

COMPUTATION OF \mathbb{Z}_3 -INVARIANTS OF REAL QUADRATIC FIELDS

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ABSTRACT. Let k be a real quadratic field and p an odd prime number which splits in k . In a previous work, the author gave a sufficient condition for the Iwasawa invariant $\lambda_p(k)$ of the cyclotomic \mathbb{Z}_p -extension of k to be zero. The purpose of this paper is to study the case $p = 3$ of this result and give new examples of k with $\lambda_3(k) = 0$, by using information on the initial layer of the cyclotomic \mathbb{Z}_3 -extension of k .

1. INTRODUCTION

Let k be a finite totally real extension of the field of rational numbers \mathbb{Q} . Let p be a fixed prime number and \mathbb{Z}_p the ring of p -adic integers. We denote by $\lambda_p(k)$, $\mu_p(k)$ and $\nu_p(k)$ the Iwasawa invariants of the cyclotomic \mathbb{Z}_p -extension of k for p (cf. [8]). In Greenberg's paper [7], both $\lambda_p(k)$ and $\mu_p(k)$ were conjectured to be zero. However, Greenberg's conjecture is not yet proven, even for real quadratic fields, although we know by the Ferrero-Washington theorem that $\mu_p(k)$ is always zero when k is an abelian extension of \mathbb{Q} (cf. [1]).

Let k be a real quadratic field and p an odd prime number which splits in k . In a previous paper [10], we gave a sufficient condition for $\lambda_p(k)$ to be zero. In the present paper, we first define two invariants $n_0^{(r)}$ and $n_2^{(r)}$ for k and p , and rewrite our previous result in terms of these invariants. Next, using this result for $p = 3$, we will give some examples of k with $\lambda_3(k) = 0$, which are of a new type. For this purpose, we will compute $n_0^{(1)}$ and $n_2^{(1)}$ for $p = 3$ by determining the unit group of the initial layer of the cyclotomic \mathbb{Z}_3 -extension of k by the method of Mäki (cf. [3, 9]).

2. A SUFFICIENT CONDITION FOR $\lambda_p(k) = 0$

Let k be a real quadratic field with class number h and fundamental unit ε , and p an odd prime number which splits in k , namely, $(p) = \mathfrak{p}\mathfrak{p}'$ in k where $\mathfrak{p} \neq \mathfrak{p}'$. Then we can choose $\alpha \in k$ such that $\mathfrak{p}'^h = (\alpha)$. Fukuda and Komatsu [5] defined two invariants $n_1, n_2 \in \mathbb{N}$ for k and p , by

$$\mathfrak{p}^{n_1} \parallel (\alpha^{p-1} - 1), \quad \mathfrak{p}^{n_2} \parallel (\varepsilon^{p-1} - 1).$$

Here $\mathfrak{p}^n \parallel \mathfrak{a}$ means that $\mathfrak{p}^n \mid \mathfrak{a}$ and $\mathfrak{p}^{n+1} \nmid \mathfrak{a}$ for an ideal \mathfrak{a} of k . Though the choice of α is not unique, n_1 is uniquely determined under the condition $n_1 \leq n_2$.

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For the cyclotomic \mathbb{Z}_p -extension

$$k = k_0 \subset k_1 \subset k_2 \subset \cdots \subset k_n \subset \cdots \subset k_\infty,$$

let \mathfrak{p}'_n be the unique prime ideal of k_n lying above \mathfrak{p}' , d_n the order of \mathfrak{p}'_n in the ideal class group of k_n and E_n the unit group of k_n . For $m \geq n \geq 0$, we denote by $N_{m,n}$ the norm map from k_m to k_n . Now we fix an integer $r \geq 0$. Then we can choose $\beta_r \in k_r$ such that $\mathfrak{p}'_r{}^{d_r} = (\beta_r)$. We define two other invariants $n_0^{(r)}, n_2^{(r)} \in \mathbb{N}$ for k and p , by

$$\mathfrak{p}^{n_0^{(r)}} \parallel (N_{r,0}(\beta_r)^{p-1} - 1), \quad p^{n_2^{(r)}} = p^{n_2}(E_0 : N_{r,0}(E_r)).$$

As in the case of n_1 , $n_0^{(r)}$ is uniquely determined under the condition $n_0^{(r)} \leq n_2^{(r)}$, though the choice of β_r is not unique. Put $n_0 = n_0^{(0)}$, noting that $n_2 = n_2^{(0)}$. It is easily seen that $n_0 \leq n_1 \leq n_2$.

Remark 1. The invariant $n_0^{(r)}$ is a generalization of m_r , which was defined in [10] under the assumptions that $p \nmid h$ and $n_2 \geq 2$.

Let A_n be the p -primary part of the ideal class group of k_n and ζ_p a primitive p th root of unity. For the CM -field $k^* = k(\zeta_p)$, we put $\lambda_p^-(k^*) = \lambda_p(k^*) - \lambda_p((k^*)^+)$, where $(k^*)^+$ is the maximal real subfield of k^* . Noting Theorem 1 of [5] and Lemma 3 of [10], we may rewrite Theorem 2 of [10] as follows, which is a generalization of the results of Fukuda and Komatsu in [2] and [5].

Theorem 1. *Let k be a real quadratic field and p an odd prime number which splits in k . Fix an integer $r \geq 0$. Assume that*

1. $A_0 = 1$,
2. $\lambda_p^-(k^*) = 1$.

If $n_0^{(r)} \neq n_2^{(r)}$, then we have $\lambda_p(k) = 0$.

Greenberg, Fukuda, Komatsu and Wada gave a number of examples of k with $\lambda_p(k) = 0$ in three cases where $n_1 = 1$, $2 \leq n_1 \neq n_2$ and $n_1 = n_2 = 2$ (cf. [2, 5, 6, 7]). However, in the case where $n_1 = n_2 \geq 3$, no examples of such k 's have been given until now. In the rest of this paper, we will give some examples of k with $\lambda_3(k) = 0$ in the cases where $n_1 = n_2 = 3, 4$ and 5 , using Theorem 1 for $p = 3$ and $r = 1$. Note that $\lambda_3^-(k^*) = \lambda_3(\mathbb{Q}(\sqrt{-3d}))$, where d is the discriminant of k (cf. the proof of Theorem 10.10 in [11]). Therefore, we will compute $n_0^{(1)}$ and $n_2^{(1)}$ for $p = 3$ to achieve our purpose.

3. COMPUTATION OF THE INVARIANTS $n_0^{(1)}$ AND $n_2^{(1)}$ FOR $p = 3$

Let m be a positive square-free integer and $k = \mathbb{Q}(\sqrt{m})$. In this section, we will explain how to compute $n_0^{(1)}$ and $n_2^{(1)}$ for $p = 3$. Let k_1 be the initial layer of the cyclotomic \mathbb{Z}_3 -extension of k and $\mathbb{Q}_1 = \mathbb{Q}(\theta)$, where $\theta = 2 \cos(2\pi/9)$. Then $k_1 = \mathbb{Q}(\sqrt{m}, \theta)$. We put $\omega = (1 + \sqrt{m})/2$ or \sqrt{m} according as $m \equiv 1 \pmod{4}$ or $2, 3 \pmod{4}$, and $\theta' = 2 \cos(4\pi/9)$. The following facts are well known:

- (a) $\{1, \theta, \theta'\}$ is an integral basis for \mathbb{Q}_1 ,
- (b) $\{1, \omega\}$ is an integral basis for k .

We further assume that m is prime to 3. Since k and \mathbb{Q}_1 are linearly disjoint over \mathbb{Q} and their discriminants are relatively prime, we obtain the following:

- (c) $\{1, \theta, \theta', \omega, \theta\omega, \theta'\omega\}$ is an integral basis for k_1 .

Since k_1 is a real cyclic extension of degree 6 over \mathbb{Q} , we can determine the unit group E_1 by Mäki's algorithm (cf. [3, 9]). We now let

$$E_1 = \langle -1, \eta_1, \eta_2, \eta_3, \eta_4, \eta_5 \rangle,$$

where the generators η_i are obtained by this algorithm and represented as linear combinations of the integral basis (c) for k_1 . Then we can compute $n_2^{(1)}$ for $p = 3$ because this algorithm enables us to determine either $N_{1,0}(E_1) = E_0$ or E_0^3 (cf. §3 in [9], also [3]).

Now we assume that $m \equiv 1 \pmod{3}$. In order to compute $n_0^{(1)}$ for $p = 3$, we need to know the \mathfrak{p} -adic valuation of $N_{1,0}(\beta_1)^2 - 1$. Thus we have to find a generator β_1 of \mathfrak{p}'^{d_1} . Note that $d_1 = d_0$ or $3d_0$.

In the case where $d_1 = 3d_0$, we have $(\beta_0) = \mathfrak{p}'^{3d_1} = (\beta_1)$. Hence, we may take $\beta_0 \in k_1$ as a generator of \mathfrak{p}'^{d_1} and can obtain β_0 explicitly by a continued fraction expansion as in solving the Pell equations.

In another case where $d_1 = d_0$, we have $(\beta_0) = \mathfrak{p}'^{3d_1} = (\beta_1^3)$. Therefore we obtain

$$\beta_1^3 = \pm \eta_1^{r_1} \eta_2^{r_2} \eta_3^{r_3} \eta_4^{r_4} \eta_5^{r_5} \beta_0 \quad \text{for some } r_i \in \mathbb{Z}.$$

Since $(\beta_0) = (\beta_1^3)$, there exists an *appropriate* system $(r_1, r_2, r_3, r_4, r_5)$, where $0 \leq r_i \leq 2$, with the property that $\eta_1^{r_1} \eta_2^{r_2} \eta_3^{r_3} \eta_4^{r_4} \eta_5^{r_5} \beta_0$ is a cube in k_1 . Then its unique real cubic root in k_1 is a generator of \mathfrak{p}'^{d_1} . To find this *appropriate* system, we check whether a given $\eta_1^{r_1} \eta_2^{r_2} \eta_3^{r_3} \eta_4^{r_4} \eta_5^{r_5} \beta_0$ is a cube in k_1 , using approximate values of its unique real cubic root and its conjugates, and this is verified rigorously afterwards (see Example 1).

Our programs have been executed on a Sun SPARC-station 2 using the C-language and on a NEC PC-9801RA using Y. Kida's UBASIC86.

Remark 2. We do not need to know the entire unit group of k_1 for our purpose. It is sufficient to know only the unit group *modulo cubic*.

4. NEW EXAMPLES OF k WITH $\lambda_3(k) = 0$

By executing our procedure mentioned in the previous section, we will give some examples of k with $\lambda_3(k) = 0$ in the cases where $n_1 = n_2 = 3, 4$ and 5 , respectively. First, we describe the following explicit example of such a k in detail.

Example 1. Let $k = \mathbb{Q}(\sqrt{8965})$ and $p = 3$. Then we can easily verify that $h = 2$ ($A_0 = 1$), $d_0 = 2$, $\varepsilon = 402390206 + 8590401\omega$, $\mathfrak{p}^2 = (890 + 19\omega)$ and $\mathfrak{p}'^2 = (890 - 19\omega)$, so $\beta_0 = 890 - 19\omega$, where ω is as in §3. It follows from the \mathfrak{p} -adic expansions of ε^2 and β_0^2 that $n_0 = n_1 = n_2 = 3$. Since $\lambda_3^-(k^*) = \lambda_3(\mathbb{Q}(\sqrt{-3 \cdot 8965}))$, it also follows from Fukuda's table [4] that $\lambda_3^-(k^*) = 1$.

On the other hand, by the method of Mäki, we obtain a system $\{\eta_1, \eta_2, \eta_3, \eta_4, \eta_5\}$ of fundamental units of k_1 as follows:

$$\begin{aligned} \eta_1 &= \varepsilon = 402390206 + 8590401\omega, \\ \eta_2 &= \theta, \\ \eta_3 &= \theta', \\ \eta_4 &= 48894 - 36464\theta + 19152\theta' + 1045\omega - 772\theta\omega + 416\theta'\omega, \\ \eta_5 &= 49939 - 19568\theta - 56804\theta' - 1045\omega + 416\theta\omega + 1188\theta'\omega, \end{aligned}$$

where θ and θ' are as in §3. We note that $\{\theta, \theta'\}$ is a system of fundamental units of \mathbb{Q}_1 , and that η_4 and η_5 are relative units of k_1 . Hence it follows that $N_{1,0}(E_1) = E_0^3$, so we obtain $n_2^{(1)} = 4$.

In this case, we have $d_1 = d_0 = 2$ by Theorem 1 for $r = 0$ of [10]. Now let us put $\gamma = \eta_1\eta_2^2\eta_3\eta_4\eta_5\beta_0$, i.e., choose the system $(1, 2, 1, 1, 1)$ (see Remark 3). Then we can get approximate values γ_i of the unique real cubic roots of the six conjugates of γ as follows:

$$\begin{aligned}\gamma_1 &\doteq -9577.8744395484425899714390649555495943 \dots, \\ \gamma_2 &\doteq -0.0578252538221422012093779803554794569 \dots, \\ \gamma_3 &\doteq -11423747.694177045764647496253250783415 \dots, \\ \gamma_4 &\doteq -0.0001895209088981839062474670241349699 \dots, \\ \gamma_5 &\doteq -13.373368069820849234894393443243885952 \dots, \\ \gamma_6 &\doteq -0.0000005612408729122976653704111340309 \dots.\end{aligned}$$

Taking the trace from k_1 to \mathbb{Q} , we have

$$\sum_{i=1}^6 \gamma_i \doteq -11433339.00000000000000000000000000000000 \dots,$$

which is close to an integer (if it is not, then this system is not *appropriate*). By solving the system of equations

$$\left\{ \begin{array}{l} \gamma_1 = x_0 + x_1\theta + x_2\theta' + (y_0 + y_1\theta + y_2\theta')\omega, \\ \gamma_2 = x_0 + x_1\theta' + x_2\theta'' + (y_0 + y_1\theta' + y_2\theta'')\omega', \\ \gamma_3 = x_0 + x_1\theta'' + x_2\theta + (y_0 + y_1\theta'' + y_2\theta)\omega, \\ \gamma_4 = x_0 + x_1\theta + x_2\theta' + (y_0 + y_1\theta + y_2\theta')\omega', \\ \gamma_5 = x_0 + x_1\theta' + x_2\theta'' + (y_0 + y_1\theta' + y_2\theta'')\omega, \\ \gamma_6 = x_0 + x_1\theta'' + x_2\theta + (y_0 + y_1\theta'' + y_2\theta)\omega', \end{array} \right.$$

where $\theta'' = 2 \cos(8\pi/9)$ and $\omega' = (1 - \sqrt{8965})/2$, we obtain the following root approximately:

$$(x_0, x_1, x_2, y_0, y_1, y_2) \doteq (-1885431, 1396449, -745160, -40251, 29812, -15908),$$

which is also close to an integer (if it is not, then this system is not *appropriate*). We then put

$$\beta_1 = -1885431 + 1396449\theta - 745160\theta' - 40251\omega + 29812\theta\omega - 15908\theta'\omega.$$

It is easy to verify that $\gamma = \beta_1^3$, so the system $(1, 2, 1, 1, 1)$ is really *appropriate*. Hence β_1 is a generator of \mathfrak{p}_1^2 . Taking the norm from k_1 to k , we have $N_{1,0}(\beta_1) = -723897967519 - 15454088423\omega$. It follows from the \mathfrak{p} -adic expansion of $N_{1,0}(\beta_1)^2$ that $n_0^{(1)} = 3 \neq n_2^{(1)}$.

Therefore we see by Theorem 1 for $r = 1$ that $\lambda_3(\mathbb{Q}(\sqrt{8965})) = 0$.

Remark 3. In Example 1, if $(r_1, r_2, r_3, r_4, r_5)$ is an *appropriate* system, then $3^2\theta^{2r_2}\theta'^{2r_3}$ is a cube in \mathbb{Q}_1 . On the other hand, it is easy to see that $3\theta^2\theta'$ is a cube in \mathbb{Q}_1 . Hence we obtain $r_2 = 2$ and $r_3 = 1$, so that it suffices for all practical purposes to search among only r_1, r_4 and r_5 , where $0 \leq r_i \leq 2$. This is often efficient in finding an *appropriate* system for other examples.

TABLE 2. The case where $n_1 = n_2 = 4$ for $p = 3$

m	h	$ A_0 $	d_0	$ D_0 $	d_1	$ D_1 $	$\lambda_3^-(k^*)$	$(n_0^{(1)}, n_2^{(1)})$	$\lambda_3(k)$
2149	1	1	1	1	1	1	1	(5,5)	*
9814	2	1	2	1	2	1	1	(5,5)	*
10849	1	1	1	1	1	1	1	(5,5)	*
16861	4	1	2	1	2	1	1	(4,5)	0
17707	1	1	1	1	1	1	1	(5,5)	*
24007	1	1	1	1	1	1	1	(4,5)	0
24985	2	1	2	1	6	3	3	(4,4)	**
25597	2	1	1	1	1	1	1	(4,5)	0
26245	16	1	4	1	4	1	1	(4,5)	0
26893	3	3	1	1	1	1	1	(4,5)	**

TABLE 3. The case where $n_1 = n_2 = 5$ for $p = 3$

m	h	$ A_0 $	d_0	$ D_0 $	d_1	$ D_1 $	$\lambda_3^-(k^*)$	$(n_0^{(1)}, n_2^{(1)})$	$\lambda_3(k)$
22333	1	1	1	1	1	1	1	(5,6)	0
42205	2	1	2	1	2	1	1	(5,6)	0

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