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POLAR GENERATION OF RANDOM VARIATES WITH THE t-DISTRIBUTION

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ABSTRACT. The "polar" method of Box and Muller uses two independent uniform variates in order to generate two independent normal variates. It can be adapted so that two variates from Student's t-distribution with parameter ν are generated, though the two variates are now not independent. An algorithm based on the polar method is exact, inexpensive, and valid for all $\nu > 0$.

Box and Muller's [1] polar method for generating random normal variates relies on two convenient properties of the normal distribution, which we may formulate as follows:

(i) Let $X \sim N(0, 1)$. Then X can be regarded as the real part of a complex random variable Z which has a *radial* distribution (the contours of the density function of Z form circles centered at the origin);

(ii) Write $Z = X + iY = Re^{i\Theta}$. Then the distribution function F_R $(\equiv 1 - G_R)$ of R is a simple algebraic expression, so simple that it is invertible; that is, given $G \equiv G_R(r)$, we can write down a closed expression for r in term of G.

The aim of this article is to show that properties (i) and (ii) are shared by the *Student t-distribution with parameter* ν (the *t_ν-distribution*) defined by the density

$$f_T(x) = B(\nu/2, 1/2)^{-1} \cdot \nu^{-1/2} \cdot (1 + x^2/\nu)^{-(\nu+1)/2}. \tag{1}$$

Thus, we are asserting that the t_ν-distribution, like the normal, has a tractable radial parent. If T has the density (1), we shall write "$T \sim t_\nu$".

Many methods have been proposed for the generation of t_ν-variates. The most important ones are described in Devroye [2, pp. 445–450], whose masterly survey we shall not attempt to emulate. The faster algorithms may require either a comparatively great programming effort, or the expensive recalculation of certain quantities required by the algorithm, whenever ν is changed. As Devroye notes, problems arise when ν is small and the departure from normality is greatest, particularly in the region $0 < \nu < 1$, where many of the algorithms fail to work at all.

So one would be interested in a theoretically simple and practically effective generator, valid for any $\nu > 0$. We now show how the polar method may be applied to the t_ν-distribution to yield such an algorithm.

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Definition. Let $Z = Re^{i\Theta}$ be a complex random variable such that

(i) R and Θ are independently distributed;
(ii) Θ is uniformly distributed on $[0, 2\pi)$;
(iii) the probability that R $(= |Z|)$ is greater than r is

$$G_R(r) = (1 + r^2/\nu)^{-\nu/2}, \qquad r \geq 0, \ \nu > 0. \tag{2}$$

Then we shall say that Z has the *radial t_ν-distribution*

The point, and justification, of this definition is the following theorem:

Theorem 1. *If $Z = X + Yi$ has the radial t_ν-distribution, then the marginal distributions of X and Y are given by $X \sim t_\nu$ and $Y \sim t_\nu$. The variates X and Y are not independent.*

Proof. By assumption, Θ has density $1/2\pi$, and R has, independently, the density $f_R(r) = -dG_R/dr = (r/\nu) \cdot (1 + r^2/\nu)^{-\nu/2-1}$. The transformation $x = r\cos\theta$, $y = r\sin\theta$ (which has Jacobian r) shows that the joint density of X, Y is

$$f_{X,Y}(x, y) = (2\pi)^{-1} \cdot (1 + (x^2 + y^2)/\nu)^{-\nu/2-1}. \tag{3}$$

Now integrate out y (note that the integrand is an even function of y and let $u = (1 + y^2/(\nu + x^2))^{-1}$, expressing the integral as a multiple of a beta integral) to confirm that the marginal distribution of $f_X(x)$ has indeed the required form (1). The result for Y follows by symmetry.

The only complex radial distribution for which X and Y are independent with continuous marginal densities is (Mathai and Pederzoli [5, pp. 9–12]) the zero-mean, equal-variance, zero-covariance bivariate normal. Hence X and Y in Theorem 1 are not independent. □

[One way of seeing this directly is to examine the distribution of Y conditional on X. The density of this distribution at y is $f_{Y|X}(y) = f_{X,Y}(x, y)/f_X(x)$, and one finds, using the functions f_X and $f_{X,Y}$ given by (1) and (3), that $f_{Y|X}(y) \propto (1 + y^2/(\nu(1 + x^2/\nu)))^{-\nu/2-1}$, so that if we define $V = Y((1 + 1/\nu)/(1 + X^2/\nu))^{1/2}$, then X and V will be independent, $X \sim t_\nu$, and $V \sim t_{\nu+1}$.]

Theorem 1 leads to our main result, the *polar method for generating t_ν-variates*:

Theorem 2. *Let G, H be* iid *variates, uniformly distributed on* $[0, 1]$. *Let $\Theta = 2\pi \cdot H$, let $R = (\nu(G^{-2/\nu} - 1))^{1/2}$ $(\nu > 0)$, let $X = R\cos\Theta$, and let $Y = R\sin\Theta$. Then $X \sim t_\nu$ and $Y \sim t_\nu$. The variates X and Y are not independent.*

Proof. Let Z have the radial t_ν-distribution. If we let $G = G_R(R)$, where G_R is given by (2), then we know that G is uniformly distributed on [0, 1]. Conversely, if G is uniformly distributed on [0, 1] and we take the inverse transformation $R = (\nu(G^{-2/\nu} - 1))^{1/2}$, then we know that $Z \equiv R\exp(2\pi i H) = Re^{\Theta i}$ has the radial t_ν-distribution. Now apply Theorem 1. □

Theorem 2 is conceptually the simplest formulation of the polar method for the t_ν-distribution. However, further improvements are possible. One can (see for instance Marsaglia and Bray [4]) avoid expensive calculation of the cosine

by using the fact that if $U + Vi$ is uniformly distributed on the unit disk, $W \equiv U^2 + V^2$, and $C \equiv U/\sqrt{W}$, then W and C are independent, W has a uniform [0, 1] distribution, and C has the same density as $\cos \Theta$, that is, $f_C(x) = 1/\pi\sqrt{(1 - x^2)}$, $-1 < x < 1$. We incorporate this modification into our proposed algorithm, and note that Y in Theorem 2 is discarded, because of dependence on X.

Polar algorithm for generating t_ν-variates.

(a) Generate iid uniform [0, 1] variates U and V. Replace U by $2U - 1$, V by $2V - 1$.

(b) Define $W \equiv U^2 + V^2$. If $W > 1$ return to (a).

(c) Let $C = U/\sqrt{W}$, $R = (\nu(W^{-2/\nu} - 1))^{1/2}$, $X = RC$.

Then $X \sim t_\nu$.

In order to sidestep one of the square-root calculations, we can rewrite (c) as

(c′) Let $C^2 = U^2/W$, $R^2 = \nu(W^{-2/\nu} - 1)$, $X = \sqrt{(R^2C^2)}$.

Thus, the only expensive steps in the polar algorithm are to calculate $W^{-2/\nu}$ and one square root, whose sign should be chosen at random.

As Neave [6] pointed out in connection with the original Box-Muller algorithm, care must be taken if the uniform random numbers required by the method (G and H in Theorem 2) are in fact pseudorandom numbers generated by the congruential method, as is currently almost invariably the case in practice. Possible cures are surveyed in Golder and Settle [3]. In particular, one can simply use two congruential generators, to different moduli (the "two-sequence method"), to generate G and H. Golder and Settle show this to be an effective cure.

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