SIMPLE PERFECT SQUARED SQUARES
AND 2×1 SQUARED RECTANGLES OF ORDER 25

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ABSTRACT. In this note tables of all simple perfect squared squares and simple 2×1 perfect squared rectangles of order 25 are presented.

1. INTRODUCTION

For describing the problem of the dissection of squares and rectangles into unequal squares in a nontrivial way we use the terminology of Brooks, Smith, Stone and Tutte [6] and Bouwkamp [1, 3, 4, 2].

A dissection of a rectangle into a finite number $N > 1$ of nonoverlapping squares is called a *squared rectangle* or a *squaring* of *order N*. The N squares are called the *elements* of the dissection. The term "elements" is also used for the (length of the) sides of the elements. If all the elements are unequal, the squaring is called *perfect* and the rectangle is called a *perfect rectangle*; otherwise the squaring is *imperfect*. A squaring that contains a smaller rectangle or square dissected in squares is called *compound*. All other squared rectangles or squares are *simple*.

The lowest-order simple perfect squared square is of order 21 and was found in March 1978 [12]. The order-22 simple perfect 2×1 squared rectangle found in August 1978 appears to be of lowest order [11]. Recently, solutions of orders 22, 23, and 24 were found by an exhaustive computer search [7]. Five simple perfect squares of order 25 were calculated by Wilson [16]. Federico found another three simple perfect squared squares of order 25 [13].

For squarings of order 25 we need c-nets of orders 26. Therefore, we first generated and identified c-nets of order 23, 24, and 25. These were stored on secondary storage. Those of order 26 were only generated and searched for the existence of perfect squared squares of order 25. I communicated my squared squares and 2×1 squared rectangles of orders 22, 23, and 24 to Bouwkamp. Bouwkamp then constructed 12 new solutions of order 25 from my results by means of transformation techniques [5].

For a historical overview of the squared-square and squared-rectangle problems, see Federico [14].

2. Mathematical theory and computer procedures

Squared rectangles and squared squares can be obtained from so-called c-nets [6]. A c-net is a three-connected planar graph. The order of a c-net is its number of edges. The *dual* of a c-net is also a c-net. The c-nets are constructed using Tutte's theorem, known since 1947 and published in 1961 [15].

Let C_n be the set of c-nets of order n. If $s \in C_n$ is not a wheel, then at least one of the nets s and its dual s' can be constructed from $\sigma \in C_{n-1}$ by addition of an edge joining two vertices. A wheel is a c-net with an even number of edges E, with one edge of degree $E/2$ and $E/2$ vertices of degree 3. The *degree* of a vertex is the number of edges joining the vertex. Generation of c-nets of order $n + 1$ out of order n gives rise to many duplicate c-nets. These can be removed using an identification method described in 1962 [8, 9] and improved in 1978 [10].

Squarings can be obtained from c-nets by considering them as electrical networks of unit resistances [6]. Basically, starting from c-nets of order n, those of order $n + 1$ are generated and identified using electronic computers. Duplicates are removed, currents are calculated and simple perfect squared squares and 2×1 squared rectangles are filtered. For details see [8, 9].

The generation of c-nets of order 25 was carried out during the Christmas vacation week 1991 using four Sun Sparc workstations of the Faculty of Computing Sciences of the University Twente connected to the university network. During the period of January 7, 1992 to March 15, 1992 the order-25 squared-square solutions were calculated by means of four HP workstations also connected to the university network. Their speed is 75 Mflops. The machines were only available to me during the nights and the weekends. From c-nets of order 25 those of order 26 were generated but kept in the machine and not stored on secondary storage. In all possible ways electromotive forces were placed in the branches and currents were calculated. Only squared squares and 2×1 squared rectangle solutions were stored on the disk.

3. Results

Results are presented in tables contained in a microfiche card attached to the inside back cover of this issue.

Table I shows the Bouwkamp codes of simple perfect squared squares of order 25. We found 19 doublets with side 273, 280, 289, 308, 322, 338, 378, 380, 404, 416, 421, 492, 512, 536, 541, 544, 550, 552, and 603. We found seven triplets with side 264, 323, 384, 392, 396, 576, and 580. Furthermore, we found two quadruplets with side 320 and 556 and one sextet with side 540. The smallest reduced side we found is 147, the largest is 661. Table II shows the Bouwkamp codes of simple 2×1 perfect squared rectangles of order 25. Among the 2×1 squared rectangles we found five doublets with sides 592×296, 604×302, 676×338, 700×350, and 830×415 and one triplet with sides 616×308.

Figure 1 shows two simple perfect squared squares with the largest element not on the boundary. Since for orders 21 to 24 no such perfect simple squared square exists, these two are of lowest order. Figures 2, 3, and 4 show three pairs of simple perfect squared squares with pairwise the same reduced sides and the same elements but differently arranged. Those of reduced sides 540
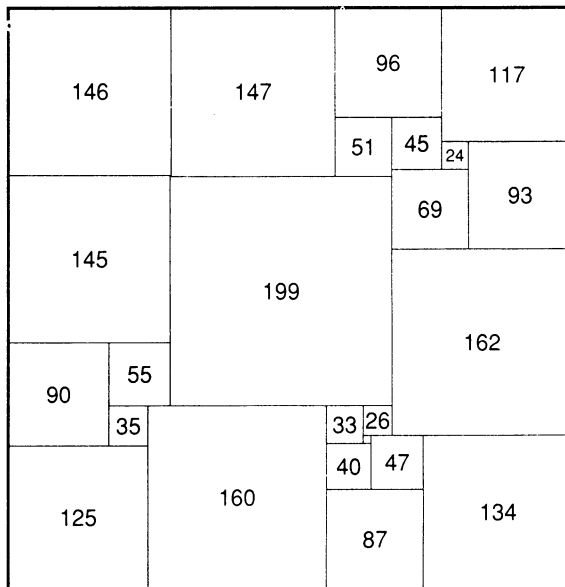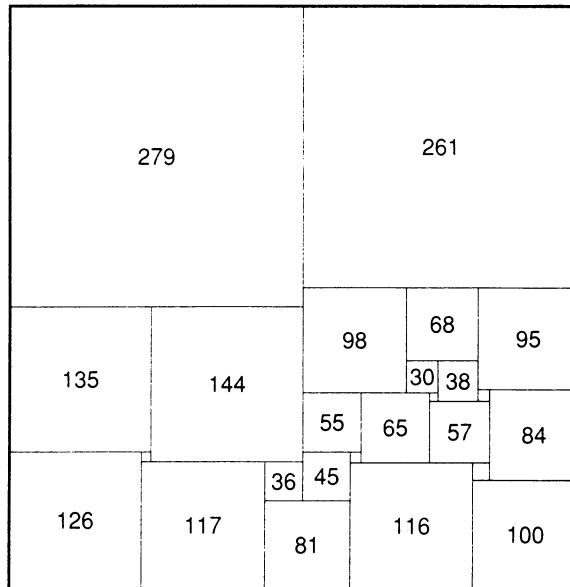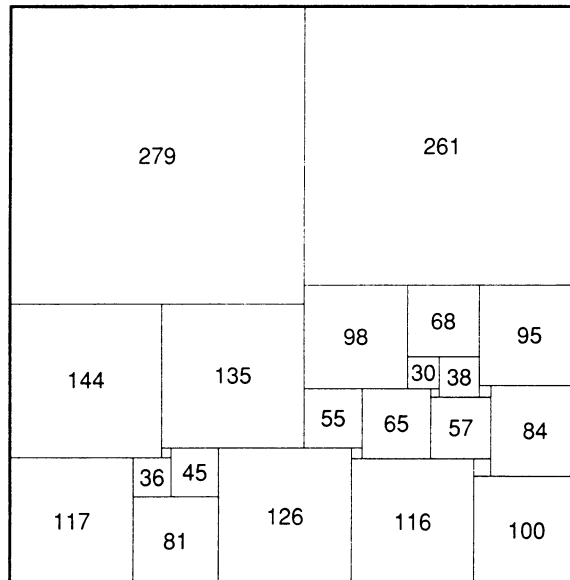
25:506
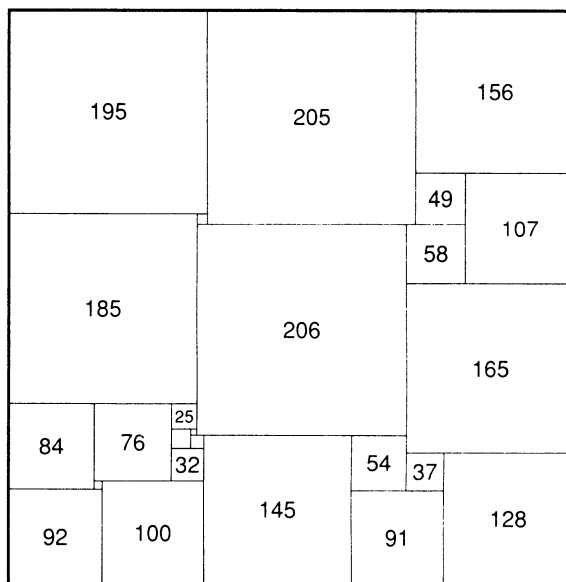
25:556

FIGURE 1. Two lowest-order simple perfect squared squares with largest element not on the boundary
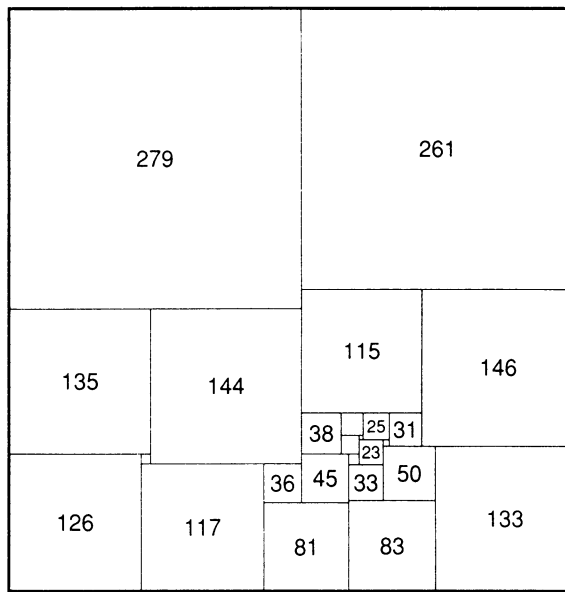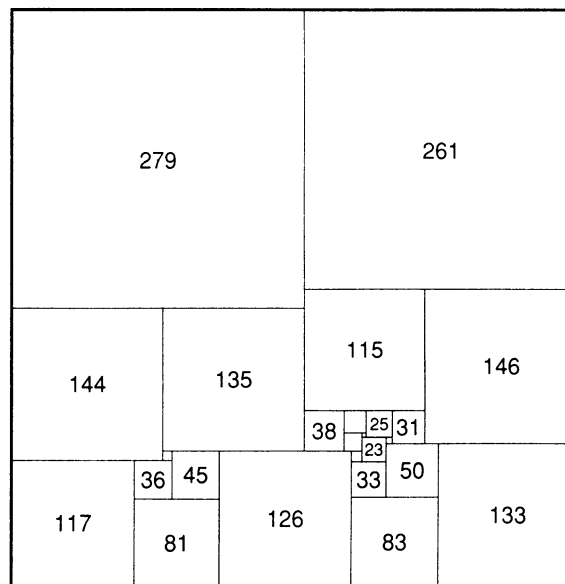
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FIGURE 2. Pair of lowest-order simple perfect squared squares with the same reduced side and elements differently arranged
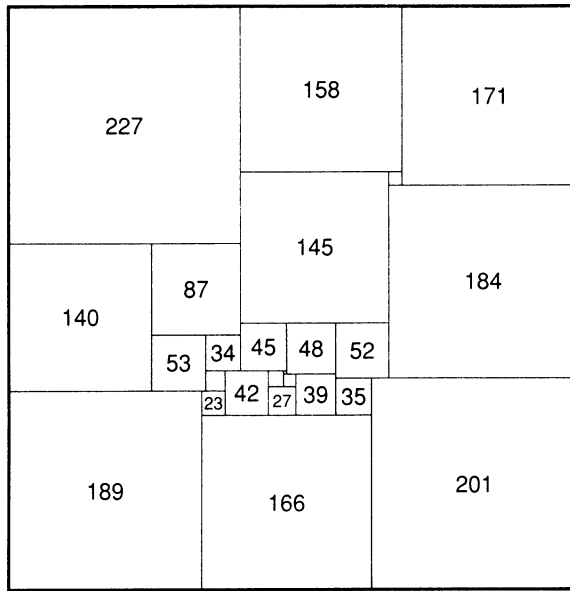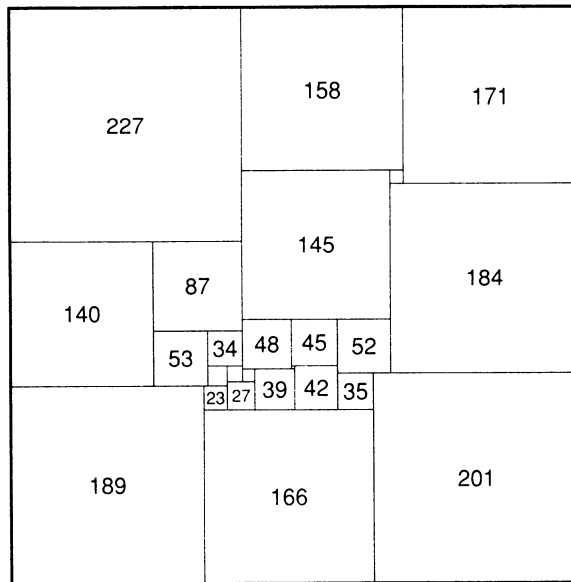
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FIGURE 3. Pair of lowest-order simple perfect squared squares
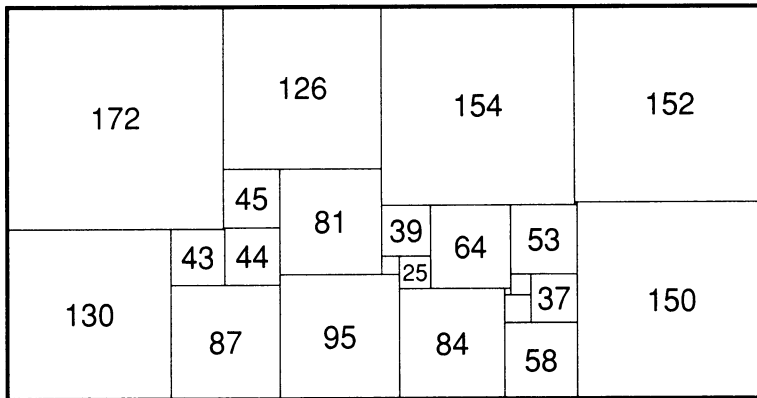with the same reduced side and elements differently arranged
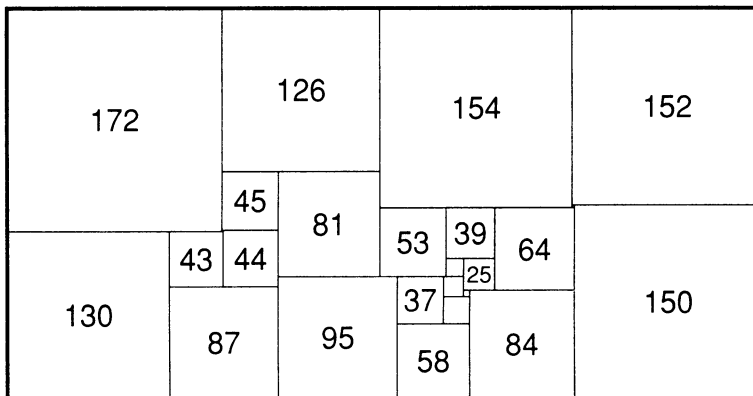
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FIGURE 4. Pair of lowest-order simple perfect squared squares with the same reduced side and elements differently arranged

25:604x302

25:604x302

FIGURE 5. Pair of lowest-order simple perfect squared 2×1 rectangles with the same reduced sides and elements differently arranged

[5] and 556 were already found by Bouwkamp and were obtained by means of transformation techniques using my results of order 24. Since for orders lower than 25 such pairs do not exist, these are of lowest order. Finally, Figure 5 shows a unique pair of simple perfect 2×1 rectangles of order 25. Since for lower order such solutions do not exist, it is of lowest order. Its existence was noticed by Bouwkamp while reading the final version of this paper.

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