

COMPUTATION OF RELATIVE CLASS NUMBERS OF CM-FIELDS

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ABSTRACT. It was well known that it is easy to compute relative class numbers of abelian CM-fields by using generalized Bernoulli numbers (see Theorem 4.17 in *Introduction to cyclotomic fields* by L. C. Washington, Grad. Texts in Math., vol. 83, Springer-Verlag, 1982). Here, we provide a technique for computing the relative class number of any CM-field.

1. STATEMENT OF THE RESULTS

Proposition 1. *Let $n \geq 1$ be an integer and $\alpha > 1$ be real. Set $P_n(x) = \sum_{k=0}^{n-1} \frac{1}{k!} x^k$,*

$$(1) \quad f_n(s) = \Gamma^n(s) A^{-2s} \left(\frac{1}{2s-1} + \frac{1}{2s-2} \right)$$

and

$$(2) \quad K_n(A) = \frac{A^2}{i\pi} \int_{\alpha-i\infty}^{\alpha+i\infty} f_n(s) ds.$$

Then, it holds

$$(3) \quad 0 \leq K_n(A) \leq 2P_n(nA^{2/n})e^{-nA^{2/n}} \leq 2n \exp(-A^{2/n}).$$

Theorem 2. *Let \mathbf{N} be a totally imaginary number field of degree $2n$ which is a quadratic extension of a totally real number field \mathbf{N}^+ of degree n , i.e. \mathbf{N} is a CM-field. Let $w_{\mathbf{N}}$ be the number of roots of unity in \mathbf{N} , $Q_{\mathbf{N}} \in \{1, 2\}$ be the Hasse unit index of \mathbf{N} , and $d_{\mathbf{N}}$, $\zeta_{\mathbf{N}}$ and $d_{\mathbf{N}^+}$, $\zeta_{\mathbf{N}^+}$ be the absolute values of the discriminants and the Dedekind zeta functions of \mathbf{N} and \mathbf{N}^+ , respectively. Let $\chi_{\mathbf{N}/\mathbf{N}^+}$ be the quadratic character associated with the quadratic extension \mathbf{N}/\mathbf{N}^+ and let ϕ_k be the coefficients of the Dirichlet series $(\zeta_{\mathbf{N}}/\zeta_{\mathbf{N}^+})(s) = L(s, \chi_{\mathbf{N}/\mathbf{N}^+}) = \sum_{k \geq 1} \phi_k k^{-s}$, $\Re(s) > 1$. Set $A_{\mathbf{N}/\mathbf{N}^+} = \sqrt{d_{\mathbf{N}}/\pi^n d_{\mathbf{N}^+}}$.*

We have

$$(4) \quad h_{\mathbf{N}}^- = \frac{Q_{\mathbf{N}} w_{\mathbf{N}}}{(2\pi)^n} \sqrt{\frac{d_{\mathbf{N}}}{d_{\mathbf{N}^+}}} \sum_{k \geq 1} \frac{\phi_k}{k} K_n(k/A_{\mathbf{N}/\mathbf{N}^+}),$$

and according to (3) this series (4) is absolutely convergent. Moreover, set

$$(5) \quad B(\mathbf{N}) \stackrel{\text{def}}{=} A_{\mathbf{N}/\mathbf{N}^+} \left(\frac{\lambda}{n} \log A_{\mathbf{N}/\mathbf{N}^+} \right)^{n/2}.$$

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Then, if $\lambda > 1$ and n are given, then the limit of $|h_{\mathbf{N}}^- - h_{\mathbf{N}}^-(M)|$ as $A_{\mathbf{N}/\mathbf{N}^+}$ approaches infinity is equal to 0, where $h_{\mathbf{N}}^-(M)$ is the approximation of the relative class number obtained by disregarding in the series occurring in (4) the indices $k > M \geq B(\mathbf{N})$.

For example, if \mathbf{N} of degree $m = 2n$ is the narrow Hilbert class field of a real quadratic number field \mathbf{L} of discriminant $d_{\mathbf{L}}$, we have

$$B(\mathbf{N}) = \left(\frac{\lambda}{4\pi}\right)^{m/4} d_{\mathbf{L}}^{m/8} \log^{m/4}(d_{\mathbf{L}}/\pi^2).$$

The following Proposition 3 explains how we compute the numerical values of the function $A \mapsto K_n(A)$ according to its series expansion:

Proposition 3. *Take $A > 0$. It holds*

$$(6) \quad K_n(A) = 1 + \pi^{n/2} A + 2A^2 \sum_{m \geq 0} \text{Res}_{s=-m}(f_n).$$

This series is absolutely convergent and for any integer $M \geq 0$ we have

$$(7) \quad \left| 2A^2 \sum_{m > M} \text{Res}_{s=-m}(f_n) \right| \leq \frac{\pi^{n/2} A^{2M+3}}{(M+1)(M!/2)^n}.$$

Finally, the following Proposition 4 explains how to compute recursively the values of the residues $\text{Res}_{s=-m}(f_n)$ occurring in (6):

Proposition 4. *We have*¹

$$(8) \quad \text{Res}_{s=-m}(f_n) = -(-1)^{nm} \frac{A^{2m}}{(m!)^n} \sum_{i=-n}^{-1} 2^{-1-i} h_i(m) ((2m+1)^i + (2m+2)^i)$$

where the $h_i(m)$'s are computed recursively from the $h_i(0)$'s by using

$$(9) \quad h_i(m+1) = \sum_{j=-n}^i h_j(m) \frac{b_{i-j}}{(m+1)^{i-j}} \quad \text{and} \quad \sum_{j=-n}^{-1} h_j(0) s^j + O(1) = \Gamma^n(s) A^{-2s},$$

where $b_k = C_{n+k-1}^{n-1} = ((k+n-1)!/k!(n-1)!)$. Thus, if

$$(10) \quad \Gamma^n(s+1) = \sum_{i=0}^{n-1} h_i s^i + O(s^n),$$

then

$$(11) \quad h_{j-n}(0) = \sum_{i=0}^j \frac{(-2 \log A)^i}{i!} h_{j-i} \quad (0 \leq j \leq n-1).$$

For proving these results, obvious questions of convergence of series and integrals, and questions of inversions of integrals and summations will not be gone into.

¹Note the misprint in the formula given in [Lou 2].

2. INTRODUCTION

Prior to the method we have developed here, the only general method for computing the relative class number of any CM-field was that developed by T. Shintani (see [Oka 1] and [Oka 2] for examples of actual relative class number computations using Shintani’s ideas). However, his method requires the knowledge of a great deal of information on the maximal totally real subfield \mathbf{N}^+ . In particular, it requires the knowledge of a system of fundamental units of the group of totally positive units of \mathbf{N}^+ . However, what makes the concept of CM-field an attractive one is that the relative class number formula

$$(12) \quad h_{\mathbf{N}}^- = \frac{Q_{\mathbf{N}}w_{\mathbf{N}}}{(2\pi)^n} \sqrt{\frac{d_{\mathbf{N}}}{d_{\mathbf{N}^+}} \frac{\text{Res}_{s=1}(\zeta_{\mathbf{N}})}{\text{Res}_{s=1}(\zeta_{\mathbf{N}^+})}} = \frac{Q_{\mathbf{N}}w_{\mathbf{N}}}{(2\pi)^n} \left(\sqrt{\frac{d_{\mathbf{N}}}{d_{\mathbf{N}^+}}} \right) L(1, \chi_{\mathbf{N}/\mathbf{N}^+})$$

enables us to get lower bounds on relative class numbers and solve class number and class group problems for CM-fields precisely because (12) does not involve any regulator (see [Lou-Oka] and [LOO]). Thus, the reader may possibly feel dissatisfied that he should have to know beforehand a good grasp of the unit group of \mathbf{N}^+ before he can compute $h_{\mathbf{N}}^-$, whereas (12) gives an expression for $h_{\mathbf{N}}^-$ which does not involve units. The reader may now possibly feel satisfied that this paper shows how using (12) he indeed gets an efficient method for computing $h_{\mathbf{N}}^-$ provided that he only knows how to compute the decomposition of any rational prime into a product of prime ideals of \mathbf{N} . The key point of our method is to establish the holomorphic continuation of $s \mapsto (\zeta_{\mathbf{N}}/\zeta_{\mathbf{N}^+})(s) = L(s, \chi_{\mathbf{N}/\mathbf{N}^+})$ in the same way Riemann did in the case of the Riemann zeta function (by using Mellin transformation) and to evaluate the resulting series at $s = 1$ (see section 4).

Finally, we note that the results of this paper are better than those of [Lou 3]. Indeed, $B(\mathbf{N})$ in (5) is $n^{n/2}$ -fold better than the one we gave in [Lou 3]. Moreover, our proof of (3) (in section 3) is more satisfactory and elegant than the one we gave in [Lou 3].

3. PROOF OF PROPOSITION 1

We use:

Lemma 5. *Let $\alpha > 1$ be real. We have*

$$\int_{\alpha-i\infty}^{\alpha+i\infty} u^s \frac{ds}{2s-1} = \begin{cases} 0 & \text{if } u < 1, \\ i\pi\sqrt{u} & \text{if } u > 1; \end{cases} \quad \text{and} \quad \int_{\alpha-i\infty}^{\alpha+i\infty} u^s \frac{ds}{2s-2} = \begin{cases} 0 & \text{if } u < 1, \\ i\pi u & \text{if } u > 1. \end{cases}$$

Now, using

$$\Gamma^n(s) = \int \int e^{-\text{Tr}(y)} y^s \frac{dy}{y}$$

where the multiple integral ranges over $(y_1, \dots, y_n) \in (\mathbf{R}_+^*)^n$ and where we set $y = y_1 y_2 \cdots y_n$ and $\text{Tr}(y) = y_1 + y_2 + \cdots + y_n$, leads to

$$K_n(A) = \frac{A^2}{i\pi} \int \int e^{-\text{Tr}(y)} \left(\int_{\alpha-i\infty}^{\alpha+i\infty} (y/A^2) \left(\frac{1}{2s-1} + \frac{1}{2s-2} \right) \right) \frac{dy}{y}.$$

Using Lemma 5 yields

$$K_n(A) = A^2 \int \int_{y \geq A^2} (\sqrt{y/A^2} + (y/A^2)) e^{-\text{Tr}(y)} \frac{dy}{y} \leq 2 \int \int_{y \geq A^2} e^{-\text{Tr}(y)} dy.$$

For example, we get $K_1(A) \leq 2e^{-A^2}$. Now, using the arithmetic-geometric mean inequality yields that $\{(y_1, \dots, y_n); y \geq A^2\}$ is included in $\{(y_1, \dots, y_n); \text{Tr}(y) \geq nA^{2/n}\}$, which yields

$$K_n(A) \leq 2 \int \int_{\text{Tr}(y) \geq nA^{2/n}} e^{-\text{Tr}(y)} dy.$$

Then, the following easily proved Lemma 6 provides us with the desired result. We finally notice that we get a shorter and more satisfactory proof of [Lou 3, Proposition 1]:

Lemma 6. *Set $P_n(x) = \sum_{k=0}^{n-1} x^k/k!$. Then*

$$P_n(\alpha)e^{-\alpha} = \int \int_{\substack{(y_1, \dots, y_n) \in \mathbf{R}_+^* \\ \text{Tr}(y) \geq \alpha}} e^{-\text{Tr}(y)} dy \leq n \int_{\alpha/n}^{+\infty} e^{-y} dy = ne^{-\alpha/n}.$$

Proof. Use

$$\begin{aligned} & \{(y_1, \dots, y_n) \in \mathbf{R}_+^*, \text{Tr}(y) \geq \alpha\} \\ & \subseteq \bigcup_{i=1}^n \left\{ (y_1, \dots, y_n), y_i \geq \frac{\alpha}{n} \text{ and } y_j \geq 0 \text{ for } j \neq i \right\}. \quad \square \end{aligned}$$

4. PROOF OF THEOREM 2

Let \mathbf{K} be a number field of degree $n = r_1 + 2r_2$, where r_1 is the number of real places of \mathbf{K} and r_2 the number of complex places of \mathbf{K} . Let $\zeta_{\mathbf{K}}$ and $\text{Reg}_{\mathbf{K}}$ be the Dedekind zeta function and regulator of \mathbf{K} . We set

$$\begin{aligned} (13) \quad A_{\mathbf{K}} &= 2^{-r_2} d_{\mathbf{K}}^{1/2} \pi^{-(r_1+2r_2)/2}, \\ \lambda_{\mathbf{K}} &= \frac{2^{r_1} h_{\mathbf{K}} \text{Reg}_{\mathbf{K}}}{w_{\mathbf{K}}} \quad \text{where } w_{\mathbf{K}} \text{ is the number of roots of unity in } \mathbf{K}, \\ F_{\mathbf{K}}(s) &= A_{\mathbf{K}}^s \Gamma\left(\frac{s}{2}\right)^{r_1} \Gamma(s)^{r_2} \zeta_{\mathbf{K}}(s). \end{aligned}$$

Hence, $F_{\mathbf{K}}$ a simple pole at $s = 1$ with residue $\lambda_{\mathbf{K}}$, and $F_{\mathbf{K}}(s) = F_{\mathbf{K}}(1 - s)$.

From now on, we let \mathbf{N} be a CM-field of degree $2n$, i.e. \mathbf{N} is a totally imaginary number field of degree $2n$ which is a quadratic extension of a totally real number field \mathbf{N}^+ of degree n . Define the ϕ_k 's by :

$$\Phi_{\mathbf{N}/\mathbf{N}^+}(s) = \frac{\zeta_{\mathbf{N}}}{\zeta_{\mathbf{N}^+}}(s) = \sum_{k \geq 1} \phi_k k^{-s} \quad (\Re(s) > 1).$$

Then, $(\zeta_{\mathbf{N}}/\zeta_{\mathbf{N}^+})(s) = L(s, \chi_{\mathbf{N}/\mathbf{N}^+})$ yields

$$(14) \quad \phi_k = \sum_{N_{\mathbf{N}^+/\mathbf{Q}}(\mathbf{I})=k} \chi_{\mathbf{N}/\mathbf{N}^+}(\mathbf{I})$$

where \mathbf{I} ranges over the integral ideals of \mathbf{N}^+ of norm k . Now,

$$\Phi_{\mathbf{N}/\mathbf{N}^+} = \zeta_{\mathbf{N}}/\zeta_{\mathbf{N}^+} \quad \text{and} \quad \Psi_{\mathbf{N}/\mathbf{N}^+} = F_{\mathbf{N}}/F_{\mathbf{N}^+}$$

are entire and $\Psi_{\mathbf{N}/\mathbf{N}^+}(s) = \Psi_{\mathbf{N}/\mathbf{N}^+}(1 - s)$. Notice that

$$(15) \quad \Psi_{\mathbf{N}/\mathbf{N}^+}(1) = \frac{\lambda_{\mathbf{N}}}{\lambda_{\mathbf{N}^+}} = \frac{h_{\mathbf{N}}^-}{Q_{\mathbf{N}} w_{\mathbf{N}}}$$

where $Q_{\mathbf{N}} \in \{1, 2\}$ is the Hasse unit index of \mathbf{N} (see [Wa, Th. 4.16]). Since

$$\Gamma(s) = \frac{2^{s-1}}{\sqrt{\pi}} \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s+1}{2}\right),$$

using (13) for \mathbf{N} and \mathbf{N}^+ leads to

$$(16) \quad \Psi_{\mathbf{N}/\mathbf{N}^+}(s) = c_{\mathbf{N}/\mathbf{N}^+} A_{\mathbf{N}/\mathbf{N}^+}^s \Gamma^n\left(\frac{s+1}{2}\right) \Phi_{\mathbf{N}/\mathbf{N}^+}(s)$$

where

$$c_{\mathbf{N}/\mathbf{N}^+} = 1/(4\pi)^{n/2} \quad \text{and} \quad A_{\mathbf{N}/\mathbf{N}^+} = \sqrt{d_{\mathbf{N}}/\pi^n d_{\mathbf{N}^+}}.$$

Note that

$$(17) \quad c_{\mathbf{N}/\mathbf{N}^+} A_{\mathbf{N}/\mathbf{N}^+} = \frac{1}{(2\pi)^n} \sqrt{\frac{d_{\mathbf{N}}}{d_{\mathbf{N}^+}}}.$$

Set

$$(18) \quad \hat{\Psi}_{\mathbf{N}/\mathbf{N}^+}(x) = \frac{1}{2i\pi} \int_{\alpha-i\infty}^{\alpha+i\infty} \Psi_{\mathbf{N}/\mathbf{N}^+}(s) x^{-s} ds \quad (\alpha > 1),$$

i.e., $\hat{\Psi}_{\mathbf{N}/\mathbf{N}^+}$ is the Mellin transform of the function $\Psi_{\mathbf{N}/\mathbf{N}^+}$. Using (18) and (16) yields

$$(19) \quad \hat{\Psi}_{\mathbf{N}/\mathbf{N}^+}(x) = c_{\mathbf{N}/\mathbf{N}^+} \sum_{k \geq 1} \phi_k H_n(kx/A_{\mathbf{N}/\mathbf{N}^+}) \quad (x > 0),$$

with

$$(20) \quad \begin{aligned} H_n(x) &= \frac{1}{2i\pi} \int_{\alpha-i\infty}^{\alpha+i\infty} \Gamma^n\left(\frac{s+1}{2}\right) x^{-s} ds \\ &= \frac{1}{i\pi} \int_{\alpha-i\infty}^{\alpha+i\infty} \Gamma^n(S) x^{1-2S} dS \quad (x > 0 \text{ and } \alpha > 0). \end{aligned}$$

Now, we move the integral (18) to the line $\Re(s) = 1 - \alpha$. Since $\Psi_{\mathbf{N}/\mathbf{N}^+}$ is entire, we do not pick up any residue. Then, we use the functional equation

$$\Psi_{\mathbf{N}/\mathbf{N}^+}(s) = \Psi_{\mathbf{N}/\mathbf{N}^+}(1 - s)$$

satisfied by $\Psi_{\mathbf{N}/\mathbf{N}^+}$ to come back to the line $\Re(s) = \alpha$. We get

$$(21) \quad x \hat{\Psi}_{\mathbf{N}/\mathbf{N}^+}(x) = \hat{\Psi}_{\mathbf{N}/\mathbf{N}^+}(1/x) \quad (x > 0).$$

Mellin's inversion formula and (21) yield

$$(22) \quad \Psi_{\mathbf{N}/\mathbf{N}^+}(s) = \int_0^\infty \hat{\Psi}_{\mathbf{N}/\mathbf{N}^+}(x) x^s \frac{dx}{x} = \int_1^\infty \hat{\Psi}_{\mathbf{N}/\mathbf{N}^+}(x) \{x^{s-1} + x^{-s}\} dx.$$

By using (22), (19) and (20) we thus get

$$\begin{aligned} &\Psi_{\mathbf{N}/\mathbf{N}^+}(s) \\ &= c_{\mathbf{N}/\mathbf{N}^+} \sum_{k \geq 1} \phi_k \int_1^\infty \left(\frac{1}{i\pi} \int_{\alpha-i\infty}^{\alpha+i\infty} \left(\frac{kx}{A_{\mathbf{N}/\mathbf{N}^+}} \right)^{1-2S} \Gamma^n(S) \{x^{s-1} + x^{-s}\} dS \right) dx \\ &= c_{\mathbf{N}/\mathbf{N}^+} \sum_{k \geq 1} \phi_k \frac{1}{i\pi} \int_{\alpha-i\infty}^{\alpha+i\infty} \Gamma^n(S) \left(\int_1^\infty \left(\frac{kx}{A_{\mathbf{N}/\mathbf{N}^+}} \right)^{1-2S} \{x^{s-1} + x^{-s}\} dx \right) dS \\ &= c_{\mathbf{N}/\mathbf{N}^+} \sum_{k \geq 1} \phi_k \frac{1}{i\pi} \int_{\alpha-i\infty}^{\alpha+i\infty} \Gamma^n(S) (k/A_{\mathbf{N}/\mathbf{N}^+})^{1-2S} \left(\frac{1}{2S-s-1} + \frac{1}{2S+s-2} \right) dS, \end{aligned}$$

and the following yields (4):

$$\begin{aligned} h_{\mathbf{N}}^- &= Q_{\mathbf{N}} w_{\mathbf{N}} \Psi_{\mathbf{N}/\mathbf{N}^+}(1) = Q_{\mathbf{N}} w_{\mathbf{N}} \Psi_{\mathbf{N}/\mathbf{N}^+}(0) \\ &= Q_{\mathbf{N}} w_{\mathbf{N}} c_{\mathbf{N}/\mathbf{N}^+} \sum_{k \geq 1} \phi_k \frac{1}{i\pi} \int_{\alpha-i\infty}^{\alpha+i\infty} \Gamma^n(S) (k/A_{\mathbf{N}/\mathbf{N}^+})^{1-2S} \left(\frac{1}{2S-2} + \frac{1}{2S-1} \right) dS \\ &= Q_{\mathbf{N}} w_{\mathbf{N}} c_{\mathbf{N}/\mathbf{N}^+} A_{\mathbf{N}/\mathbf{N}^+} \sum_{k \geq 1} \frac{\phi_k}{k} K_n(k/A_{\mathbf{N}/\mathbf{N}^+}) \\ &= \frac{Q_{\mathbf{N}} w_{\mathbf{N}}}{(2\pi)^n} \sqrt{\frac{d_{\mathbf{N}}}{d_{\mathbf{N}^+}}} \sum_{k \geq 1} \frac{\phi_k}{k} K_n(k/A_{\mathbf{N}/\mathbf{N}^+}). \end{aligned}$$

Now, we prove the assertion below (5).

To start with we quote some elementary facts we will need.

1) We have

$$(23) \quad P_n(x) = \sum_{k=0}^{n-1} \frac{1}{k!} x^k \leq \sum_{k=0}^{n-1} \frac{1}{k!} x^{n-1} \leq e x^{n-1} \quad (x \geq 1).$$

2) The derivative of

$$(24) \quad g(x) = x^{\frac{2n-2}{n}} e^{-n x^{2/n}}$$

is

$$g'(x) = \frac{1}{n} \left((2n-2) - 2n x^{2/n} \right) x^{\frac{n-2}{n}} e^{-n x^{2/n}}$$

and we have $g'(x) \leq 0$ if $x \geq 1$ and

$$(25) \quad |g'(x)| \leq 2x e^{-n x^{2/n}} \quad (x \geq 1).$$

Moreover,

$$g''(x) = \frac{1}{n^2} \left(4n^2 x^{4/n} - (6n^2 - 4n) x^{2/n} + (2n^2 - 6n + 4) \right) x^{-2/n} e^{-n x^{2/n}},$$

the second derivative of g , satisfies $g''(x) \geq 0$ if $x \geq 2^{n/2}$. Note that (3) and (24) yield

$$(26) \quad K_n(x) \leq 2en^{n-1} g(x) \quad (x \geq 1).$$

3) If $g(x) \geq 0$, $g'(x) \leq 0$ and $g''(x) \geq 0$ on $[\alpha, +\infty[$, then $\alpha \leq a \leq b$ implies

$$(27) \quad 0 \leq g(a) - g(b) \leq (a - b)g'(a).$$

4) If $A((n + 1)/2)^{n/2} \geq 1$, then the derivative of

$$(28) \quad h(x) = x \log^n(ex)e^{-n(x/A)^{2/n}}$$

is

$$h'(x) = (n - (2(x/A)^{2/n} - 1) \log(ex)) \log^{n-1}(ex)e^{-n(x/A)^{2/n}}$$

and we have $h'(x) \leq 0$ provided that $x \geq A((n + 1)/2)^{n/2}$ and $x \geq 1$, hence provided that $x \geq A((n + 1)/2)^{n/2}$ if $A((n + 1)/2)^{n/2} \geq 1$.

Now, we set $A = A_{\mathbf{N}/\mathbf{N}^+}$, $S_n(k) = \sum_{i=1}^k \frac{d_n(i)}{i}$ where $d_n(i)$ is the number of ways of writing i as an ordered product of n positive integers, and

$$R_M = \sum_{k>M} \frac{\phi_k}{k} K_n(k/A).$$

We want an upper bound on R_M . We note that (14) yields $|\phi_k| \leq d_n(k)$. Moreover,

$$S_n(k) = \sum_{i=1}^k \frac{d_n(i)}{i} \leq \left(\sum_{i=1}^k \frac{1}{i} \right)^n \leq \log^n(ek).$$

Thus, we have

$$\begin{aligned} |R_M| &\leq \sum_{k>M} \frac{d_n(k)}{k} K_n(k/A) \\ &\leq 2en^{n-1} \sum_{k>M} (S_n(k) - S_n(k-1))g(k/A) \quad (\text{if } M \geq A) \\ &\text{(by using (26))} \\ &\leq 2en^{n-1} \sum_{k>M} S_n(k)(g(k/A) - g((k+1)/A)) \\ &\leq \frac{2en^{n-1}}{A} \sum_{k>M} S_n(k)g'(k/A) \quad (\text{if } M \geq 2^{n/2}A) \\ &\text{(by using (27))} \\ &\leq \frac{4en^{n-1}}{A^2} \sum_{k>M} k \log^n(ek)e^{-n(k/A)^{2/n}} \\ &\text{(by using (25))} \\ &= \frac{4en^{n-1}}{A^2} \sum_{k>M} h(k) \leq \frac{4en^{n-1}}{A^2} \int_M^\infty h(x)dx \quad (\text{if } M \geq \left(\frac{n+1}{2}\right)^{n/2} A \geq 1) \\ &\text{(by using (28)).} \end{aligned}$$

Now, we set $B = (eA)^{2/n}$ and we change the variable by setting $x = Ay^{n/2}$. We get

$$|R_M| \leq 2e(n^2/2)^n \int_{(M/A)^{2/n}}^\infty y^n \log^n(By) e^{-ny} \frac{dy}{y}.$$

Since $H(y) = y^{n+1} \log^n(By) e^{-ny}$ decreases on $[(M/A)^{2/n}, +\infty[$ if $M \geq \left(\frac{2n+2}{n}\right)^{n/2} A \geq e^{(n/2)-1}$ (since its derivative

$$H'(y) = ((n + 1 - ny) \log(By) + n)y^n \log^{n-1}(By) e^{-ny}$$

satisfies $H'(y) \leq 0$ if $y \geq (2n + 2)/n$ and $B^{\frac{2n+2}{2}} \geq e$), we get

$$\begin{aligned} |R_M| &\leq 2e(n^2/2)^n \int_{(M/A)^{2/n}}^\infty H(y) \frac{dy}{y^2} \\ &\leq 2e(n^2/2)^n H((M/A)^{2/n}) \int_{(M/A)^{2/n}}^\infty \frac{dy}{y^2} \\ &= 2e(n^2/2)^n H((M/A)^{2/n}) / (M/A)^{2/n}, \end{aligned}$$

i.e., if $M \geq \left(\frac{2n+2}{n}\right)^{n/2} A \geq e^{(n/2)-1}$, then we have the following explicit upper bound :

(29)

$$|R_M| \leq 2e \left(\frac{n^2}{2} G((M/A)^{2/n}) \right)^n \quad \text{where } G(y) = y \log(By) e^{-y} \quad \text{and } B = (eA)^{2/n}.$$

Now, we choose $M \approx B(\mathbf{N}) = A \left(\frac{\lambda}{n} \log A\right)^{n/2}$ and note that

$$G\left(\frac{\lambda}{n} \log A\right) = O_n(A^{-\lambda/n} \log^2 A)$$

yields the desired result :

$$(30) \quad |h_{\mathbf{N}}^- - h_{\mathbf{N}}^-(M)| = \frac{Q_{\mathbf{N}} w_{\mathbf{N}}}{2^n \pi^{n/2}} A |R_M| = O_n \left(\frac{\log^{2n} A}{A^{\lambda-1}} \right).$$

5. PROOF OF PROPOSITION 3

Let $M \geq 0$ be a given integer. Shifting the integral (2) to the left to the line $\Re(s) = -M - \frac{1}{2}$, we pick a residue at $s = 1$, a residue at $s = 1/2$, and a residue $\text{Res}_{s=-m}(f_n)$ at each nonpositive integer $-m \leq 0$. Hence, by using $\Gamma(1/2) = \sqrt{\pi}$ we get

$$(31) \quad K_n(A) = 1 + \pi^{n/2} A + 2A^2 \sum_{m=0}^M \text{Res}_{s=-m}(f_n) + \frac{A^2}{i\pi} \int_{-M-\frac{1}{2}-i\infty}^{-M-\frac{1}{2}+i\infty} f_n(s) ds.$$

Now, it is well known that for any nonnegative integer $l \geq 0$ we have

$$\left| \Gamma \left(\frac{2l+1}{2} + it \right) \right|^2 = \frac{\pi}{\text{ch}(\pi t)} \prod_{k=0}^{l-1} \left| \frac{2k+1}{2} + it \right|^2,$$

where $\text{ch}(x) = (e^x + e^{-x})/2$. Hence, using the functional equation $\Gamma(s)\Gamma(1-s) = \pi/\sin(\pi s)$ leads to

$$(32) \quad \left| \Gamma \left(-\frac{2l+1}{2} + it \right) \right|^2 = \frac{\pi}{\text{ch}(\pi t)} \prod_{k=0}^l \left| \frac{2k+1}{2} + it \right|^{-2},$$

and

$$\begin{aligned} & \left| f_n\left(-M - \frac{1}{2} + it\right) \right| \\ & \leq \left(\frac{\pi}{\text{ch}(\pi t)} \right)^{n/2} \frac{A^{2M+1}}{\left(\prod_{k=0}^M \left| \frac{2k+1}{2} + it \right| \right)^n} \left(\frac{1}{|2M+2+it|} + \frac{1}{|2M+3+it|} \right) \\ & \leq \frac{1}{(\text{ch}(\pi t))^{n/2}} \frac{\pi^{n/2} A^{2M+1}}{(M!/2)^n} \frac{2}{2M+2}. \end{aligned}$$

Set

$$c_n = \int_{-\infty}^{+\infty} \frac{dt}{(\text{ch}(\pi t))^{n/2}} = \frac{2}{\pi} \int_0^1 \left(\frac{2}{u+u^{-1}} \right)^{n/2} \frac{du}{u}$$

(note that the sequence $(c_n)_{n \geq 0}$ decreases, that $c_1 = \frac{4\sqrt{2}}{\pi} \int_0^1 \frac{dv}{\sqrt{v^4+1}} \leq 4\sqrt{2}/\pi$ and $c_2 = 1$). Then,

$$(33) \quad \left| \frac{A^2}{i\pi} \int_{-M-\frac{1}{2}-i\infty}^{-M-\frac{1}{2}+i\infty} f_n(s) ds \right| \leq \frac{c_n}{\pi} \frac{\pi^{n/2} A^{2M+3}}{(M+1)(M!/2)^n}.$$

Note that the greater the value of n , the faster the series (6) converges.

6. PROOF OF PROPOSITION 4

We have

$$\begin{aligned} & \text{Res}_{s=-m}(f_n) \\ & = -A^{2m} \text{Res}_{s=0} \left(s \mapsto \Gamma^n(-m+s) A^{-2s} \left(\frac{1}{2m+1-2s} + \frac{1}{2m+2-2s} \right) \right). \end{aligned}$$

If we set

$$(34) \quad \Gamma^n(-m+s) A^{-2s} = \sum_{i=-n}^{-1} a_i(m) s^i + O(1),$$

then we get

$$(35) \quad \text{Res}_{s=-m}(f_n) = -A^{2m} \sum_{i=-n}^{-1} a_i(m) 2^{-1-i} ((2m+1)^i + (2m+2)^i).$$

Now, $\Gamma(s) = \frac{1}{s} \Gamma(s+1)$ yields

$$\sum_{i=-n}^{-1} a_i(m+1) s^i + O(1) = (-1)^n (m+1-s)^{-n} \left(\sum_{j=-n}^{-1} a_j(m) s^j + O(1) \right)$$

and

$$(m+1-s)^{-n} = \frac{1}{(m+1)^n} \sum_{k=0}^{n-1} C_{k+n-1}^{n-1} \frac{s^k}{(m+1)^k} + O(s^n)$$

yields

$$(36) \quad a_i(m+1) = \frac{(-1)^n}{(m+1)^n} \sum_{j=-n}^i \frac{a_j(m)}{(m+1)^{i-j}} C_{i-j+n-1}^{n-1}.$$

Thus, in order to simplify the recursion relation (36), we define

$$h_i(m) = (-1)^{nm} (m!)^n a_i(m).$$

Then, using (35) yields (8), and using (34) and (36) yields (9). Note that (10) makes it easy to compute the numerical values of the h_i 's by using Maple, for example.

7. EXAMPLES OF RELATIVE CLASS NUMBERS COMPUTATIONS

In order to use (4) to compute relative class numbers, it remains to explain how we compute the ϕ_k 's. Since

$$\phi_k = \sum_{N_{\mathbf{N}^+/\mathbf{Q}}(\mathbf{I})=k} \chi_{\mathbf{N}/\mathbf{N}^+}(\mathbf{I})$$

(see (14)), then $k \mapsto \phi_k$ is multiplicative and we only have to explain how we compute the ϕ_{p^m} where p is prime and $m \geq 1$. We will only explain this when \mathbf{N} is normal over \mathbf{Q} . In that case, let e and f be the inertia and residual degrees of p in \mathbf{N}_+ . Set $g = n/(ef)$. Then in \mathbf{N}^+ we have $(p) = (\mathcal{P}_1 \cdots \mathcal{P}_g)^e$ and

$$\chi_{\mathbf{N}/\mathbf{N}^+}(\mathcal{P}_1) = \cdots = \chi_{\mathbf{N}/\mathbf{N}^+}(\mathcal{P}_g),$$

and we let ϵ_p be the common value of these g symbols. Now, $N_{\mathbf{N}^+/\mathbf{Q}}(\mathbf{I}) = p^m$ if and only if $\mathbf{I} = \prod_{i=1}^g \mathcal{P}_i^{e_i}$ with $f \sum_{i=1}^g e_i = m$. Set

$$C_i^j = \frac{i!}{j!(i-j)!}.$$

Since the equation $\sum_{i=1}^g e_i = K$ has C_{K+g-1}^{g-1} solutions in nonnegative integers e_i , we easily get

$$(37) \quad \phi_{p^m} = \begin{cases} 0 & \text{if } f \text{ does not divide } m, \\ \epsilon_p^k C_{k+g-1}^{g-1} & \text{if } f \text{ divides } m \text{ and } m = kf. \end{cases}$$

This formula (37) makes it easy to compute the ϕ_{p^m} . We refer the reader to [Lou 1], [Lou 2], [Lou 3], [Lou-Oka] and [LOO] for actual computations of relative class numbers.

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