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STABILITY RESULTS FOR ONE-STEP DISCRETIZED COLLOCATION METHODS IN THE NUMERICAL TREATMENT OF VOLTERRA INTEGRAL EQUATIONS

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ABSTRACT. This paper is concerned with the stability analysis of the discretized collocation method for the second-kind Volterra integral equation with degenerate kernel. A fixed-order recurrence relation with variable coefficients is derived, and local stability conditions are given independent of the discretization. Local stability and stability with respect to an isolated perturbation of some methods are proved. The reliability of the derived stability conditions is shown by numerical experiments.

1. INTRODUCTION

We consider the second-kind Volterrra integral equation

$$y(t) = g(t) + \int_{t_0}^{t} k(t, s, y(s))\,ds, \qquad t \in [t_0, T], \tag{1.1}$$

where g and k are given continuous functions on (t_0, T) and $S \times R$, respectively, with $S = \{(t, s): t_0 \le s \le t \le T\}$. Let

$$\Pi_N: t_0 < t_1 < \cdots < t_N = T$$

denote a partition of the integration range, with $t_{i+1} - t_i = h$, and let $0 \le c_1 < c_2 < \cdots < c_m \le 1$ be m parameters. The one-step discretized collocation method [4, 9] (hereafter referred to as VDC) approximates the solution $y(t)$ of (1.1) by a spline function $u(t)$ defined by

$$u(t_i + sh) = \sum_{k=1}^{m} L_k(s) u_{ik}, \qquad s \in [0, 1], \tag{1.2}$$

where

$$L_k(s) = \prod_{\substack{j=1 \\ j \ne k}}^{m} (s - c_j)/(c_k - c_j) \tag{1.3}$$

and the u_{ik} are solutions of the system

$$u_{ij} = g(t_{ij}) = \Psi_j^i + \phi_j^i, \qquad j = 1, \dots, m, \tag{1.4}$$

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where $t_{ij} = t_i + c_j h$ and ψ_j^i, ϕ_j^i are quadrature sums approximating

$$\int_{t_0}^{t_i} k(t_{ij}, s, u(s))\, ds \quad \text{and} \quad \int_{t_i}^{t_{ij}} k(t_{ij}, s, u(s))\, ds,$$

respectively. Following [9] (to which the reader is referred to for more details), the most common choices for ψ_j^i and ϕ_j^i are

(a)

$$\Psi_j^i = h \sum_{\nu=0}^{i-1} \sum_{k=1}^{m-\vartheta} w_k k(t_{ij}, t_{\nu k}, u_{\nu k}), \tag{1.5}$$

$$\phi_j^i = h \sum_{k=1}^{m-\vartheta} w_{jk} k(t_{ij}, t_{ik}, u_{ik}), \tag{1.6}$$

where

$$w_k = \int_0^1 L_k(s)\, ds, \qquad k = 1, \ldots, m, \tag{1.7}$$

$$w_{jk} = \int_0^{c_j} L_k(s)\, ds, \qquad k, j = 1, \ldots, m, \tag{1.8}$$

$$\vartheta = \begin{cases} 0 & \text{for the "implicit discretization,"} \\ 1 & \text{for the "discretization using m} - 1 \text{ points,"} \\ j-1 & \text{for the "explicit discretization,"} \\ j & \text{for the "diagonally explicit discretization."} \end{cases}$$

(b)

"Fully implicit discretization": the ψ_j^i given by (1.5) with $\vartheta = 0$,

$$\phi_j^i = h \sum_{k=1}^{m} c_j w_k k\left(t_{ij}, t_i + c_j c_k h, \sum_{\nu=1}^{m} L_\nu(c_j c_k) u_{i\nu}\right), \tag{1.9}$$

so that the kernel k is computed only in S.

VDC methods, until now, have been studied mainly with regard to convergence order [4, 7, 8], and less so with regard to stability properties. In [10] and [11] we analyzed stability of the exact collocation method by applying it to the basic and the convolution test equation, and to the second-kind integral equation with degenerate kernel, respectively. Here we perform the stability analysis of the VDC method for a linear second-kind integral equation with degenerate kernel of rank n [3, 5],

$$y(t) = g(t) + \int_{t_0}^{t} \sum_{l=1}^{n} a_l(t) b_l(s) y(s)\, ds, \qquad t \in [t_0, T]. \tag{1.10}$$

From the Stone-Weierstrass theorem it follows that the class of degenerate kernels,

$$k(t, s) = \sum_{l=1}^{n} a_l(t) b_l(s), \tag{1.11}$$

is dense in the class of all continuous kernels, so that (1.10) can be considered a significant test equation for stability analysis.

Conditions ensuring the stability of the analytical solution of (1.1) are given in [1, 5, 9].

In order to analyze the stability properties of VDC methods, a finite-length recurrence relation is constructed in §2 for a vector containing the numerical solution. This recursion is then used to prove local stability conditions. The conditions obtained are independent of the kernel decomposition and require the localization of the roots of a polynomial whose degree is the minimum of the rank of the kernel and the number of the collocation points. In the particular case of the degenerate convolution kernel, the above conditions are shown to furnish a priori stability conditions. In §3, conditions for the stability of the trapezoidal rule are given. Then, for particular degenerate kernels, a class of VDC methods is determined, including those whose collocation parameters are the Legendre and Radau points, which satisfy the local stability conditions, and a bound for h is found which ensures the bounded propagation of an isolated perturbation in these methods. Finally, in §5, numerical results are reported, showing the reliability of the stability conditions derived.

2. Recurrence relation and local stability conditions

In this section, stability theorems are derived for the VDC method applied to the second-kind Volterra integral equation with degenerate kernel (1.10). These theorems are independent of the choice of discretizations, such as those suggested in §1. The results obtained are analogous to those derived for the exact collocation method in [11], and since they are based on the same technique, only a short outline of the proofs is given.

First we derive a finite-length recurrence relation for the method. To do so, we set

$$\alpha_{jk}^{i} = a_k(t_{ij}), \quad j = 1, \dots, m, \ k = 1, \dots, n, \qquad A^i = [\alpha_{jk}^{i}]. \tag{2.1}$$

In correspondence to the chosen quadrature formulae (1.6), (1.9) we denote by z_j^i a quadrature sum for the integral $\int_{t_0}^{t_i} b_j(s)u(s)\,ds$ and we set

$$s_{jk}^{i} = \begin{cases} w_{jk}k(t_{ij}, t_{ik}), & k = 1, \dots, m-\vartheta, \ j = 1, \dots, m \\ & \text{for the discretization (a)}, \\ 0 & k = m-\vartheta+1, \dots, m, \ j = 1, \dots, m, \\ c_j w_k \sum_{\nu=1}^{m} k(t_i, & t_i + c_j c_\nu h) L_k(c_j c_\nu), \\ & j, k = 1, \dots, m, \ \text{for the discretization (b)}, \end{cases} \tag{2.2}$$
$$S^i = (s_{jk}^{i}),$$

$$\beta_{jk}^{i} = \begin{cases} w_k b_j(t_{ik}), & k = 1, \dots, m-\vartheta, \ j = 1, \dots, n, \\ 0, & k = m-\vartheta+1, \dots, m, \ j = 1, \dots, n, \end{cases} \tag{2.3}$$
$$B^i = (\beta_{jk}^{i}).$$

Let I_k be the identity matrix of order k and

$$v_i = [u(t_i), u_{i-1,1}, \dots, u_{i-1,m}, z_1^i, \dots, z_n^i]^{\mathrm{T}}, \tag{2.4}$$

$$g_i = [g(t_{i1}), \dots, g(t_{im})]^{\mathrm{T}}, \tag{2.5}$$

$$L = [L_1(1), \dots, L_m(1)]^{\mathrm{T}}, \tag{2.6}$$

$$M_0^i = \begin{bmatrix} 1 & -L & 0 \\ 0 & I_m - hS^i & 0 \\ 0 & -hB^i & I_n \end{bmatrix}, \qquad M_1^i = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & A^i \\ 0 & 0 & I_n \end{bmatrix}.$$

Then the following theorem holds.

Theorem 2.1. *The application of the VDC method to a second-kind Volterra integral equation with degenerate kernel leads to the finite-length recurrence relation*

$$M_0^i v_{i+1} - M_1^i v_i = \begin{pmatrix} 0 \\ g_i \\ 0 \end{pmatrix}. \tag{2.7}$$

Proof. The proof is outlined only for the case of the implicit discretization. Using the above notations, we have

$$\psi_j^i = \sum_{l=1}^{n} a_l(t_{ij}) z_l^i,$$

and the assertion follows from (1.2) and (1.4) by observing that in the case considered, z_l^i satisfies

$$z_l^{i+1} = z_l^i + h \sum_{k=1}^{m} w_k b_l(t_{ik}) u_{ik}. \quad \square \tag{2.8}$$

Remark 2.1. In the particular case of the polynomial convolution kernel, it can be shown by tedious algebraic manipulations that the recurrence formula (2.7) reduces to the recursion derived in [2] for an extended Runge-Kutta method. $\square$

From the applicability of the method it follows that $I_m - hS^i$ is invertible, so that (2.7) can be written as

$$v_{i+1} = M^i v_i + \rho_i, \tag{2.9}$$

where

$$\rho_i = \begin{pmatrix} 0 \\ (I_m - hS^i)^{-1} g_i \\ hB^i (I_m - hS^i) g_i \end{pmatrix}, \tag{2.10}$$

and the stability matrix is given by

$$M^i = (M_0^i)^{-1}(M_1^i) = \begin{bmatrix} 0 & 0 & L(I_m - hS^i)^{-1} A^i \\ 0 & 0 & (I_m - hS^i)^{-1} A^i \\ 0 & 0 & hB^i (I_m - hS^i)^{-1} A^i + I_n \end{bmatrix}. \tag{2.11}$$

Since the elements of M^i depend on the step number i, it is possible to derive conditions only for local stability [9, p. 432].

Put

$$D^i = I_n + hB^i (I_m - hS^i)^{-1} A^i. \tag{2.12}$$

Then the following theorem can be established.

Theorem 2.2. *The VDC method is*

(i) *locally stable in the strong sense if all eigenvalues of* D^i *are inside the unit circle;*

(ii) *locally stable if the eigenvalues of* M^i *are inside the unit circle and those on the boundary are weakly stable, i.e., the corresponding Jordan blocks are of order* 1.

Remark 2.2. When $h \to 0$, the matrix $D^i \to I$, and hence the VDC method is locally stable as $h \to 0$. □

Defining the matrix

$$F(x) = (x-1)(I_m - hS^i) - hA^iB^i \tag{2.13}$$

and the polynomial

$$C(x) = \det[F(x)], \tag{2.14}$$

we can easily prove the following theorem.

Theorem 2.3. *The VDC method is locally stable in* t_i *in the strong sense if all zeros of*

$$(x-1)^{n-m}C(x) = 0 \tag{2.15}$$

are inside the unit circle; it is locally stable if they are inside or on the unit circle and those on the unit circle correspond to weakly stable eigenvalues of M^i.

Remark 2.3. In the case of the implicit and fully implicit discretization, which are the most common, the polynomial $C(x)$ does not depend on the kernel decomposition. In fact, the (j, k) element of the matrix $F(x)$ defined in (2.13) can be written as

$$f_{jk} = (x-1)(\delta_{jk} - hs^i_{jk} - hw_k k(t_{ij}, t_{ik})),$$

and so the stability conditions derived from the above theorem can be applied under the only hypothesis that the kernel is separable, even if the decomposition is unknown.

Remark 2.4. The calculation of the n eigenvalues of (2.12) is reduced to the determination of the roots of a polynomial whose degree is the minimum of n and m.

Corollary 2.1. *If* $n > m$, *then the VDC method is not locally stable in the strong sense.*

Theorem 2.4. *If* $n > m$ *and* $\det(A^iB^i) \neq 0$, *then the VDC method is locally stable in* t_i *when all the zeros of* $C(x)$ *are inside the unit circle.*

Theorem 2.5. *If* $k(t, s)$ *is a convolution degenerate kernel, then the eigenvalues of the matrix* M^i *do not depend on* i.

Note that for this kernel, Theorem 2.3 yields a priori stability conditions.

Remark 2.5. We recall [9] that if $c_m = 1$, then the VDC method is equivalent to an extended Pouzet Runge-Kutta method for the discretization (a), and to a de Hoog and Weiss Runge-Kutta method for the discretization (b). Therefore, all theorems so far stated are valid also for these Runge-Kutta methods.

3. Stability theorems for the implicit discretization method

This section is concerned with stability properties of the VDC method applied to equation (1.10). We first deal with the particular case of two collocation parameters $c_1 = 0$, $c_2 = 1$. For such methods the discretizations of kind (a) and (b) coincide; moreover, the VDC method is equivalent to the trapezoidal rule.

The following theorem holds.

Theorem 3.1. *The trapezoidal rule is locally stable in* t_i *if and only if each of the following three conditions holds*:

(i) $k(t_i, t_i)k(t_{i+1}, t_{i+1}) - k(t_i, t_{i+1})k(t_{i+1}, t_i) > 0$;

(ii) $k(t_{i+1}, t_{i+1}) + k(t_i, t_i) < 0$;

(iii)

$$4 + hk(t_i, t_i) - hk(t_{i+1}, t_{i+1}) - h^2/4[k(t_i, t_i)k(t_{i+1}, t_{i+1}) - k(t_i, t_{i+1})k(t_{i+1}, t_i)] > 0.$$

Proof. The theorem is easily proved by applying Theorem 2.3 and the Routh-Hurwitz conditions to the coefficients of the second-degree polynomial $C(x)$ given in (2.14). □

From Theorem 3.1, the following corollaries can be readily obtained. Let

$$\gamma = \max_i \sup_{(t,s)\in[t_i, t_{i+1}]^2} |k(t, s)|. \tag{3.1}$$

Corollary 3.1. *If conditions* (i) *and* (ii) *of the previous theorem hold, then the trapezoidal rule is locally stable in* t_i *for* $h\gamma < 1$.

Corollary 3.2. *If the kernel* $k(t, s)$ *is of convolution type,* $k(t, s) = k(t - s)$, *and if conditions* (i) *and* (ii) *hold, then the trapezoidal rule is locally stable in each* t_i *if*

$$h < 4/\sqrt{k(0)^2 - k(h)k(-h)}.$$

We now present some results which will be useful later. Define the following matrix:

$$\Gamma(t) = (\gamma_{jk}) = (a_k(t)b_j(t)), \qquad j, k = 1, \dots, n,$$

and vector:

$$P(t) = [b_1(t)g(t), \dots, b_n(t)g(t)]^{\mathrm{T}}.$$

It is known that the solution $y(t)$ of the integral equation (1.10) can be written as

$$y(t) = \sum_{k=1}^{n} a_k(t)z_k(t),$$

where the vector $z(t) = [z_1(t) \cdots z_n(t)]^{\mathrm{T}}$ is the solution of the associated differential system:

$$z'(t) = \Gamma(t)z(t) + P(t), \tag{3.2}$$

$$z(t_0) = 0. \tag{3.2'}$$

Theorem 3.2. *The differential system* (3.2) *is dissipative with respect to the Euclidean norm if and only if a function* $f(t) > 0$ *on* $[t_0, \infty)$ *exists such that* $a_k(t) = -f(t)b_k(t)$, $k = 1, \dots, n$.

Proof. The condition is sufficient, as it can be proved by a trivial extension of Theorem 3.4 in [1]. It is also necessary; in fact, if the system (3.2) is dissipative with respect to the Euclidean norm, the eigenvalues of $\frac{1}{2}(\Gamma + \Gamma^T)$ are negative or equal to 0. These eigenvalues are 0 (of multiplicity $n-2$) and

$$\chi_1 = \sum_{l=1}^{n} a_l(t)b_l(t) + \sqrt{\sum_{l=1}^{n} a_l^2(t) \sum_{l=1}^{n} b_l^2(t)},$$

$$\chi_2 = \sum_{l=1}^{n} a_l(t)b_l(t) - \sqrt{\sum_{l=1}^{n} a_l^2(t) \sum_{l=1}^{n} b_l^2(t)}.$$

The assertion now follows from the Cauchy-Schwarz inequality. □

Definition 3.1. The degenerate kernel (1.11) is of class A if it satisfies the following conditions:

(i) There exist p, q such that $1/p + 1/q = 1$ and

$$\left(\sum_{l=1}^{n} |a_l(t)|^p\right)^{1/p} \int_{t_0}^{t} \left(\sum_{l=1}^{n} |b_l(s)|^q\right)^{1/q} ds < \infty.$$

(ii) There exists a function $f(t) > 0$ on $[t_0, \infty)$ such that

$$a_l(t) = -f(t)b_l(t), \qquad l = 1, \dots, n.$$

From Theorem 3.2 and the results in [2] it follows that integral equations with kernel of class A are stable.

A VDC method with implicit discretization can be written in compact form using the "Butcher array"

$$\begin{array}{c|c} c & W \\ \hline & w \end{array}$$

where the elements of W and w are defined in (1.8) and (1.7), respectively.

Definition 3.2. A VDC method and a Runge-Kutta method for ordinary differential equations characterized by the same Butcher array are said to be "corresponding."

Then we can prove:

Theorem 3.3. *A VDC method corresponding to an algebraically stable Runge-Kutta method is, on kernels of class* A, *locally stable in each* t_i *and for each stepsize* h.

Proof. Setting

$$z_{lj}^i = z_l^i + h \sum_{k=1}^{m} w_{jk} b_l(t_{ik}) u_{ik},$$

we can write (2.8) as

$$z_l^i = z_l^i + h\sum_{k=1}^{m} w_k b_l(t_{ik})\left[g(t_{ik}) + \sum_{r=1}^{n} a_r(t_{ik}) z_{rk}^i\right], \qquad l = 1, \ldots, n,$$

$$z_{lj}^i = z_l^i + h\sum_{k=1}^{m} w_{jk} b_l(t_{ik})\left[g(t_{ik}) + \sum_{r=1}^{n} a_r(t_{ik}) z_{rk}^i\right], \qquad l = 1, \ldots, n, \quad j = 1, \ldots, m. \tag{3.3}$$

This can be viewed as the corresponding Runge-Kutta method applied to the dissipative system (3.2) with initial conditions $z_j(t_i) = z_j^i$, $j = 1, \ldots, n$.

With $Z_i = [z_1^i, \ldots, z_n^i]$, it follows from the algebraic stability of the Runge-Kutta methods that

$$\|Z_{i+1} - Z_{i+1}^*\|_2 \le \|Z_i - Z_i^*\|_2 \quad \text{for every } Z_i,\, Z_i^*, \tag{3.4}$$

where $\|\cdot\|_2$ denotes the Euclidean norm. Since, from (2.9), (2.12), one has

$$Z_{i+1} = D^i Z_i + hB^i(I_m - hS^i)^{-1} g_i, \tag{3.5}$$

the inequality (3.4) implies

$$\|D^i\|_2 \le 1, \qquad i \ge 0, \tag{3.6}$$

and the assertion follows. □

The previous theorem allows us immediately to state the following result about two of the most common VDC methods.

Corollary 3.3. *The VDC method whose collocation parameters are the Gauss points is, on kernels of class A, locally stable in each t_i and for each stepsize h.*

Corollary 3.4. *The Runge-Kutta method Radau* IIA *is, on kernels of class A, locally stable in each t_i and for each stepsize h.*

Now we wish to investigate the conditions for bounded propagation of isolated perturbations [9, p. 428], since it is known that local stability is only a necessary condition for this kind of stability.

Theorem 3.4. *If the corresponding Runge-Kutta method is algebraically stable, then the VDC method involves, on kernels of class A, a bounded propagation of isolated perturbations, provided that $h < 1/\gamma\|W\|_\infty$, where γ is given by* (3.1) *and W by* (1.8).

Proof. Denote by $\{v_i\}$ and $\{v_i^*\}$ the solution of (2.9) obtained for the inhomogeneous terms $\{\rho_i\}$ and $\{\rho_i^*\}$, respectively. Since, from the definition of an isolated perturbation, we have $\|\rho_i - \rho_i^*\|_2 \le \delta$ for $i = 0, \ldots, r$, we get from (3.5) that

$$\begin{aligned}\|Z_{i+1} - Z_{i+1}^*\|_2 &\le \|D_i\|_2\|Z_i - Z_i^*\|_2 \\ &\quad + \|hB^i(I_m - hS^i)^{-1} - (hB^i(I_m - hS^i)^{-1})^*\|_2 \\ &\le \|D_i\|_2\|Z_i - Z_i^*\|_2 + \delta.\end{aligned} \tag{3.7}$$

Setting

$$\Omega_1 = \max_{0\le i\le r}\left\{1 + \sum_{k=0}^{i}\prod_{j=1}^{k}\|D^{i-j}\|_2\right\},$$

we obtain from (3.7), (3.4)

$$\|Z_{i+1} - Z^*_{i+1}\|_2 \leq \Omega_1\delta, \qquad i \geq 0. \tag{3.8}$$

Analogously, for the vector $U_{i+1} = [u_{i1}, u_{i2}, \dots, u_{im}]^{\mathrm{T}}$, (2.9) yields

$$\|U_{i+1} - U^*_{i+1}\| \leq \Omega_2\delta, \qquad i \leq r, \tag{3.9}$$

where

$$\Omega_2 = \max_{0\leq i\leq r} [1 + \|(I_m - hS^i)^{-1}A^i\|_2\Omega_1],$$

whereas, for $i > r$, from (2.9) and (3.8), observing that the hypothesis $h\|W\|_\infty\gamma < 1$ implies

$$\|(I_m - hS^i)^{-1}\|_\infty \leq 1/(1 - h\gamma\|W\|_\infty),$$

we obtain

$$\|U_{i+1} - U^*_{i+1}\| \leq \Omega_3\delta, \qquad i \geq r+1, \dots, \tag{3.10}$$

where

$$\Omega_3 = \sqrt{n}/(1 - h\gamma\|W\|_\infty) \sup_{t\in[t_0, \infty)} \sum_{l=1}^{n} |a_l(t)|\Omega_1.$$

Finally, from (1.2),

$$|u(t_{i+1}) - u^*(t_{i+1})| \leq \|L\|_2\|U_{i+1} - U^*_{i+1}\|, \tag{3.11}$$

where L is defined in (2.6). Therefore, putting

$$\Omega^* = \max\{\Omega_1, \Omega_2, \Omega_3, \|L\|_2\Omega_2, \|L\|_2\Omega_3\},$$

we get

$$\|v_i - v^*_i\|_2 \leq \Omega^*\delta, \qquad i \geq 0,$$

as asserted. □

4. Numerical experiments

In this section we report on numerical results intended to show the reliability of the stability conditions previously derived.

We have chosen the following integral equation problems, whose solutions have, respectively, a constant, oscillating, and decreasing behavior:

$$\text{(A)}\quad y(t) = 1 + 120t - 100(1 - e^{-t}) + \int_0^t [100e^{(s-t)} - 120]y(s)\,ds, \qquad t \in [0, 20],$$

exact solution: $y(t) = 1$;

$$\text{(B)}\quad \begin{aligned} y(t) = {} & \sin 10t + 480/101(\sin 10t - 10\cos 10t + 10e^{-t}) - 50(\cos 10t - 1) \\ & + \int_0^t [500e^{(s-t)} + 480]y(s)\,ds, \qquad t \in [0, 20], \end{aligned}$$

exact solution: $y(t) = \sin 10t$;

$$\text{(C)}\quad \begin{aligned} y(t) = {} & e^{-t} - 3000(t + 0.1)/(1 + t)(e^{-2t} - 1) \\ & - 6000\int_0^t (t + 0.1)/(1 + t)e^{-s}y(s)\,ds, \qquad t \in [0, 20], \end{aligned}$$

exact solution: $y(t) = e^{-t}$.

Moreover, in order to test the reliability of the stability conditions in nonlinear problems, we consider the following equations:

$$\text{(D)} \qquad y(t) = t - 1 + 2e^{-t} + \int_0^t (2e^{s-t} - 1)y^2(s)\,ds\,, \qquad t \in [0\,,\,40]\,,$$

exact solution: $y(t) = 1$;

$$\text{(E)} \qquad y(t) = 1 + t + t^2/2 - \int_0^t (1 + t - s)y^2(s)\,ds\,, \qquad t \in [0\,,\,40]\,,$$

exact solution: $y(t) = 1$.

As to the nonlinear problems, we apply the stability theorems to their linearized versions.

Problems (A)–(E) have been solved with the following VDC methods:

(1) $m = 2\,,\ c_1 = 0\,,\ c_2 = 1$;
(2a) $m = 3\,,\ c_1 = 0\,,\ c_2 = 1/2\,,\ c_3 = 1$ (Lobatto points) with implicit discretization;
(2b) $m = 3$ (Lobatto points) with fully implicit discretization;
(3a) $m = 4\,,\ c_1 = 0.08858\,,\ c_2 = 0.40947\,,\ c_3 = 0.78766\,,\ c_4 = 1$ (Radau IIA points) with implicit discretization;
(3b) $m = 4$ (Radau IIA points) with fully implicit discretization.

For the convolution problems (A), (B), (D), and (E), the application of the stability theorems derived in §§2 and 3 requires the calculation of the roots of the polynomial (2.14), which in this case is of second degree. In virtue of Theorem 2.5, this calculation is made only in the first step. In problem (C), the calculation of the root of the polynomial (2.14) whose degree, in this case, is one, must be made for every step. Of course, since the kernel is of class A, for the method (3a), this computation could be avoided in virtue of Theorem 3.3.

In Tables 1–5 we report relative and absolute errors in selected points for the above collocation methods applied to the integral problems (A)–(E). The quantities $X_{\min}$ and $X_{\max}$ are respectively the minimum and the maximum of the values assumed in the integration range by the maximum modulus root of the polynomial (2.14).

We observe that, in all cases, the VDC methods have a stable behavior every time the stability condition is satisfied. However, for problem (C), the numerical solution decreases more slowly than the true solution, and thus the relative error can be large; this is particularly true for the VDC methods (1) and (2). In some cases (method (1), problem (C), $h = 0.1$; method (2a), problem (C), $h = 0.5$; method (3a), problem (A), $h = 0.5$; methods (3a), (3b), problem (D), $h = 0.05$) the method has a stable behavior even if the stability condition is not satisfied; this is because the stability condition holds for the vector v_i and not only for $u(t_i)$.

In other cases (for example method (1), problem (A), $h = 0.25$; method (2a), problem (B), $h = 0.5$; problem (D), $h = 0.05$ and so on), the stability condition is not satisfied and the method is unstable.

Note that, for method (1) applied to problems (A) and (B), as expected, $X_{\max}$ is less than 1 if and only if the hypothesis of Theorem 3.3 holds.

Finally, we wish to stress that numerical results, more numerous than those reported here, show that the fully implicit discretization is more accurate than the implicit one; and the latter has the expected stable behavior in the cases covered by Theorem 3.3.

TABLE 1
Collocation parameters: $c_1 = 0, \; c_2 = 1$

Problem	h	t	Rel.Err.	Abs.Err.	X_{min}	X_{max}
A	.25	10	.70E 05	.70E 05	.29E 01	.29E 01
		20	.25E 12	.25E 12		
	.1	10	.48E-14	.48E-14	.84E 00	.84E 00
		20	.18E-13	.18E-13		
B	.1	10	.54E 14	.27E 14	.20E 01	.20E 01
		20	.28E 28	.25E 28		
	.05	10	.47E-01	.23E-01	.57E 00	.57E 00
		20	.24E-01	.21E-01		
C	.1	10	.17E-01	.77E-06	.22E-01	.10E 01
		20	.31E 03	.64E-06		
	.06	10	.62E-02	.27E-06	.43E-02	.99E 00
		20	.11E 03	.23E-06		
D	.15	10	.29E 00	.29E 00	.13E 01	.13E 01
		40	.10E 06	.10E 06		
	.05	10	.38E-01	.38E-01	.11E 01	.11E 01
		20	.37E 01	.37E 01		
E	.25	10	.00E 00	.00E 00	.99E 00	.99E 00
		20	.00E 00	.00E 00		
	.1	10	.47E-15	.47E-15	.99E 00	.99E 00
		20	.83E-16	.83E-16		

TABLE 2

Collocation parameters: $c_1 = 0,\ c_2 = 1/2,\ c_3 = 1$ (*impl. discr.*)

Problem	h	t	Rel.Err.	Abs.Err.	X_{min}	X_{max}
A	.5	10	.30E 10	.30E 05	.99E 01	.99E 01
		20	.10E 22	.10E 22		
	.25	10	.14E-02	.14E-02	.65E 00	.65E 00
		20	.14E-02	.14E-02		
B	.5	10	.16E 19	.83E 18	.12E 03	.12E 03
		20	overflow	overflow		
	.1	10	.81E-01	.41E-01	.27E 00	.27E 00
		20	.53E-01	.46E-01		
C	.5	10	.10E 01	.46E-04	.10E-00	.14E 01
		20	.18E 05	.38E-04		
	.1	10	.11E-04	.53E-09	.79E-01	.99E 00
		20	.21E 00	.43E-09		
D	. 5	10	.49E-01	.49E-01	.26E 01	.26E 01
		18*	.61E 01	.61E 06		
	.05	10	.44E-05	.44E-05	.11E 01	.11E 01
		20	.57E-03	.57E-03		
E	. 5	10	.27E-15	.27E-15	.97E 00	.97E 00
		20	.51E-14	.51E-14		
	.1	10	.27E-14	.27E-14	.99E 00	.99E 00
		20	.17E-13	.17E-13		

*In the next step, the nonlinear system of the collocation equations cannot be solved.

TABLE 3

Collocation parameters: $c_1 = 0,\ c_2 = 1/2,\ c_3 = 1$ (*fully impl. discr.*)

Problem	h	t	Rel.Err.	Abs.Err.	$X_{\min}$	$X_{\max}$
A	.5	10	.75E-03	.75E-03	.55E 00	.55E 00
		20	.75E-03	.75E-03		
	.1	10	.10E-06	.10E-06	.95E 00	.95E 00
		20	.10E-06	.10E-06		
B	.5	10	.21E 10	.10E 10	.12E 03	.29E 02
		20	.39E 19	.34E 19		
	.05	10	.30E-03	.10E-02	.87E 00	.87E 00
		20	.54E-04	.47E-04		
C	.5	10	.69E-03	.31E-07	.74E-01	.99E 00
		20	.13E 02	.27E-07		
	.06	10	.14E-06	.65E-11	.72E-01	.99E 00
		20	.27E-02	.55E-11		
D	. 5	10	.16E-02	.16E-02	.26E 01	.26E 01
		40	.25E 04	.25E 04		
	.05	10	.28E-06	.28E-06	.11E 01	.11E 01
		20	.68E-04	.68E-04		
E	. 5	10	.10E-14	.10E-14	.97E 00	.97E 00
		20	.17E-14	.17E-14		
	.1	10	.27E-14	.27E-14	.99E 00	.99E 00
		20	.17E-13	.17E-13		

TABLE 4
Collocation parameters: $c_1 = .08858$, $c_2 = .40947$, $c_3 = .78766$, $c_4 = 1$ (*impl. discr.*)

Problem	h	t	Rel.Err.	Abs.Err.	X_{min}	X_{max}
A	.5	10	.36E-03	.36E-03	.14E 01	.14E 01
		20	.36E-03	.36E-03		
	.1	10	.28E-08	.28E-08	.95E 00	.95E 00
		20	.28E-08	.24E-08		
B	.5	10	.17E 01	.86E 00	.99E 00	.99E 00
		20	.11E 01	.10E 01		
	.05	10	.76E-06	.38E-06	.87E 00	.87E 00
		20	.38E-06	.33E-06		
C	.5	10	.36E-05	.16E-09	.30E-02	.99E 00
		20	.65E-01	.13E-09		
	.06	10	.49E-08	.22E-12	.90E-03	.99E 00
		20	.57E-05	.11E-13		
D	. 5	10	.11-05	.11E-05	.26E 01	.26E 01
		35*	.29E 01	.20E 01		
	.05	10	.16E-10	.16E-10	.11E 01	.11E 01
		20	.39E-08	.39E-08		
E	. 5	10	.44E-15	.44E-15	.97E 00	.97E 00
		20	.41E-14	.41E-14		
	.1	10	.44E-15	.44E-15	.99E 00	.99E 00
		20	.54E-14	.54E-14		

*In the next step, the nonlinear system of the collocation equations cannot be solved.

TABLE 5
Collocation parameters: $c_1 = .08858$, $c_2 = .40947$, $c_3 = .78766$, $c_4 = 1$ (*fully impl. discr.*)

Problem	h	t	Rel.Err.	Abs.Err.	X_{min}	X_{max}
A	.5	10	.12E-07	.12E-07	.24E 00	.24E 00
		20	.12E-07	.12E-07		
	.1	10	.63E-11	.63E-11	.95E 00	.95E 00
		20	.64E-11	.64E-11		
B	.5	10	.43E-02	.22E-02	.17E 01	.17E 01
		20	.19E 00	.17E 00		
	.05	10	.94E-07	.51E-07	.87E 00	.87E 00
		20	.54E-07	.54E-07		
C	.5	10	.12E-06	.57E-11	.34E-02	.99E 00
		20	.19E-02	.40E-11		
	.06	10	.49E-08	.22E-12	.61E-03	.99E 00
		20	.57E-05	.11E-13		
D	. 5	10	.94-07	.94E-07	.26E 01	.26E 01
		40	.63E 00	.63E 00		
	.05	10	.14E-10	.14E-10	.11E 01	.11E 01
		20	.35E-08	.35E-08		
E	. 5	10	.55E-16	.55E-16	.97E 00	.97E 00
		20	.21E-14	.21E-14		
	.1	10	.11E-14	.11E-14	.99E 00	.99E 00
		20	.11E-13	.11E-13		

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