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THE Γ-FUNCTION REVISITED: POWER SERIES EXPANSIONS AND REAL-IMAGINARY ZERO LINES

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ABSTRACT. Explicit power series expansions of the gamma function are given close to $-10, -2, -1, 0, 1, 2, 3, 4$, and 10 together with formulas that can be used in other integer points. Further, curves along which the real or imaginary part of the function vanish are presented.

1. INTRODUCTION

When just one isolated value of the Γ-function is needed, it may be computed in a straightforward fashion by Stirling's formula, followed by repeated use of the functional relation $\Gamma(z+1) = z\Gamma(z)$. However, if more detailed knowledge of the function is desired within a small area close to the real axis, a different approach might be reasonable. For $n = 2, 3, 4, 10$ we give expansions of the form

$$\Gamma(n+1+z) = n!(1 + d_1 z + d_2 z^2 + \cdots),$$

and for $n = 0, 1, 2, 10$,

$$\begin{aligned}(-1)^n n!\Gamma(-n+z) = n/(1-z) - 1/((n+1)(1+z))\\ + z^{-1}(1 + f_1 z + f_2 z^2 + \cdots).\end{aligned}$$

Reasonably fast convergence is obtained, at least for $|z| < 1$.

2. A USEFUL ALGORITHM

We will propose an algorithm for computation of certain finite products (also infinite when convergent). This will give us a tool for determining various sets of coefficients. Let

$$P = \prod_{k=1}^{n} \left(1 + \frac{x}{p_k}\right)^{m_k} = 1 + u_1 x + u_2 x^2 + \cdots, \tag{1}$$

where p_k and m_k are given real numbers $\neq 0$ and the coefficients u_k are to be determined. Taking logarithms and differentiating, we obtain

$$\ln P = \sum_{k=1}^{n} m_k \ln\left(1 + \frac{x}{p_k}\right), \qquad \frac{P'}{P} = \sum_{k=1}^{n} \frac{m_k}{x + p_k}.$$

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Now we define

$$(2) \qquad f_r(x) = (-1)^{r+1} \sum_{k=1}^{n} m_k (x+p_k)^{-r}, \qquad r = 1, 2, 3, \dots,$$

with the obvious property

$$(3) \qquad f_r'(x) = r f_{r+1}(x).$$

Further, we put

$$(4) \qquad f_r(0) = w_r = (-1)^{r+1} \sum_{k=1}^{n} m_k p_k^{-r}.$$

Repeated differentiation gives

$$\begin{aligned} P' &= P f_1, \\ P'' &= P' f_1 + P f_2, \\ P''' &= P'' f_1 + 2P' f_2 + 2P f_3, \\ P^{\text{iv}} &= P''' f_1 + 3P'' f_2 + 6P' f_3 + 6P f_4, \\ &\dots\dots\dots \end{aligned}$$

Setting $x = 0$ and observing that $P^{(r)}(0) = r! u_r$ with $P(0) = 1$, we get the system

$$(5) \qquad \begin{aligned} u_1 &= w_1, \\ 2u_2 &= w_1 u_1 + w_2, \\ 3u_3 &= w_1 u_2 + w_2 u_1 + w_3 \\ 4u_4 &= w_1 u_3 + w_2 u_2 + w_3 u_1 + w_4, \\ &\dots\dots\dots \end{aligned}$$

Note that if the m_k are positive integers, then P is a polynomial, and all u_k with $k > \sum_{r=1}^{n} m_r$ are zero. Using Cramer's rule, we find after some manipulation:

$$(6) \qquad u_n = \frac{1}{n!} \begin{vmatrix} w_1 & -1 & 0 & \cdots & 0 \\ w_2 & w_1 & -2 & \cdots & 0 \\ w_3 & w_2 & w_1 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & -(n-1) \\ w_n & w_{n-1} & \cdots & \cdots & w_1 \end{vmatrix}.$$

In practical computation the system (5) should be solved recursively. However, we also present the first few values explicitly:

$$(7) \qquad \begin{aligned} u_1 &= w_1, \\ u_2 &= \tfrac{1}{2}(w_1^2 + w_2), \\ u_3 &= \tfrac{1}{6}(w_1^3 + 3w_1 w_2 + 2w_3), \\ u_4 &= \tfrac{1}{24}(w_1^4 + 6w_1^2 w_2 + 8w_1 w_3 + 3w_2^2 + 6w_4), \\ &\dots\dots\dots\dots \end{aligned}$$

It is interesting to observe that these expressions also appear in a quite different context. Consider the permutations of, e.g., four objects, which we denote

1, 2, 3, and 4. Let (1)(2)(3)(4) be the identical permutation; (1 3)(2 4) the permutation where 1 and 3 switch places, and the same for 2 and 4; (3)(1 4 2) the permutation where 3 keeps its place, while 1 goes over into 4, 4 into 2, and 2 into 1; etc. Denote a part permutation involving k objects by s_k; then, e.g., (1)(3)(2 4) is written $s_1^2 s_2$, (2)(1 4 3) as $s_1 s_3$, and so on. Writing down all $24 = 4!$ permutations and dividing by 24, we get the so-called *cycle index*

$$S_4 = \tfrac{1}{24}(s_1^4 + 6s_1^2 s_2 + 8s_1 s_3 + 3s_2^2 + 6s_4),$$

an expression with exactly the same structure as u_4 in (7).

3. Power series expansion for $\Gamma(z+2)$

As is well known, the Γ-function is defined in product form by

$$\Gamma(z+1) = e^{-\gamma z} \prod_{n=1}^{\infty} \frac{e^{z/n}}{1 + z/n}, \tag{8}$$

where as usual γ is the Euler constant, $\gamma = 0.57721\,56649\ldots$. Taking logarithms, we obtain

$$\begin{aligned} \ln \Gamma(z+1) &= -\gamma z + \sum_{n=1}^{\infty} \left[\frac{z}{n} - \ln\left(1 + \frac{z}{n}\right)\right] \\ &= (1-\gamma)z - \ln(1+z) + \sum_{n=2}^{\infty} \frac{(-1)^n(\zeta(n)-1)z^n}{n}, \end{aligned}$$

where $\zeta(n) = \sum_{k=1}^{\infty} k^{-n}$, $n > 1$, is the Riemann zeta function. Now we introduce the notations

$$\begin{cases} g_1 = 1 - \gamma, \\ g_r = (-1)^r(\zeta(r) - 1), \qquad r = 2, 3, 4, \ldots, \end{cases} \tag{9}$$

and obtain $\Gamma(z+2) = \exp(g_1 z + \frac{1}{2}g_2 z^2 + \frac{1}{3}g_3 z^3 + \cdots)$. We can then construct the corresponding power series

$$\begin{aligned} G(z) = \Gamma(z+2) &= \exp(g_1 z + \tfrac{1}{2}g_2 z^2 + \cdots) \\ &= 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots. \end{aligned} \tag{10}$$

Differentiation gives

$$G'(z) = G(z) \cdot (g_1 + g_2 z + g_3 z^2 + \cdots) = c_1 + 2c_2 z + 3c_3 z^2 + \cdots.$$

Hence, we obtain a linear system

$$\begin{aligned} c_1 &= g_1, \\ 2c_2 &= g_1 c_1 + g_2, \\ 3c_3 &= g_1 c_2 + g_2 c_1 + g_3, \\ &\cdots\cdots\cdots \end{aligned}$$

which has the same structure as (5). The solution is

$$c_n = \frac{1}{n!} \begin{vmatrix} g_1 & -1 & 0 & 0 & \cdots & 0 \\ g_2 & g_1 & -2 & 0 & \cdots & 0 \\ g_3 & g_2 & g_1 & -3 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ g_n & g_{n-1} & \cdots & \cdots & \cdots & g_1 \end{vmatrix}. \tag{11}$$

TABLE 1

Numerical values of g_n (9) *and* c_n (11). *For larger values of* k, $(-1)^k(2^{-(k+1)} - 3^{-(k+1)})$ *is a remarkably good approximation of* c_k.

n	g_n	c_n	n	g_n	c_n
1	0.42278 43351 +0	0.42278 43351 +0	11	−0.49418 86041 −3	−0.24094 14358 −3
2	0.64493 40668 +0	0.41184 03304 +0	12	0.24608 65533 −3	0.12167 38065 −3
3	−0.20205 69032 +0	0.81576 91925 −1	13	−0.12271 33476 −3	−0.60792 89132 −4
4	0.82323 23371 −1	0.74249 01075 −1	14	0.61248 13506 −4	0.30453 55703 −4
5	−0.36927 75514 −1	−0.26698 20687 −3	15	−0.30588 23631 −4	−0.15234 93590 −4
6	0.17343 06198 −1	0.11154 04572 −1	16	0.15282 25941 −4	0.76217 79696 −5
7	−0.83492 77382 −2	−0.28526 45821 −2	17	−0.76371 97638 −5	−0.38121 10400 −5
8	0.40773 56198 −2	0.21039 33341 −2	18	0.38172 93265 −5	0.19064 91658 −5
9	−0.20083 92826 −2	−0.91957 38388 −3	19	−0.19082 12717 −5	−0.95338 77803 −6
10	0.99457 51278 −3	0.49038 84508 −3	20	0.95396 20339 −6	0.47674 16946 −6

The coefficients c_n are well known; they were given by Davis [1, p. 186] and were denoted there by E_n. On the other hand, only very few power series expansions are reported in the *Handbook of Mathematical Functions* [2]; the most interesting is one for $1/\Gamma(z)$. We note here that $z = \pm 1$ leads to the relation $c_1 + c_3 + c_5 + \cdots = c_2 + c_4 + c_6 + \cdots = \frac{1}{2}$.

The numerical values of c_n are obtained recursively from the linear system. They are displayed in Table 1 together with the constants g_n.

4. POWER SERIES EXPANSIONS FOR $\Gamma(n+1+z)$

Using the fundamental relation $\Gamma(z+1) = z\Gamma(z)$ repeatedly, we find, when $n \geq 2$,

$$\Gamma(n+1+z) = (z+n)(z+n-1)\cdots(z+2)\Gamma(z+2)$$

or, after division by $\Gamma(n+1) = n!$,

$$\frac{\Gamma(n+1+z)}{\Gamma(n+1)} = \left(1+\frac{z}{2}\right)\left(1+\frac{z}{3}\right)\cdots\left(1+\frac{z}{n}\right)(1+c_1 z + c_2 z^2 + \cdots). \tag{12}$$

The product $(1+\frac{z}{2})(1+\frac{z}{3})\cdots(1+\frac{z}{n}) = 1 + a_1 z + \cdots + a_{n-1} z^{n-1}$ is computed as in (5) with $w_r = (-1)^{r+1}\sum_{k=2}^{n} k^{-r}$ (note that w_r depends on n). Finally, we compute the product

$$(1 + a_1 z + \cdots + a_{n-1} z^{n-1})(1 + c_1 z + c_2 z^2 + \cdots) = 1 + d_1 z + d_2 z^2 + \cdots \tag{13}$$

and obtain the result

$$d_k = \sum_{r=0}^{k} a_r c_{k-r} \tag{14}$$

with $a_0 = c_0 = 1$ and $a_k = 0$ when $k \geq n$. It is obvious that also the coefficients $d_1, d_2, d_3, \ldots$ depend on n. The case $n = 1$ is trivial, since then $d_k = c_k$. Also the case $n = 0$ is easy to handle, since the relation $\Gamma(z+1) = \Gamma(z+2)/(z+1)$ leads to $d_k = c_k - d_{k-1}$ with $d_0 = 1$. For $n = 2, 3, 4$ and 10, numerical values of d_k are given in Table 2.

TABLE 2

The coefficients d_k for computation of $\Gamma(n+1+z) = \Gamma(n+1)\cdot(1+d_1z+d_2z^2+\cdots)$. When $n=1$ we have $d_k = c_k$ (see Table 1). When $n=0$, use $\Gamma(1+z) = (1+c_1z+c_2z^2+\cdots)/(1+z)$.

k	$n=2$	$n=3$	$n=4$	$n=10$
1	0.92278 43351 +00	0.12561 17668 +01	0.15061 17668 +01	0.23517 52589 +01
2	0.62323 24980 +00	0.93082 72763 +00	0.12448 56693 +01	0.28129 53288 +01
3	0.28749 70845 +00	0.49524 12505 +00	0.72794 80695 +00	0.22782 17646 +01
4	0.11503 74704 +00	0.21086 98319 +00	0.33468 01445 +00	0.14037 86967 +01
5	0.36857 52331 −01	0.75203 34677 −01	0.12792 08047 +00	0.70121 89188 +00
6	0.11020 55468 −01	0.23306 39579 −01	0.42107 23248 −01	0.29552 29663 +00
7	0.27243 77038 −02	0.63978 95266 −02	0.12224 49421 −01	0.10799 66380 +00
8	0.67761 04301 −03	0.15857 36109 −02	0.31852 09926 −02	0.34911 52784 −01
9	0.13239 28315 −03	0.35826 29749 −03	0.75469 70023 −03	0.10135 47328 −01
10	0.30601 53141 −04	0.74732 47525 −04	0.16429 82190 −03	0.26742 17142 −02
11	0.42527 89587 −05	0.14453 30006 −04	0.33136 41887 −04	0.64741 28057 −03
12	0.12030 88620 −05	0.26206 85149 −05	0.62340 10164 −05	0.14494 79266 −03
13	0.44011 95495 −07	0.44504 14950 −06	0.11002 12782 −05	0.30209 44116 −04
14	0.57111 38227 −07	0.71782 03392 −07	0.18304 24077 −06	0.58937 70496 −05
15	−0.81573 76050 −08	0.10879 75137 −07	0.28825 25985 −07	0.10815 35153 −05
16	0.43117 48695 −08	0.15926 23345 −08	0.43125 61188 −08	0.18745 04774 −06
17	−0.12205 52013 −08	0.21669 75524 −09	0.61485 33886 −09	0.30796 97095 −07
18	0.43645 75319 −09	0.29606 86106 −10	0.83781 24916 −10	0.48116 78892 −08
19	−0.14195 14757 −09	0.35343 68324 −11	0.10936 08359 −10	0.71695 06577 −09
20	0.47804 44902 −10	0.48729 04734 −12	0.13708 82554 −11	0.10213 91242 −09

5. POWER SERIES EXPANSION RELATED TO $\Gamma(-n+z)$

By straightforward calculation we find

$$\Gamma(z-n) = \frac{(-1)^n\Gamma(z+2)}{n!z}\left\{(1+z)(1-z)\left(1-\frac{z}{2}\right)\cdots\left(1-\frac{z}{n}\right)\right\}^{-1}. \tag{15}$$

Hence, $m_k = -1$ and $p_k = -k$, if we add an extra value $p_0 = 1$. In this way we obtain

$$w_r = (-1)^r + \sum_{k=1}^{n} k^{-r}. \tag{16}$$

The coefficients b_r for the reciprocal of the expression within braces are again obtained from (5). The final coefficients e_r defined through

$$\Gamma(z-n) = ((-1)^n/n!z)(1+e_1z+e_2z^2+\cdots)$$

can be computed as in (14). When $n=0$, we have

$$\begin{aligned}\Gamma(z) = \frac{\Gamma(z+2)}{z(z+1)} &= \frac{1}{z}(1+c_1z+c_2z^2+\cdots)(1-z+z^2-\cdots)\\ &= \frac{1}{z}(1+e_1z+e_2z^2+\cdots),\end{aligned}$$

and hence $e_r = c_r - e_{r-1}$ with $e_0 = 1$. Obviously, the coefficients e and d are identical when $n=0$.

TABLE 3

Coefficients f_k for $n = 0, 1, 2, 10$ and $k = 1(1)20$ defined by $(-1)^n n!\Gamma(-n+z) = n/(1-z) - 1/(n+1)(1+z) + \frac{1}{z}(1 + f_1 z + f_2 z^2 + \cdots)$.

k	$n = 0$	$n = 1$	$n = 2$	$n = 10$
1	0.42278 43351 +0	−0.77215 66490 −1	−0.74388 23316 +0	−0.75573 38320 +1
2	−0.10944 00467 −1	−0.88159 66957 −1	−0.46010 08354 +0	−0.57281 88072 +1
3	0.92520 92392 −1	0.43612 54346 −2	−0.22568 91633 +0	−0.39862 02989 + 1
4	−0.18271 91317 −1	−0.13910 65882 −1	−0.12675 52405 +0	−0.25857 89587 +1
5	0.18004 93110 −1	0.40942 72277 −2	−0.59283 34797 −1	−0.15806 26439 +1
6	−0.68508 85379 −2	−0.27566 13102 −2	−0.32398 28709 −1	−0.92200 23064 +0
7	0.39982 39558 −2	0.12416 26456 −2	−0.14957 51709 −1	−0.51762 70823 +0
8	−0.18943 06217 −2	−0.65267 97612 −3	−0.81314 38305 −2	−0.28225 77798 +0
9	0.97473 23780 −3	0.32205 26168 −3	−0.37436 66536 −2	−0.15036 97153 +0
10	−0.48434 39272 −3	−0.16229 13104 −3	−0.20341 24578 −2	−0.78757 12620 −1
11	0.24340 24914 −3	0.81111 18100 −4	−0.93595 11081 −3	−0.40699 95193 −1
12	−0.12172 86849 −3	−0.40617 50386 −4	−0.50859 30579 −3	−0.20839 00601 −1
13	0.60935 79356 −4	0.20318 28969 −4	−0.23397 82393 −3	−0.10591 68640 −1
14	−0.30482 23652 −4	−0.10163 94683 −4	−0.12715 30665 −3	−0.53582 39971 −2
15	0.15247 30062 −4	0.50833 53797 −5	−0.58493 17943 −4	−0.27001 07474 −2
16	−0.76255 20927 −5	−0.25421 67130 −5	−0.31788 75684 −4	−0.13576 96113 −2
17	0.38134 10527 −5	0.12712 43397 −5	−0.14623 13503 −4	−0.68127 30636 −3
18	−0.19069 18869 −5	−0.63567 54719 −6	−0.79472 42984 −5	−0.34155 98177 −3
19	0.95353 10888 −6	0.31785 56169 −6	−0.36557 65875 −5	−0.17104 44012 − 3
20	−0.47678 93943 −6	−0.15893 37773 −6	−0.19868 16715 −5	−0.85635 72098 −4

It turned out that the coefficients e_k for large values of k essentially behave as $n + (-1)^k/(n+1)$. For this reason, we tabulate in Table 3 other coefficients f_k defined by

$$f_k = e_k - (n + (-1)^k/(n+1)), \tag{17}$$

giving much better accuracy. If a complete power series is wanted, it is an easy matter to recompute the coefficients e_k. Using the new coefficients, we find

$$\Gamma(-n+z) = \frac{(-1)^n}{(n-1)!(1-z)} - \frac{(-1)^n}{(n+1)!(1+z)} + \frac{(-1)^n}{n!z}(1 + f_1 z + f_2 z^2 + \cdots). \tag{18}$$

This formula will give reasonable results, even when $|z|$ is only slightly less than 1, and when $|z| < \frac{1}{2}$, we will get 10-digit accuracy. In the case $n = 0$ we get

$$f_k = d_k + (-1)^{k-1} = e_k + (-1)^{k-1}, \qquad k = 1, 2, 3, \ldots, \tag{19}$$

with

$$\Gamma(z) = -\frac{1}{1+z} + \frac{1}{z}(1 + f_1 z + f_2 z^2 + \cdots)$$

and, of course, $\Gamma(1+z) = z\Gamma(z)$. For larger values of k, the coefficients f_k behave approximately as $-2^{-k}(n(n-1) + (-1)^k/(n+1)(n+2))$.

Since the coefficients vary in a very regular manner for larger values of k, at least 8-digit accuracy is easily attainable when $|z| \le 1$.

6. Zero lines for the real and the imaginary part

As is well known, the Γ-function has no zeros. However, the function values are real along the real axis, and we now ask if there are curves in the complex plane where the real or the imaginary part vanishes. It will then be helpful to consider points $z = x + \varepsilon e^{iv}$ with x real. We start with the imaginary part and find

$$\operatorname{Im} \Gamma(z) = \varepsilon \Gamma'(x) \sin v + \tfrac{1}{2}\varepsilon^2 \Gamma''(x) \sin 2v + O(\varepsilon^3).$$

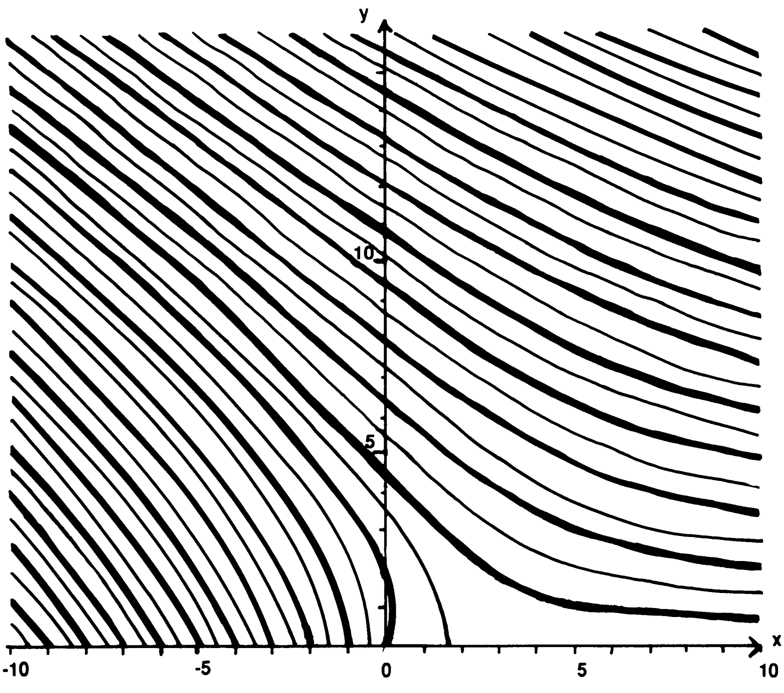
This part will vanish if $\Gamma'(x) = 0$, and further we see that we must have $\sin 2v = 0$, giving $v = \pi/2$. The points defined by $\Gamma'(x) = 0$ are well known and (except for the first one) lie between the poles. A few of them are given in [2] (1.46163, −0.504, −1.573, −2.611, −3.635, −4.658, −5.667, −6.678, etc.).

Since the real part cannot vanish in a regular point on the real axis, we have to investigate the situation close to the poles. We have

$$\Gamma(z - n) = \frac{(-1)^n}{n!z}(1 + e_1 z + e_2 z^2 + \cdots)$$

and define

$$R = \Gamma(-n + \varepsilon e^{iv}) = \frac{(-1)^n}{n!}(\varepsilon^{-1} e^{-iv} + e_1 + e_2 \varepsilon e^{iv} + \cdots).$$



$\operatorname{Im} \Gamma(z) = 0$; $\mathbf{Re}\,\boldsymbol{\Gamma}(\mathbf{z}) = 0$

FIGURE 1. *Real and imaginary zero lines*

The real part is

$$\mathrm{Re}(R) = \frac{(-1)^n}{n!}\left[\varepsilon^{-1}\cos v + e_1 + O(\varepsilon)\right] = 0$$

with $e_1 = f_1 + n - 1/(n+1)$, $n = 0, 1, 2, \ldots$, which gives $\cos v = -e_1\varepsilon$, i.e., $v \simeq \pi/2 + e_1\varepsilon$. Hence, both types of curves pass vertically over the real axis (note, however, that this vertical direction changes very rapidly owing to the large values of e_1). We see that there is one curve with vanishing real part passing through every pole.

The practical computation is tedious but straightforward, and the resulting figure is only intended to give a general impression of the family of curves under discussion.

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