_h u)_{kj}^n + \frac{G(u_{kj}^{n+1}) - G(u_{kj}^{n-1})}{u_{kj}^{n+1} - u_{kj}^{n-1}} = 0 \tag{8}$$

with the initial values

$$u_{kj}^0 = \phi_{kj}, \qquad u_{kj}^1 = \phi_{kj} + \Delta t \cdot \psi_{kj} + \tfrac{1}{2}\Delta t^2 \Phi_{kj}, \tag{9}$$

where

$$\Phi_{kj} = h^{-2}[\phi_{k+1,j} + \phi_{k-1,j} + \phi_{k,j+1} + \phi_{k,j-1} - 4\phi_{kj}] - \phi_{kj}^3.$$

Thus we are required to solve a nonlinear implicit equation at every time step. Solvability follows provided one has uniform control on the discrete solution (see §2), and each $u_{kj}^n$ will have compact support.

In §4 we show that the solution of the scheme converges to the exact solution in energy norm. We derive there certain discrete Sobolev type inequalities as well.

The problem remains of establishing $L^\infty$-convergence to the exact solution. In §3 we write a representation for the solution of a nonhomogeneous discrete wave equation; this comes from [3]. Other properties of the kernel $S$, and an $L^2$-estimate, are also given. Finally, in §5 we apply this representation to the scheme (8) to derive the required uniform inequalities.

In order to state our main theorem, we introduce the following standard notation. Given a sequence $\{u_{kj}^n\}$, $k, j \in \mathbb{Z}$, $n \in \mathbb{N}$, we write

$$\|u^n\|_\infty = \sup_{k,j} |u_{kj}^n|, \tag{10}$$

$$\|u^n\|_p = \left( \sum_{k,j} |u_{kj}^n|^p h^2 \right)^{1/p} \tag{11}$$

for $1 \le p < \infty$, $h$ as above. The *energy density* $\varepsilon^n$ is defined by

$$\begin{aligned} \varepsilon_{ij}^n = \frac{1}{4\Delta t^2}[&(u_{ij}^{n+1} - u_{i+1,j}^n)^2 + (u_{ij}^n - u_{i+1,j}^{n+1})^2 \\ &+ (u_{ij}^{n+1} - u_{i,j+1}^n)^2 + (u_{ij}^n - u_{i,j+1}^{n+1})^2] \end{aligned} \tag{12}$$

and the *energy norm* by

$$|||u^n|||^2 = h^2 \sum_{i,j} \varepsilon_{ij}^n . \tag{13}$$

All sums will in fact be finite, since $u_{kj}^n$ will have compact support.

**Theorem.** *Let $T > 0$ be arbitrary, and let the exact solution $u$ of* (1), (2) *be approximated by the solutions $u_{kj}^n$ of the scheme* (8), (9). *Assume the data $\phi, \psi \in C_0^\infty(\mathbb{R}^2)$. Let $n\Delta t = T$, $h = \sqrt{2}\Delta t$. Then there exist constants $c_T$ and $k_T$, depending only on the data and on $T$, with the property that whenever $k_T \cdot \Delta t < 1$, we have*

$$|||u(t^n, \cdot) - u^n||| \le c_T \Delta t^2,$$

$$\sup_{k,j} |u(t^n, x_k, y_j) - u_{kj}^n| \le c_T \Delta t^2 \cdot \left[\ln \frac{1}{\Delta t}\right]^{1/2}.$$

Constants will change from line to line and will be denoted by $c$. Those which depend on $T$ will be written $c_T$, etc. All sums, e.g. $\sum_{i,j} u_{ij}$, are taken over all of $\mathbb{Z}^2$ unless otherwise noted.

## 2. The scheme and the exact solution

We first cite some properties of the exact solution $u$. Thus consider the equation

$$u_{tt} - \Delta u + u^3 = 0 \qquad (x, y \in \mathbb{R},\ t > 0) \tag{1}$$

with $(u, u_t)$ given by $(\phi, \psi) \in C_0^\infty$ at $t = 0$. Here, $\Delta = \partial_x^2 + \partial_y^2$.

**Lemma 1.** *Given data $(\phi, \psi) \in C_0^\infty$ and an arbitrary time $T > 0$, there exists a unique global $C_0^\infty$-solution $u$ of* (1), (2) *enjoying the following properties*:

(i) $\frac{1}{2}\int (u_t^2 + |\nabla u|^2)\,dx + \frac{1}{4}\int u^4\,dx = \text{const}$,

(ii) $\sup_{T \ge t \ge 0, x \in \mathbb{R}^2} |u(t, x)| + \sup_{0 \le t \le T} \|D^\alpha u(t, \cdot)\|_{L^2} \le c_{T,\alpha} < \infty$

*for any multi-index $\alpha$.*

This situation has been treated by many authors; we refer to [1, 7]. The uniform norm is actually uniformly bounded and, in fact, decays (cf. [2, 4]). Once an $L^\infty$-estimate is known, $L^2$-bounds on higher derivatives follow easily by differentiating the equation and applying energy estimates.

We introduce here also the *free solution* $v(t, x)$:

$$(\partial_t^2 - \Delta)v = 0 \qquad (t > 0,\ x, y \in \mathbb{R}), \tag{14}$$

$$v(0, x, y) = \phi(x, y), \qquad v_t(0, x, y) = \psi(x, y). \tag{15}$$

$v$ and $u$ share the same data.

Now consider the scheme (8), (9) whose solutions $u_{kj}^n$ are to approximate $u(t^n, x_k, y_j)$. Since $\Delta t = h/\sqrt{2}$ by (5), we may rewrite (8) as

$$\begin{aligned} u_{kj}^{n+1} = &- u_{kj}^{n-1} + \frac{1}{2}[u_{k+1,j}^n + u_{k-1,j}^n + u_{k,j+1}^n + u_{k,j-1}^n] \\ &- \Delta t^2 \left[\frac{G(u_{kj}^{n+1}) - G(u_{kj}^{n-1})}{u_{kj}^{n+1} - u_{kj}^{n-1}}\right] \end{aligned} \tag{16}$$

with $u^0, u^1$ given by (9).

We define for $u \neq v$

$$H(u, v) = \frac{G(u) - G(v)}{u - v} = \frac{(u+v)}{4}(u^2 + v^2). \tag{17}$$

Then (16) is

$$\begin{aligned} u_{kj}^{n+1} = &- u_{kj}^{n-1} + \frac{1}{2}[u_{k+1,j}^n + u_{k-1,j}^n + u_{k,j+1}^n + u_{k,j-1}^n] \\ &- \frac{\Delta t^2}{4}(u_{kj}^{n-1})^3 - \frac{\Delta t^2}{4} u_{kj}^{n+1}[(u_{kj}^{n+1})^2 + (u_{kj}^{n-1})^2 + (u_{kj}^{n+1})(u_{kj}^{n-1})], \end{aligned}$$

or

$$\left[1 + \frac{\Delta t^2}{4}\{(u_{kj}^{n+1})^2 + (u_{kj}^{n-1})^2 + u_{kj}^{n+1} u_{kj}^{n-1}\}\right] u_{kj}^{n+1} = b_{kj}^n, \tag{18}$$

where

$$b_{kj}^n = -u_{kj}^{n-1} + \frac{1}{2}[u_{k+1,j}^n + u_{k-1,j}^n + u_{k,j+1}^n + u_{k,j-1}^n] - \frac{\Delta t^2}{4}(u_{kj}^{n-1})^3. \tag{19}$$

We can write (18) as an implicit equation

$$s = g(s) \tag{20}$$

to be solved for $s = u_{kj}^{n+1}$, with

$$g(s) = \frac{b_{kj}^n}{1 + \Delta t^2\{s^2 + s u_{kj}^{n-1} + (u_{kj}^{n-1})^2\}/4}. \tag{21}$$

**Lemma 2.** *Assume that there is a constant* $c_T$ *such that*

$$\max_{0\le l\le n}\sup_{k,j}|u^l_{kj}|\le c_T\,.$$

*Then, if* $\Delta t\cdot c_T$ *is small enough, we have*

(i) (20) *is uniquely solvable for* $s=u^{n+1}_{kj}$ *on, say, the set* $|s|\le 8c_T$.

(ii) *Let* $\phi_{kj}=\psi_{kj}=0$ *for* $|k|+|j|\ge R$. *Then for every* $n$,

$$u^{n+1}_{kj}=0\quad \textit{for } |k|+|j|\ge R+n\,.$$

*Proof.* To (20) we apply a standard fixed point theorem (cf. [5, p. 86ff]). Thus, take $s_0=0$ and define

$$s_{\nu+1}=g(s_\nu)\qquad(\nu=0,1,\dots).$$

By hypothesis, $|b^n_{kj}|\le 4c_T$ for $\Delta t$ small enough. Thus, on the set $\{|s|\le 8c_T\}$ we have

$$\begin{aligned}|g'(s)|&\le\frac{|b^n_{kj}|}{[1+\Delta t^2\{s^2+su^{n-1}_{kj}+(u^{n-1}_{kj})^2\}/4]^2}\left[\frac{|s|\Delta t^2}{2}+\frac{\Delta t^2|u^{n-1}_{kj}|}{4}\right]\\&\le\frac{4c_T}{1}\cdot\left[\frac{8c_T\Delta t^2}{2}+\frac{c_T\Delta t^2}{4}\right]\le 17c_T^2\Delta t^2\le\frac{1}{2}\equiv\lambda\end{aligned}$$

for $\Delta t$ small enough. If we can show that

$$|s_0-g(s_0)|\le(1-\lambda)\cdot 8c_T\,,$$

we can apply Theorem 1, pp. 86–87 of [5]. However, $s_0=0$ and

$$|g(0)|=\frac{|b^n_{kj}|}{1+(\Delta t^2/4)\cdot(u^{n-1}_{kj})^2}\le 4c_T\,,$$

and this proves (i).

(ii) is easily proved by induction. The data have compact support by hypothesis. Assume that $u^l_{kj}$ has this support property for $l\le n$. We see from the definition (19) of $b^n_{kj}$ that

$$b^n_{kj}=0\quad\text{if } |k|+|j|\ge R+n\,.$$

Since the coefficient of $u^{n+1}_{kj}$ in (18) always exceeds 1, we are done. □

## 3. The representation

We cite a number of results from [3]. There, the solution to the problem

$$(L_hu)^n_{kj}=f^n_{kj}\,,\tag{22}$$

$$u^0_{kj}=\phi_{kj}\,,\qquad u^1_{kj}=\phi_{kj}+\Delta t\cdot\psi_{kj}+\tfrac{1}{2}\Delta t^2\Phi_{kj}\tag{23}$$

is represented as follows:

$$u_{kj}^{n+1} = v_{kj}^{n+1} + (Sf)_{kj}^{n}. \tag{24}$$

Here, $v_{kj}^{n}$ is the "discrete free solution", and $Sf$ is a linear operator whose action on $f$ is given by a convolution of $f$ with a discrete kernel $S_{pk}^{l}$ defined by

$$S_{pk}^{l} = \sum_{m=0}^{p} (-4)^m \binom{l-m}{m} \binom{l-2m}{p-m} \binom{l-2m}{|k|+p-m}. \tag{25}$$

**Lemma 3** (from [3]). (i) *We have* $S_{pk}^{l} \geq 0$; *in fact, for* $k \geq 0$,

$$S_{pk}^{l} = \frac{\binom{l-p}{k}}{\binom{k+p}{p}} \cdot 4^p \cdot [P_p^{(k,\, l-2p-k)}(0)]^2,$$

*where* $P_p^{(\alpha,\beta)}(x)$ *is the Jacobi polynomial of degree* $p$ *with parameters* $\alpha$, $\beta$ (*cf.* [6]). *Here, the parameters range in* $|k| = 0, 1, 2, \ldots$, $p = 0, 1, \ldots, [l/2]$, $l = 0, 1, \ldots$. *Outside these ranges,* $S_{pk}^{l}$ *is defined to be zero.*

(ii) *There holds* $\sum_{p,k} S_{pk}^{l} = (1+l) \cdot 2^l$.

(iii) *Let* $\theta = (\theta_1, \theta_2)$ *be the dual Fourier variables in the discrete Fourier Transform* $\mathscr{F}$:

$$\hat{u}(\theta_1, \theta_2) \equiv \sum_{k,j} e^{-i(k\theta_1 + j\theta_2)} u_{kj} \equiv \mathscr{F} u(\theta).$$

*Further, define an angle* $\tilde{\psi}$ *by*

$$\cos\tilde{\psi} = \tfrac{1}{2}(\cos\theta_1 + \cos\theta_2). \tag{26}$$

*Then*

$$\mathscr{F}^{-1}\left[\frac{\sin(n+1)\tilde{\psi}}{\sin\tilde{\psi}}\right]_{kj} = 2^{-n} \sum_{p=0}^{[n/2]} \delta_{|k|+|j|,\, n-2p} S_{pk}^{n}.$$

*Proof*. (i) is Theorem 2 of [3]. There, it is also shown that $S$ satisfies the recursion

$$S_{p,|k|}^{l} = S_{p,|k|}^{l-1} + S_{p-1,|k|+1}^{l-1} + S_{p,|k|-1}^{l-1} + S_{p-1,|k|}^{l-1} - 4S_{p-1,|k|}^{l-2}. \tag{27}$$

Since $S_{pk}^{l} \geq 0$, we may sum this over $p$, $k$ to obtain (ii) (cf. the last page of [3]). For (iii), we obtain from the material following Lemma 1 of [3],

$$\begin{aligned} \mathscr{F}^{-1}\left[\frac{\sin(n+1)\tilde{\psi}}{\sin\tilde{\psi}}\right]_{kj} &= \sum_{m=0}^{[n/2]} 2^{-n}(-4)^m \binom{n-m}{m} \\ &\quad \cdot \sum_{p=0}^{[(n-2m)/2]} \binom{n-2m}{p} \binom{n-2m}{|k|+p} \delta_{|k|+|j|,\, n-2m-2p}. \end{aligned} \tag{28}$$

We replace $p$ by $p+m$ and then invert the $m, p$ summations. When use is made of (25), (iii) results. □

We do not require the explicit form of $S^l_{pk}$.

Given these facts about $S^l_{pk}$, we can write the free solution $v^{n+1}_{kj}$ in (24) as follows:

$$
\begin{aligned}
v^{n+1}_{kj} = & -2^{1-n} \sum_{p=0}^{[(n-1)/2]} \sum_{\substack{\alpha,\beta \\ |\alpha-k|+|\beta-j|=n-1-2p}} S^{n-1}_{p,k-\alpha}\phi_{\alpha\beta} \\
& + 2^{-n} \sum_{p=0}^{[n/2]} \sum_{\substack{\alpha,\beta \\ |\alpha-k|+|\beta-j|=n-2p}} S^{n}_{p,k-\alpha}(\phi_{\alpha\beta} + \Delta t\psi_{\alpha\beta} + \tfrac{1}{2}\Delta t^2\Phi_{\alpha\beta}).
\end{aligned} \tag{29}
$$

This is the Corollary to Theorem 1 in [3]. Theorem 1 of [3] itself gives the form of the operator $Sf$:

$$
(Sf)^n_{kj} = \Delta t^2 \sum_{l=0}^{n-1} 2^{-l} \sum_{p=0}^{[l/2]} \sum_{\substack{\alpha,\beta \\ |\alpha-k|+|\beta-j|=l-2p}} S^{l}_{p,k-\alpha} f^{n-l}_{\alpha\beta}. \tag{30}
$$

We require one more piece of information about the kernel $S^l_{pk}$.

**Lemma 4.** *There is a constant $c$ such that*

$$
\sum_{p=0}^{[n/2]} \sum_{k} [S^n_{pk}]^2 \le c \cdot 4^n \cdot \ln(n+2)
$$

*for all $n = 1, 2, \ldots$.*

*Proof.* Consider the result of squaring the expression in Lemma 3, part (iii). When this is carried out, one obtains many "cross-terms" involving terms of the form

$$
\delta_{|k|+|j|,\,n-2p}\,\delta_{|k|+|j|,\,n-2q} S^n_{pk} S^n_{qk}
$$

for $p \neq q$, each of which vanishes. Thus, the Kronecker $\delta$'s in (iii) of Lemma 3 act as if they were an orthogonal set, and we get

$$
\left[\mathscr{F}^{-1}\left(\frac{\sin(n+1)\tilde{\psi}}{\sin\tilde{\psi}}\right)_{kj}\right]^2 = 4^{-n} \sum_{p=0}^{[n/2]} \delta_{|k|+|j|,\,n-2p} [S^n_{pk}]^2. \tag{31}
$$

We sum this over all $k$, $j$. Performing the $j$-summation first, we see that there are at most two nonzero terms on the right, each of which is unity. Hence, from (31) and the Parseval equality we get

$$
4^{-n} \sum_{p=0}^{[n/2]} \sum_{k} [S^n_{pk}]^2 \le c \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \left[\frac{\sin(n+1)\tilde{\psi}}{\sin\tilde{\psi}}\right]^2 d\theta_1\, d\theta_2. \tag{32}
$$

We call $\mathscr{I}_n$ the integral on the right here. By (26),

$$\cos\tilde{\psi} = \frac{1}{2}(\cos\theta_1 + \cos\theta_2) = \cos\left(\frac{\theta_1+\theta_2}{2}\right)\cos\left(\frac{\theta_1-\theta_2}{2}\right).$$

Introducing new variables $x = (\theta_1+\theta_2)/2$, $y = (\theta_1-\theta_2)/2$, and using elementary considerations, we obtain

$$\mathscr{I}_n = c\int_0^{\pi/2}\int_0^{\pi/2}\left[\frac{\sin(n+1)\tilde{\psi}}{\sin\tilde{\psi}}\right]^2 dx\,dy,$$

where now $\cos\tilde{\psi} = \cos x\cos y$, and hence

$$\tilde{\psi} = \frac{\pi}{2} - \sin^{-1}(\cos x\cos y), \qquad 0 \le \tilde{\psi} \le \pi. \tag{33}$$

Denote by $A = \{(x, y): 0 \le x, y \le \pi/4\}$, and let $A^c$ be the relative complement of $A$ in $[0, \pi/2]\times[0, \pi/2]$. On $A^c$, at least one of $x, y$ lies in the interval $[\pi/4, \pi/2]$, and hence $\cos x\cos y \le 1\cdot 1/\sqrt{2}$. It follows that $\sin^{-1}(\cos x\cos y) \le \sin^{-1}(1/\sqrt{2}) = \pi/4$, and hence that $\tilde{\psi} \ge \pi/4$ on $A^c$. Therefore, $\sin\tilde{\psi}$ is bounded below, and the integral over $A^c$ is bounded uniformly.

On $A$ itself we use the inequality

$$\left(\frac{\sin(n+1)\tilde{\psi}}{\sin\tilde{\psi}}\right)^2 \le c\min((n+1)^2, \tilde{\psi}^{-2})$$

and these observations: First, for $0 \le \phi \le \pi/2$ we have

$$\cos\phi \ge 1 - \tfrac{1}{2}\phi^2 \tag{34}$$

by simple Taylor series. Secondly, on $A$ we have

$$\cos\tilde{\psi} = \cos x\cos y \ge 1/\sqrt{2}\cdot 1/\sqrt{2} = \tfrac{1}{2},$$

and hence

$$0 \le \tilde{\psi} = \pi/2 - \sin^{-1}(\cos\tilde{\psi}) \le \pi/2 - \sin^{-1}(\tfrac{1}{2}) = \pi/3.$$

By (34), then,

$$\cos\tilde{\psi} \ge 1 - \tfrac{1}{2}\tilde{\psi}^2. \tag{35}$$

An elementary series argument shows that $\cos x \le 1 - x^2/4$ for $0 \le x \le \pi/4$. Thus, on $A$ we have

$$\cos\tilde{\psi} = \cos x\cos y \le \left(1-\frac{x^2}{4}\right)\left(1-\frac{y^2}{4}\right) = 1 - \frac{1}{4}(x^2+y^2) + \frac{x^2y^2}{16}$$

$$\le 1 - \frac{1}{4}(x^2+y^2) + \frac{x^2}{16}\cdot\left(\frac{\pi}{4}\right)^2 \le 1 - c(x^2+y^2) \quad \text{with } c > 0.$$

Combining this with (35), and defining $\rho^2 = x^2+y^2$, we get

$$1 - \tfrac{1}{2}\tilde{\psi}^2 \le \cos\tilde{\psi} \le 1 - c\rho^2,$$

i.e.,

$$\tilde{\psi} \geq c\rho \quad \text{on } A, \text{ with } c > 0. \tag{36}$$

Therefore, we have

$$\begin{aligned}\int\int_A &\leq c\int\int_A \min\{(n+1)^2, \rho^{-2}\}\,dx\,dy \leq c\int_0^{\pi/2\sqrt{2}} \min\{(n+1)^2, \rho^{-2}\}\rho\,d\rho \\ &= c\left[\int_0^{1/(n+1)} (n+1)^2\rho\,d\rho + \int_{1/(n+1)}^{\pi/2\sqrt{2}} \frac{d\rho}{\rho}\right] \leq c\ln(n+1),\end{aligned}$$

and the proof of the lemma is complete. □

## 4. Energy estimates

In this section we make error estimates in the energy norm. We begin with the scheme in the form (8):

$$(L_h u)_{kj}^n + \frac{G(u_{kj}^{n+1}) - G(u_{kj}^{n-1})}{u_{kj}^{n+1} - u_{kj}^{n-1}} = 0.$$

Multiply this by $(u_{kj}^{n+1} - u_{kj}^{n-1})$ and sum over all $k, j$. We sum by parts that term which approximates $\Delta u$. If we call

$$\begin{aligned}\hat{e}^n = \sum_{k,j} \Bigg[ &\frac{(u_{kj}^{n+1} - u_{kj}^n)^2}{\Delta t^2} + \frac{1}{h^2}\Big\{(u_{k+1,j}^{n+1} - u_{kj}^{n+1})(u_{k+1,j}^n - u_{kj}^n) \\ &+ (u_{k,j+1}^{n+1} - u_{kj}^{n+1})(u_{k,j+1}^n - u_{kj}^n)\Big\} \\ &+ G(u_{kj}^{n+1}) + G(u_{kj}^n)\Bigg],\end{aligned} \tag{37}$$

then we have the following

**Lemma 5.** (i) *We have* $\hat{e}^n = \hat{e}^{n-1}$, *and hence,* $\hat{e}^n = \hat{e}^0$. *Thus, the scheme preserves a discrete energy.*

(ii) *For every* $n$, $\hat{e}^n$ *is nonnegative and can be expressed by*

$$\begin{aligned}\hat{e}^n = \frac{1}{4\Delta t^2}\sum_{k,j} &[(u_{kj}^{n+1} - u_{k+1,j}^n)^2 + (u_{kj}^n - u_{k+1,j}^{n+1})^2 \\ &+ (u_{kj}^{n+1} - u_{k,j+1}^n)^2 + (u_{kj}^n - u_{k,j+1}^{n+1})^2] \\ + \sum_{k,j} &[G(u_{kj}^{n+1}) + G(u_{kj}^n)].\end{aligned}$$

(iii) *Pure spatial differences evaluated at the same time can be bounded*:

$$h^{-2}\sum_{k,j} [(u_{kj}^{n+1} - u_{k+2,j}^{n+1})^2 + (u_{kj}^{n+1} - u_{k,j+2}^{n+1})^2] \leq c\cdot\hat{e}^n.$$

*Remark.* Recall the energy density definition $\varepsilon_{kj}^n$ in (12). We see that

$$\hat{e}^n = \sum_{k,j} \varepsilon_{kj}^n + \sum_{k,j} [G(u_{kj}^{n+1}) + G(u_{kj}^n)], \tag{38}$$

as expected.

*Proof of Lemma* 5. Part (i) is a standard calculation and is omitted. The summation by parts produces no boundary terms, since $u_{kj}^n$ has compact support for each $n$. As for (ii), we write $\hat{e}^n$ (using $\Delta t = h/\sqrt{2}$) as

$$\begin{aligned}
\Delta t^2 &\left[\hat{e}^n - \sum_{k,j} \{G(u_{kj}^{n+1}) + G(u_{kj}^n)\}\right] \\
&= \sum_{k,j} \left[\frac{1}{2}(u_{kj}^{n+1} - u_{kj}^n)^2 + \frac{1}{4}(u_{k+1,j}^{n+1} - u_{k+1,j}^n)^2 \right. \\
&\qquad\qquad \left. + \frac{1}{4}(u_{k,j+1}^{n+1} - u_{k,j+1}^n)^2\right] \\
&\quad + \frac{1}{2}\sum_{k,j} [(u_{k+1,j}^{n+1} - u_{kj}^{n+1})(u_{k+1,j}^n - u_{kj}^n) \\
&\qquad\qquad + (u_{k,j+1}^{n+1} - u_{kj}^{n+1})(u_{k,j+1}^n - u_{kj}^n)] \\
&= \frac{1}{4}\sum_{k,j} [(u_{kj}^{n+1} - u_{kj}^n)^2 + (u_{k+1,j}^{n+1} - u_{k+1,j}^n)^2 \\
&\qquad\qquad + 2(u_{k+1,j}^{n+1} - u_{kj}^{n+1})(u_{k+1,j}^n - u_{kj}^n)] \\
&\quad + \frac{1}{4}\sum_{k,j} [(u_{kj}^{n+1} - u_{kj}^n)^2 + (u_{k,j+1}^{n+1} - u_{k,j+1}^n)^2 \\
&\qquad\qquad + 2(u_{k,j+1}^{n+1} - u_{kj}^{n+1})(u_{k,j+1}^n - u_{kj}^n)].
\end{aligned} \tag{39}$$

Now we use the elementary relation

$$(A-B)^2 + (C-D)^2 + 2(C-A)(D-B) = (A-D)^2 + (B-C)^2$$

in each of the above lines, and (ii) results.

For the proof of (iii), we have from (ii) that, e.g.,

$$\begin{aligned}
4\Delta t^2 \hat{e}^n &\geq \sum_{k,j} [(u_{kj}^{n+1} - u_{k+1,j}^n)^2 + (u_{kj}^n - u_{k+1,j}^{n+1})^2] \\
&= \sum_{k,j} [(u_{kj}^{n+1} - u_{k+1,j}^n)^2 + (u_{k+1,j}^n - u_{k+2,j}^{n+1})^2].
\end{aligned}$$

However, since

$$(a-b)^2 + (b-c)^2 \geq \tfrac{1}{2}(a-c)^2$$

(as can be established by elementary calculus), we obtain

$$4\Delta t^2 \cdot \hat{e}^n \geq \frac{1}{2} \sum_{k,j} (u_{kj}^{n+1} - u_{k+2,j}^{n+1})^2$$

and this is (iii). □

Now denote by $\tau_{kj}^n$ the truncation error, i.e., the amount by which the exact solution $u$ fails to satisfy the approximate scheme:

$$\begin{aligned} \tau_{kj}^n = {} & \frac{u(t^{n+1}, x_k, y_j) - 2u(t^n, x_k, y_j) + u(t^{n-1}, x_k, y_j)}{\Delta t^2} \\ & - \frac{1}{h^2}[u(t^n, x_{k+1}, y_j) + u(t^n, x_{k-1}, y_j) + u(t^n, x_k, y_{j+1}) \\ & \qquad + u(t^n, x_k, y_{j-1}) - 4u(t^n, x_k, y_j)] \\ & + \frac{G(u(t^{n+1}, x_k, y_j)) - G(u(t^{n-1}, x_k, y_j))}{u(t^{n+1}, x_k, y_j) - u(t^{n-1}, x_k, y_j)}. \end{aligned} \tag{40}$$

Simple Taylor series arguments give us

**Lemma 6.** *On any strip* $[0, T] \times \mathbb{R}^2$, *there is a constant* $c_T > 0$ *such that* $|\tau_{kj}^n| \leq c_T \cdot \Delta t^2$.

In the proof we simply take enough $L^2$-derivatives in Lemma 1 so that

$$\sup_{t \leq T} \|D^4 u(t, \cdot)\|_\infty \leq c_T$$

via the Sobolev inequality. Incidentally, we have an estimate for $\|u(t)\|_2$ in terms of $c_T \|\nabla u(t)\|_2$ by the support property and the Poincaré inequality.

We will need discrete versions of certain of these classical inequalities, which take into account the special type of square integrability which arises from our energy estimates.

**Lemma 7.** *Let the sequence* $\{w_{kj}^n\}$ *have compact support for each fixed* $n$. *Let* $[0, T]$ *be an arbitrary time interval with* $n\Delta t = T$, $h = \sqrt{2}\Delta t$. *Then*

$$\sum_{k,j} (w_{kj}^{n+1})^2 \leq cT^2 \sum_{k,j} (w_{kj}^{n+1} - w_{k-2,j}^{n+1})^2 \Delta t^{-2}.$$

*Remark.* This is a discrete Poincaré inequality.

*Proof of Lemma* 7. Using the support property, we write

$$w_{kj}^{n+1} = \sum_{\substack{l=-n-c \\ \text{Step } 2}}^{k} (w_{lj}^{n+1} - w_{l-2,j}^{n+1})$$

for some constant $c$. Then

$$|w_{kj}^{n+1}| \leq cn^{1/2} \left( \sum_l (w_{lj}^{n+1} - w_{l-2,j}^{n+1})^2 \right)^{1/2},$$

and hence

$$\sum_{k,j} (w_{kj}^{n+1})^2 \le cn^2 \sum_{l,j} (w_{l,j}^{n+1} - w_{l-2,j}^{n+1})^2 = cT^2 \sum_{l,j} \frac{(w_{lj}^{n+1} - w_{l-2,j}^{n+1})^2}{\Delta t^2}. \quad \square$$

**Lemma 8.** *Let $h > 0$, and let $\{w_{kj}\}$ be any sequence of compact support. Define*

$$\delta_1 w_{kj} = \frac{w_{kj} - w_{k-2,j}}{h}, \qquad \delta_2 w_{kj} = \frac{w_{kj} - w_{k,j-2}}{h}. \tag{41}$$

*As in* (13), *define, for $i = 1, 2$,*

$$\|\delta_i w\|_1 = \sum_{k,j} |\delta_i w_{kj}| h^2, \qquad \|w\|_2 = \left( \sum_{k,j} (w_{kj})^2 h^2 \right)^{1/2}.$$

*Then*:

(i) *There holds* $\|w\|_2^2 \le \|\delta_1 w\|_1 \|\delta_2 w\|_1$.

(ii) *For any integer $N$ we have*

$$\|\delta_i (w)^N\|_1 \le c_N \|\delta_i w\|_2 \|(w)^{N-1}\|_2.$$

*Remark.* (i) is a discrete analogue of the Sobolev inequality $\|u\|_{n/(n-1)} \le c\|\nabla u\|_1$ for $x \in \mathbb{R}^n$.

*Proof of Lemma* 8. By the support property we can write

$$w_{kj} = \sum_{\substack{l=-n-c \\ \text{Step 2}}}^{k} (w_{lj} - w_{l-2,j})$$

for some constant $c$. Similarly,

$$w_{kj} = \sum_{\substack{\nu=-n-c \\ \text{Step 2}}}^{j} (w_{k\nu} - w_{k,\nu-2}).$$

Therefore,

$$|w_{kj}|^2 \le \left[ \sum_l |w_{lj} - w_{l-2,j}| \right] \left[ \sum_\nu |w_{k\nu} - w_{k,\nu-2}| \right].$$

Summing this over $k$, $j$, we obtain

$$\begin{aligned} \sum_{k,j} |w_{kj}|^2 &\le \left[ \sum_{l,j} |w_{lj} - w_{l-2,j}| \right] \left[ \sum_{k,\nu} |w_{k\nu} - w_{k,\nu-2}| \right] \\ &= \left[ h \sum_{l,j} |\delta_1 w_{lj}| \right] \left[ h \sum_{k,\nu} |\delta_2 w_{k\nu}| \right], \end{aligned}$$

and this proves (i).

For the proof of (ii) we recall the identity

$$x^N - y^N = (x - y) \sum_{k=0}^{N-1} x^k y^{N-1-k},$$

from which we get

$$|x^N - y^N| \le c|x-y|(|x|^{N-1} + |y|^{N-1}).$$

Thus,

$$\begin{aligned}|\delta_1(w_{kj})^N| &\equiv \frac{|(w_{kj})^N - (w_{k-2,j})^N|}{h} \\ &\le c\frac{|w_{kj} - w_{k-2,j}|}{h}(|w_{kj}|^{N-1} + |w_{k-2,j}|^{N-1}).\end{aligned}$$

We multiply by $h^2$ and sum over $k, j$:

$$\begin{aligned}\|\delta_1(w^N)\|_1 &\le c\sum_{k,j} h^2|\delta_1 w_{kj}|(|w_{kj}|^{N-1} + |w_{k-2,j}|^{N-1}) \\ &\le 2c\left(\sum_{k,j} h^2|\delta_1 w_{kj}|^2\right)^{1/2}\left(\sum_{k,j}|w_{kj}|^{2(N-1)}h^2\right)^{1/2} \\ &= 2c\|\delta_1 w\|_2\|(w)^{N-1}\|_2,\end{aligned}$$

as desired. □

**Corollary.** *Denote by* $\{u^n_{kj}\}$ *the solution of the discrete scheme* (8), (9). *Define*

$$\delta u^n = |\delta_1 u^n| + |\delta_2 u^n|.$$

*Then the following estimates hold*:

(i) $\|u^n\|_4 + \|\delta u^n\|_2 \le \text{const}$.

(ii) $\|u^n\|_6 \le \text{const}$, $\|u^n\|_8 \le \text{const}$.

*Proof*. From the energy equality in Lemma 5, part (ii), we get the bound on $\|u^n\|_4$. Then, using part (iii) of Lemma 5, the bound on $\|\delta u^n\|_2$ follows. For part (ii) here, we apply parts (i) and (ii) of Lemma 8 with $N = 3$ and $w_{kj} = u^n_{kj}$:

$$\begin{aligned}\|(u^n)^3\|_2^2 &\le \|\delta_1(u^n)^3\|_1\|\delta_2(u^n)^3\|_1 \\ &\le c\|\delta_1 u^n\|_2\|(u^n)^2\|_2 \cdot \|\delta_2 u^n\|_2\|(u^n)^2\|_2 \\ &\le c\|\delta u^n\|_2^2\|u^n\|_4^4 \le \text{const}.\end{aligned}$$

The bound on $\|u^n\|_8$ is similar. □

*Remark.* It is clear that $\|u^n\|_p \le \text{const}$ for any $p$, $4 \le p < \infty$.

Now we can estimate the error in the energy norm. Define

$$e^n_{kj} = u(t^n, x_k, y_j) - u^n_{kj}. \tag{42}$$

We recall the definitions

$$H(u, v) = \frac{G(u) - G(v)}{u - v} = \frac{1}{4}(u+v)(u^2 + v^2), \tag{17}$$

$$|||u^n|||^2 = h^2\sum_{k,j}\varepsilon^n_{kj}, \tag{13}$$

where $\varepsilon^n_{kj}$ is given by (12) as a sum of squares.

**Theorem 9.** *Consider an arbitrary time interval* $[0, T]$ *with* $n\Delta t = T$. *Denote by* $\widetilde{\mathscr{E}}^n$ *the energy of the error* $e^n$ *in* (42):

$$\widetilde{\mathscr{E}}^n \equiv \frac{1}{4\Delta t^2} \sum_{k,j} [(e_{kj}^{n+1} - e_{k+1,j}^n)^2 + (e_{kj}^n - e_{k+1,j}^{n+1})^2 + (e_{kj}^{n+1} - e_{k,j+1}^n)^2 + (e_{kj}^n - e_{k,j+1}^{n+1})^2]. \tag{43}$$

*Then there exists a constant* $c_T$ *such that for* $\Delta t$ *sufficiently small,*

$$\widetilde{\mathscr{E}}^n \le c_T[\widetilde{\mathscr{E}}^0 + \Delta t^2].$$

*Remark.* The square of the *energy norm* of the error $e^n$ is thus given by $h^2\widetilde{\mathscr{E}}^n$.

*Proof of Theorem* 9. As is standard, we write the scheme (8) for $u_{kj}^n$ and subtract from it the equation (40) defining the truncation error to get

$$(L_h e)_{kj}^n + H(u(t^{n+1}, x_k, y_j), u(t^{n-1}, x_k, y_j)) - H(u_{kj}^{n+1}, u_{kj}^{n-1}) - \tau_{kj}^n = 0 \tag{44}$$

with the initial values

$$\begin{aligned} e_{kj}^0 &= u(0, x_k, y_j) - u_{kj}^0 = 0, \\ e_{kj}^1 &= u(t^1, x_k, y_j) - \left(\phi_{kj} + \frac{\Delta t^2}{2}\Phi_{kj} + \Delta t \cdot \psi_{kj}\right) = O(\Delta t^3) \end{aligned} \tag{45}$$

uniformly on $[0, T] \times \mathbb{Z}^2$.

Applying the mean-value theorem, we can write the nonlinear term above (i.e., the difference of the $H$'s) as

$$\overline{H}_u \cdot e_{kj}^{n+1} + \overline{H}_v \cdot e_{kj}^{n-1}, \tag{46}$$

where the overbar means the gradient of $H$, $\nabla H = (H_u, H_v)$, is evaluated at some intermediate point on the line segment joining

$$(u(t^{n+1}, x_k, y_j), u(t^{n-1}, x_k, y_j)) \quad \text{and} \quad (u_{kj}^{n+1}, u_{kj}^{n-1}).$$

We use the expression (46) in (44), multiply the result by $e_{kj}^{n+1} - e_{kj}^{n-1}$, and then sum over all $k, j$. As in (37) and Lemma 5, we then get

$$\begin{aligned} \widetilde{\mathscr{E}}^n = \widetilde{\mathscr{E}}^{n-1} &+ \sum_{k,j} \{\overline{H}_u e_{kj}^{n+1}(e_{kj}^{n+1} - e_{kj}^{n-1}) + \overline{H}_v e_{kj}^{n-1}(e_{kj}^{n+1} - e_{kj}^{n-1})\} \\ &- \sum_{k,j} \tau_{kj}^n (e_{kj}^{n+1} - e_{kj}^{n-1}) = 0. \end{aligned} \tag{47}$$

In each of the last three terms there appears the expression $e_{kj}^{n+1} - e_{kj}^{n-1}$, for

which we have the $l_2$-estimate

$$\begin{aligned}\sum_{k,j}(e_{kj}^{n+1}-e_{kj}^{n-1})^2 &= \sum_{k,j}[(e_{kj}^{n+1}-e_{k+1,j}^{n})+(e_{k+1,j}^{n}-e_{kj}^{n-1})]^2\\ &\le 2\sum_{k,j}(e_{kj}^{n+1}-e_{k+1,j}^{n})^2+2\sum_{k,j}(e_{k+1,j}^{n}-e_{kj}^{n-1})^2\\ &\le 8\Delta t^2\cdot\tilde{\mathscr{E}}^n+8\Delta t^2\cdot\tilde{\mathscr{E}}^{n-1}.\end{aligned}\tag{48}$$

Hence, for the truncation error term, we have by Lemma 6 and the support property,

$$\begin{aligned}\left|\sum_{k,j}\tau_{kj}^{n}(e_{kj}^{n+1}-e_{kj}^{n-1})\right| &\le c_T\Delta t^2\sum_{\substack{k,j\\|k|+|j|\le cn}}|e_{kj}^{n+1}-e_{kj}^{n-1}|\\ &\le c_T\Delta t^2\cdot(cn^2)^{1/2}\left(\sum_{k,j}(e_{kj}^{n+1}-e_{kj}^{n-1})^2\right)^{1/2}\\ &\le c_T\Delta t^2\cdot n\cdot\Delta t(\tilde{\mathscr{E}}^n+\tilde{\mathscr{E}}^{n-1})^{1/2}\\ &= c_T\Delta t^2(\tilde{\mathscr{E}}^n+\tilde{\mathscr{E}}^{n-1})^{1/2}\le c_T\Delta t^3+c_T\Delta t(\tilde{\mathscr{E}}^n+\tilde{\mathscr{E}}^{n-1}),\end{aligned}\tag{49}$$

where we have used (48).

Next, for the $\overline{H}_u$-term in (47), we write

$$\begin{aligned}&\left|\sum_{k,j}\overline{H}_u\cdot e_{kj}^{n+1}(e_{kj}^{n+1}-e_{kj}^{n-1})\right|\\ &\le\left(\sum_{k,j}|\overline{H}_u|^4\right)^{1/4}\left(\sum_{k,j}|e_{kj}^{n+1}|^4\right)^{1/4}\left(\sum_{k,j}(e_{kj}^{n+1}-e_{kj}^{n-1})^2\right)^{1/2}\\ &\le c\Delta t(\tilde{\mathscr{E}}^n+\tilde{\mathscr{E}}^{n-1})^{1/2}\left(\sum_{k,j}|\overline{H}_u|^4\right)^{1/4}\left(\sum_{k,j}|e_{kj}^{n+1}|^4\right)^{1/4},\end{aligned}\tag{50}$$

where we have used (48). For the last term, we have by Lemma 7 and the definition of the norm in Lemma 8, and from the statements (i), (ii) there,

$$\begin{aligned}\sum_{k,j}|e_{kj}^{n+1}|^4 &= c\Delta t^{-2}\|e^{n+1}\|_4^4=c\Delta t^{-2}\|(e^{n+1})^2\|_2^2\\ &\le c\Delta t^{-2}\|\delta_1(e^{n+1})^2\|_1\|\delta_2(e^{n+1})^2\|_1\\ &\le c\Delta t^{-2}\|\delta(e^{n+1})^2\|_1^2\le c\Delta t^{-2}\|\delta e^{n+1}\|_2^2\|e^{n+1}\|_2^2\\ &\le c\Delta t^{-2}\cdot\Delta t^2\cdot\tilde{\mathscr{E}}^n\cdot cT^2\tilde{\mathscr{E}}^n\Delta t^2\le c_T\Delta t^2(\tilde{\mathscr{E}}^n)^2.\end{aligned}\tag{51}$$

Inserting this in (50), we get

$$\begin{aligned}(52)\qquad &\left|\sum_{k,j} \overline{H}_u \cdot e_{kj}^{n+1}(e_{kj}^{n+1} - e_{kj}^{n-1})\right| \\ &\le c_T \Delta t^{3/2}(\tilde{\mathscr{E}}^n + \tilde{\mathscr{E}}^{n-1}) \left[\sum_{k,j} |\overline{H}_u|^4\right]^{1/4}.\end{aligned}$$

Now by definition (17), $H_u$ (and $H_v$ as well) grow at most quadratically:

$$(53)\qquad |H_u| + |H_v| \le c(u^2 + v^2).$$

Hence,

$$(54)\qquad |\overline{H}_u|^4 \le c[(u(t^{n+1}, x_k, y_j))^8 + (u(t^{n-1}, x_k, y_j))^8 + (u_{kj}^{n+1})^8 + (u_{kj}^{n-1})^8].$$

Now by the Corollary to Lemma 8, the discrete $l_8$-norm of $u_{kj}^n$ is bounded. The $L^8$-norm of the exact solution is bounded by the energy bound on the $L^4$-norm and by Sobolev. Hence,

$$(55)\qquad \sum_{k,j} |\overline{H}_u|^4 \le c \cdot \Delta t^{-2},$$

which, when inserted into (52), gives us

$$(56)\qquad \left|\sum_{k,j} |\overline{H}_u|\, |e_{kj}^{n+1}|(e_{kj}^{n+1} - e_{kj}^{n-1})\right| \le c_T \Delta t(\tilde{\mathscr{E}}^n + \tilde{\mathscr{E}}^{n-1}).$$

The term in (47) involving $\overline{H}_v$ can clearly be estimated in the same way. Inserting the estimates (49), (56) into (47), we get

$$(57)\qquad \tilde{\mathscr{E}}^n \le \tilde{\mathscr{E}}^{n-1} + c_T \Delta t(\tilde{\mathscr{E}}^n + \tilde{\mathscr{E}}^{n-1}) + c_T \Delta t^3,$$

or

$$(58)\qquad (1 - c_T \Delta t)\tilde{\mathscr{E}}^n \le (1 + c_T \Delta t)\tilde{\mathscr{E}}^{n-1} + c_T \Delta t^3.$$

The constant $c_T$ here depends only on the data $\phi$, $\psi$ and on $T$. For $\Delta t$ small enough we have $(1 - c_T \Delta t)^{-1} \le 1 + 2c_T \Delta t$, and so (58) yields

$$(59)\qquad \tilde{\mathscr{E}}^n \le (1 + 4c_T \Delta t)\tilde{\mathscr{E}}^{n-1} + c_T \Delta t^3.$$

Iterating this, we get

$$\begin{aligned}(60)\qquad \tilde{\mathscr{E}}^n &\le (1 + 4c_T \Delta t)^n \tilde{\mathscr{E}}^0 + c_T \Delta t^3 \sum_{l=0}^{n-1} (1 + 4c_T \Delta t)^l \\ &\le (1 + 4c_T \Delta t)^n [\tilde{\mathscr{E}}^0 + c_T \Delta t^2] \\ &= \left(1 + \frac{4c_T T}{n}\right)^n [\tilde{\mathscr{E}}^0 + c_T \Delta t^2] \le c_T[\tilde{\mathscr{E}}^0 + \Delta t^2],\end{aligned}$$

which is the claim of Theorem 9. □

We can now eliminate the $\widetilde{\mathscr{E}}^0$-term as follows. By choice of data for $u^0_{kj}$, $u^1_{kj}$ we have

$$e^0_{kj} = 0, \qquad e^1_{kj} = O(\Delta t^3),$$

the order bound being uniform on $[0, T] \times \mathbb{Z}^2$. Hence,

$$\widetilde{\mathscr{E}}^0 = \frac{1}{\Delta t^2} \sum_{k,j} (e^1_{kj})^2 \le c_T \Delta t^{-2} \sum_{\substack{k,j \\ |k|+|j| \le c/\Delta t}} \Delta t^6 \le c_T \Delta t^2. \tag{61}$$

**Corollary.** *Denote the energy norm of the error* $e^n_{kj} \equiv u(t^n, x_k, y_j) - u^n_{kj}$ *by* $|||e^n|||^2 = \Delta t^2 \cdot \widetilde{\mathscr{E}}^n$. *Then on any time interval* $[0, T]$ *with* $n\Delta t = T$, $h = \sqrt{2}\Delta t$, *there exists a constant* $c_T$, *depending only on* $T$ *and the data, such that* $|||e^n|||$ $\le c_T \Delta t^2$ *for* $\Delta t$ *small enough.*

## 5. Uniform estimates

In this section we will show that the errors $u(t^n, x_k, y_j) - u^n_{kj}$ converge uniformly to zero at (essentially) the rate $\Delta t^2$. Thus, the hypothesis of Lemma 2 will be superfluous, since $\sup_{x,t \le T} |u(t, x)|$ is known to be bounded. We begin by estimating the convergence for the free solution. Let $\widetilde{\Phi}_{kj} = \Phi_{kj} + \phi^3_{kj}$.

**Lemma 10.** *Consider a discrete free solution:* $(L_h v)^n_{kj} = 0$, $v^0_{kj} = \phi_{kj}$, $v^1_{kj} = \phi_{kj} + \Delta t \psi_{kj} + \Delta t^2 \widetilde{\Phi}_{kj}/2$. *On any interval* $[0, T]$ *with* $n\Delta t = T$, $h = \sqrt{2}\Delta t$, *there is a constant* $c_T$, *depending only on the data and on* $T$, *such that*

$$\sup_{k,j} |v(t^n, x_k, y_j) - v^n_{kj}| \le c_T \Delta t^2.$$

*Proof.* Let the truncation error $\overline{\tau}^n_{kj}$ be defined by (40), with $u$ replaced by $v$ and the nonlinear terms dropped (i.e., $G(\cdot) \equiv 0$). We define

$$\rho^n_{kj} = v(t^n, x_k, y_j) - v^n_{kj}. \tag{62}$$

Then $\rho^n_{kj}$ satisfies the recursion

$$\rho^{n+1}_{kj} = -\rho^{n-1}_{kj} + \tfrac{1}{2}[\rho^n_{k+1,j} + \rho^n_{k-1,j} + \rho^n_{k,j+1} + \rho^n_{k,j-1}] + \Delta t^2 \cdot \overline{\tau}^n_{kj}$$

in analogy to (16). Since $v \in C^4[[0, T] \times \mathbb{R}^2]$ and is bounded in that space,

$$|\overline{\tau}^n_{kj}| \le c_T \Delta t^2.$$

The initial values are

$$\rho^0_{kj} = 0, \qquad \rho^1_{kj} = v(\Delta t, x_k, y_j) - \left[\phi_{kj} + \Delta t \psi_{kj} + \frac{\Delta t^2}{2} \widetilde{\Phi}_{kj}\right], \tag{63}$$

and hence

$$|\rho^1_{kj}| = O(\Delta t^3) \tag{64}$$

uniformly on $[0, T] \times \mathbb{Z}^2$. The representation formulas (29), (30) then give us

$$\begin{aligned} \rho_{kj}^{n+1} &= 2^{-n} \sum_{p=0}^{[n/2]} \sum_{\substack{\alpha, \beta \\ |\alpha-k|+|\beta-j|=n-2p}} S_{p,k-\alpha}^{n} \rho_{\alpha\beta}^{1} \\ &\quad + \Delta t^2 \sum_{l=0}^{n-1} 2^{-l} \sum_{p=0}^{[l/2]} \sum_{\substack{\alpha, \beta \\ |\alpha-k|+|\beta-j|=l-2p}} S_{p,k-\alpha}^{l} \overline{\tau}_{\alpha\beta}^{n-l} . \end{aligned} \tag{65}$$

By Lemma 3, part (ii), we have $\sum_{p,k} S_{pk}^{l} = (1+l) \cdot 2^{l}$. Therefore,

$$\begin{aligned} |\rho_{kj}^{n+1}| &\le c_T 2^{-n} \cdot \Delta t^3 \sum_{p=0}^{[n/2]} \sum_{|\alpha-k|+|\beta-j|=n-2p} S_{p,k-\alpha}^{n} \\ &\quad + c_T \Delta t^4 \sum_{l=0}^{n-1} 2^{-l} \sum_{p=0}^{[l/2]} \sum_{|\alpha-k|+|\beta-j|=l-2p} S_{p,k-\alpha}^{l} \\ &= c_T \Delta t^3 (1+n) + c_T \Delta t^4 \sum_{l=0}^{n-1} (1+l) \le c_T \Delta t^2 , \end{aligned} \tag{66}$$

as desired. □

**Theorem 11.** *Consider an arbitrary time interval* $[0, T]$. *Let* $n\Delta t = T$, $h = \sqrt{2}\Delta t$. *Then there exists a constant* $c_T$, *depending only on* $T$ *and the data, such that*

$$\sup_{k,j} |u(t^n, x_k, y_j) - u_{kj}^n| \le c_T \Delta t^2 \left[ \ln \frac{1}{\Delta t} \right]^{1/2}$$

*for* $\Delta t$ *sufficiently small.*

*Proof*. As before, we define the errors by

$$e_{kj}^n = u(t^n, x_k, y_j) - u_{kj}^n \tag{42}$$

and the truncation error $\tau_{kj}^n$ by (40).

The error equation (44) can be written

$$\begin{aligned} e_{kj}^{n+1} &= -e_{kj}^{n-1} + \tfrac{1}{2}[e_{k+1,j}^n + e_{k-1,j}^n + e_{k,j+1}^n + e_{k,j-1}^n] \\ &\quad + \Delta t^2 \cdot \tau_{kj}^n + \Delta t^2 \{\overline{H}_u e_{kj}^{n+1} + \overline{H}_v e_{kj}^{n-1}\}, \end{aligned} \tag{67}$$

where we have used (46). The initial values are

$$e_{kj}^0 = 0, \qquad e_{kj}^1 = O(\Delta t^3) \tag{68}$$

uniformly on $[0, T] \times \mathbb{Z}^2$. We represent the solution $e_{kj}^{n+1}$ of (67) using (24):

$$e_{kj}^{n+1} = (e_{\text{free}})_{kj}^{n+1} + (e_{NL})_{kj}^{n} . \tag{69}$$

Here, $\|e_{\text{free}}^{n+1}\|_\infty \le c_T \Delta t^2$ from Lemma 10, and

$$(70)\qquad (e_{NL})_{kj}^n \equiv \Delta t^2 \sum_{l=0}^{n-1} 2^{-l} \sum_{p=0}^{[l/2]} \cdot \sum_{|\alpha-k|+|\beta-j|=l-2p} S_{p,k-\alpha}^l [\tau_{\alpha\beta}^{n-l} + \overline{H}_u e_{\alpha\beta}^{n+1-l} + \overline{H}_v \cdot e_{\alpha\beta}^{n-1-l}].$$

By Lemma 6, $\tau_{kj}^n = O(\Delta t^2)$ uniformly on $[0, T] \times \mathbb{Z}^2$. Hence, the truncation error term in $e_{NL}$ is dominated by

$$(71)\qquad c_T \Delta t^4 \sum_{l=0}^{n-1} 2^{-l} \sum_{p=0}^{[l/2]} \sum_{|\alpha-k|\le l-2p} S_{p,k-\alpha}^l = c_T \Delta t^4 \sum_{l=0}^{n-1} (1+l) \le c_T \Delta t^2,$$

where we have used Lemma 3 (ii).

Now consider the nonlinear term involving $\overline{H}_u$ in the expression $e_{NL}$; call it $\mathscr{H}_u$. Our method of estimation will be such that the term $\mathscr{H}_v$ is handled in the same way. We write, for indices $\beta$ with $|\beta - j| = l - 2p - |\alpha - k|$,

$$(72)\qquad |\mathscr{H}_u| \le \Delta t^2 \sum_{l=0}^{n-1} 2^{-l} \left( \sum_{p,\alpha} (S_{p\alpha}^l)^2 \right)^{1/2} \left( \sum_{p,\alpha,\beta} |\overline{H}_u|^4 \right)^{1/4} \cdot \left( \sum_{p,\alpha,\beta} |e_{\alpha\beta}^{n+1-l}|^4 \right)^{1/4}.$$

In view of Lemma 4, the square norm of $S$ appearing here is less than $c \cdot 2^l (\ln(l+2))^{1/2}$. The sum involving $|\overline{H}_u|^4$ has been dealt with in (53), (54); the result was given in (55):

$$(55)\qquad \sum_{k,j} |\overline{H}_u|^4 \le c \Delta t^{-2}.$$

As for the last term in (72), we recall (51):

$$(51)\qquad \sum_{k,j} |e_{kj}^{n+1}|^4 \le c_T \Delta t^2 (\tilde{\mathscr{E}}^n)^2 \le c_T \Delta t^2 [\tilde{\mathscr{E}}^0 + \Delta t^2]^2 \le c_T \Delta t^6,$$

by Theorem 9 and (61). Thus, from (72) we get

$$(73)\qquad \begin{aligned} |\mathscr{H}_u| &\le c_T \Delta t^2 \cdot \sum_{l=0}^{n-1} \ln^{1/2}(l+2) [c \Delta t^{-2}]^{1/4} [c_T \Delta t^6]^{1/4} \\ &\le c_T \Delta t^3 \sum_{l=0}^{n-1} \ln^{1/2}(l+2) \\ &\le c_T \Delta t^3 \cdot \ln^{1/2}(n+1) \cdot n \le c_T \Delta t^2 \left[ \ln \frac{1}{\Delta t} \right]^{1/2} \end{aligned}$$

for $\Delta t$ sufficiently small. As mentioned above, the $\mathscr{H}_v$-term is estimated similarly. Therefore, from (69) we conclude that

$$\sup_{k,j} |e_{kj}^{n+1}| \leq c_T \Delta t^2 \left[ \ln \frac{1}{\Delta t} \right]^{1/2},$$

which proves the result. □

*Concluding remarks.* 1. Suppose the nonlinear term were $u^p$ instead of $u^3$, where $p$ is odd, $p \geq 5$. The energy (both continuous and discrete) gives a bound on $u^{p+1}$ in $L^1$, and Sobolev shows that $u^q$ is then bounded in $L^1$, for $p+1 \leq q < \infty$. Thus, the preceding analysis can be carried through for other power functions.

2. We point out that we approximate $u(\Delta t, x_k, y_j)$ by $u_{kj}^1$ to *third-order accuracy*. This allows us to use only $L^1$ and $L^2$ estimates on the kernel $S_{pk}^l$, and seems to be where the "loss of derivatives" problem arises.

## Bibliography


1. J. Ginibre and G. Velo, *The global Cauchy problem for the nonlinear Klein-Gordon equation*, Math. Z. **189** (1985), 487–505.
2. R. Glassey, *An elementary proof of time decay for solutions to semilinear wave equations in low space dimension*, Brown University LCDS Report #82-24, 1982.
3. R. Glassey, E. Horst, A. Lenard, and J. Schaeffer, *On the inverse of the discrete two-dimensional wave operator at CFL*, Preprint.
4. R. Glassey and H. Pecher, *Time decay for nonlinear wave equations in two space dimensions*, Manuscripta Math. **38** (1982), 387–400.
5. E. Isaacson and H. B. Keller, *Analysis of numerical methods*, Wiley, New York, 1966.
6. W. Magnus, F. Oberhettinger, and R. P. Soni, *Formulas and theorems for the special functions of mathematical physics*, Springer-Verlag, New York, 1966.
7. W. Strauss, *Nonlinear wave equations*, CBMS Lectures, George Mason University, January, 1989.
8. W. Strauss and L. Vasquez, *Numerical solution of a nonlinear Klein-Gordon equation*, J. Comput. Phys. **28** (1978), 271–278.
9. L. Vasquez and G. Benyu, *A numerical scheme for nonlinear Klein-Gordon equations*, Chinese J. Appl. Sci. **1** (1983), 25–32.



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