

SEVEN CONSECUTIVE PRIMES IN ARITHMETIC PROGRESSION

HARVEY DUBNER AND HARRY NELSON

ABSTRACT. It is conjectured that there exist arbitrarily long sequences of *consecutive* primes in arithmetic progression. In 1967, the first such sequence of 6 consecutive primes in arithmetic progression was found. Searching for 7 consecutive primes in arithmetic progression is difficult because it is necessary that a prescribed set of at least 1254 numbers between the first and last prime all be composite. This article describes the search theory and methods, and lists the only known example of 7 consecutive primes in arithmetic progression.

1. INTRODUCTION

It is conjectured that the number of primes in arithmetic progression can be as large as you like [2]. A prodigious amount of computer time has been used to search for long strings of primes in arithmetic progression, with the current record being 22 [6]. A related conjecture is the following: there exist arbitrarily long sequences of *consecutive* primes in arithmetic progression [2]. In 1967, Lander and Parkin [4] reported finding the first and smallest sequence of 6 consecutive primes in AP, where the starting prime is 121174811 and the common difference is 30. Since then many other sequences of 6 such primes have been found, as well as other sets of 6 consecutive primes with common differences of 60 and 90 [9].

It is easy to show that 7 primes in AP must have a common difference that is at least 210 (excluding the singular case where the first prime is 7) [7]. But if these are also to be consecutive, this implies that there is a prescribed set of 1254 numbers between the first and last prime that must be composite. This simultaneous requirement for having 7 primes in AP and also a prescribed set of 1254 composite numbers, is the reason for the difficulty in finding 7 consecutive primes in AP. However, by extending a method of Nelson for assuring that a given set of numbers would be composite, such a search became practical [5].

In this article we describe the search theory and methods, and give the only known example of 7 consecutive primes in arithmetic progression.

2. DENSITY OF CONSECUTIVE PRIMES

In order to help estimate the search time for finding 7 consecutive primes in AP we can use the Prime Number Theorem to estimate the density of prescribed sets of primes and prescribed sets of composites as a function of the size of the numbers.

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TABLE 1. Density of consecutive primes in arithmetic progression

number consec primes	number of composites	digits for max density	approx probabilities		
			primes	compos	both
5	116	11	$9 * 10^{-8}$	$9 * 10^{-3}$	$9 * 10^{-10}$
6	145	11	$4 * 10^{-9}$	$3 * 10^{-3}$	$1 * 10^{-11}$
7	1254	78	$1 * 10^{-16}$	$1 * 10^{-3}$	$2 * 10^{-19}$
8	1463	80	$8 * 10^{-19}$	$3 * 10^{-4}$	$3 * 10^{-22}$
9	1672	81	$4 * 10^{-21}$	$1 * 10^{-4}$	$5 * 10^{-25}$
10	1881	82	$2 * 10^{-23}$	$4 * 10^{-5}$	$8 * 10^{-28}$
11	23090	910	$5 * 10^{-37}$	$1 * 10^{-5}$	$5 * 10^{-42}$

Assuming statistical independence, the approximate probability of finding k primes near a number N is,

$$(1) \quad P(k) \approx \left(\frac{1}{\log N} \right)^k,$$

while the minimum number, r , of required composites is,

$$(2) \quad r = (k-1)(d-1), \quad \begin{cases} d = 30 & \text{for } k = 5, 6, \\ d = 210 & \text{for } k = 7, 8, 9, 10, \\ d = 2310 & \text{for } k = 11. \end{cases}$$

Similarly, the approximate probability of finding r composites near N is,

$$(3) \quad C(k) \approx \left(1 - \frac{1}{\log N} \right)^r.$$

Thus, for a random search near N , the approximate probability of finding k consecutive primes in AP is the product of equations (1) and (3),

$$(4) \quad Q(k) \approx \left(\frac{1}{\log N} \right)^k \cdot \left(1 - \frac{1}{\log N} \right)^r.$$

For each k we can calculate the value of N that optimizes the probability, $Q(k)$. Near such an N is where the density of k consecutive primes in AP should be at a maximum. Table 1 summarizes these results.

While simply searching the sequence of primes up to about 10^{10} would be a viable way to look for 6 consecutive primes in AP, the size of the likely candidate values for 7 such primes preclude this approach. However we could search for 7 primes with a common difference of 210 near 78 digit numbers using appropriate sieving methods, and when such a prime set is found, test the intermediate 1254 numbers. About 1 time in 1000 they should all be composite. We estimated that such a search would take about 10 computer-years on a PC 486/66, which was much too long to be practical. The factor of 1000 contributed by the 1254 composites was a major source of difficulty.

3. COMPOSITE SEQUENCES

In 1975 one of the authors solved a problem in the Journal of Recreational Mathematics by developing a method that used a system of simultaneous modular

TABLE 2. Composite-set covering – 1254 numbers

Numbers left vs. number of equations							
n	prime	$b(n)$	left	n	prime	$b(n)$	left
1	2	0	624	43	191	45	57
2	3	0	414	44	193	112	55
3	5	0	330	45	197	4	53
4	7	0	282	46	199	157	51
5	11	0	255	47	211	84	49
6	13	7	233	48	223	69	47
7	17	5	217	49	227	54	45
8	19	7	204	50	229	15	43
9	23	6	191	51	233	86	41
10	29	9	183	52	239	186	39
11	31	19	175	53	241	225	37
12	37	9	168	54	251	123	35
13	41	4	161	55	257	178	33
14	43	40	154	56	263	213	31
15	47	22	148	57	269	7	30
16	53	2	143	58	271	0	29
17	59	57	137	59	277	204	27
18	61	2	133	60	281	21	26
19	67	44	128	61	283	28	25
20	71	7	124	62	293	190	23
21	73	33	120	63	307	56	22
22	79	33	116	64	311	72	21
23	83	5	113	65	313	105	20
24	89	7	109	66	317	304	18
25	97	25	105	67	331	8	17
26	101	2	102	68	337	1	16
27	103	20	99	69	347	2	15
28	107	22	96	70	349	50	14
29	109	77	93	71	353	92	13
30	113	11	91	72	359	5	12
31	127	80	88	73	367	170	11
32	131	2	86	74	373	247	10
33	137	5	83	75	379	267	9
34	139	50	80	76	383	285	8
35	149	12	77	77	389	0	7
36	151	136	74	78	397	27	6
37	157	6	72	79	401	5	5
38	163	54	70	80	409	339	4
39	167	143	67	81	419	352	3
40	173	35	65	82	421	380	2
41	79	16	62	83	431	405	1
42	181	80	59	84	433	427	0

equations to guarantee that a specific sequence of numbers will be composite [5]. This method with modifications was applicable to our 7 prime problem.

Consider the system of modular equations,

$$(5) \quad x \equiv b_j \pmod{p_j}, \quad p_j = j\text{th prime.}$$

The Chinese Remainder Theorem [8] states that there is always a solution for x satisfying (5), with

$$(6) \quad 0 \leq x < m, \quad m = \prod p_j.$$

It is easy to write a computer program to solve for such an x .

For each j , starting at a number determined by x and b_j , every p_j th number is divisible by p_j and therefore composite. For r equations, the set of b 's determine a particular pattern of composite numbers. By adding to the number of equations and selecting appropriate b 's, more numbers join the composite pattern. In this manner any selection of numbers can be made composite. While assuring that these numbers are composite, by properly restricting b , numbers,

$$(7) \quad q(s) = x + 210 \cdot s + 1, \quad s = 0, 1, 2, 3, 4, 5, 6$$

can be kept free of any small factors, p_j , so that these remain candidates for the primes in arithmetic progression.

We start by setting $b_j = 0$ for $p_j = 2, 3, 5, 7$ and then add additional equations until all desired numbers are forced to be composite. When a new equation is added, we determine the new b by trying every possible b and selecting the one that guarantees the most additional composite numbers in our target set while keeping the candidates for primes in AP free of small factors. Usually there are many such "optimum" b 's and we arbitrarily chose the smallest one. 7 primes require 1254 composite numbers. Table 2 shows how many numbers are left of the 1254 which are not certain to be composite, as a function of the number of equations with the b 's chosen as described. For all 1254 to be composite we needed 84 equations. Note that from equation (6), the size of the solution, x , tends to increase with each new equation.

Other rules were tried which gave different sets of b 's, but this did not result in any significant improvement for the number of equations, so that the time estimate for finding 7 primes in AP was hardly affected. It does not seem computationally feasible to determine the absolute optimum set of b 's.

4. SEARCH TIME

From equation (6), let

$$(8) \quad y_N = x + m \cdot N, \quad N = 0, 1, 2, 3, \dots$$

Thus, each y is a potentially good starting point for finding 7 primes in AP since, for the small primes p_j , adding the term, $m \cdot N$, has no effect on the divisibility properties of the composite numbers or the 7 possible primes $q(s)$. An array representing the 7 values of $q(s)$ for each y_N can be sieved to eliminate all N for which any $q(s)$ has a factor less than some maximum, p_{\max} . This dramatically reduces the number of q sets that need to be tested for probable primality using Fermat tests.

Our original plan was to use all 84 equations, appropriately sieve N , and test the remaining sets of $q(s)$ until 7 primes were found. However this meant that the

size of the primes would be about 185 digits, and the time for a prime test was large enough so that the search time was still quite long. We then investigated the possibility of having fewer equations so that the primes would be smaller, and depend on probability for making the remaining numbers composite. By modifying (1) and (3) to allow for sieving we estimated the search time as a function of the number of equations.

When a random number near N is free of factors up to p , the probability of it being prime becomes approximately,

$$(9) \quad \text{prob} \approx \frac{\log p}{.562 \cdot \log N}.$$

This is based on Merten's theorem [3, p. 351] and a short discussion in Merten's [1], and is valid for p relatively small and N relatively large as is true in this article.

Equations (1) and (3) now become,

$$(10) \quad P(7) \approx \left(\frac{\log(\text{pmax})}{.562 \cdot \log N} \right)^7, \quad C(7) \approx \left(1 - \frac{\log p_c}{.526 \cdot \log N} \right)^t,$$

where p_c and t depend on the number of equations. The results are shown in Table 3. The test times should be considered approximate because of the use of some simplifying assumptions and because the times are heavily machine dependent, but the relative values are significant. Using 42 instead of 84 equations reduces the expected search time by about a factor of 35.

5. RESULTS

Based on Table 3 we used up to 5 PC's, 486/66 or equivalent, to search for the 7 consecutive probable primes in AP in the appropriate range of equations numbering from 36 to 52. After about a total of 70 computer-days (about 20 days of calendar time) the search was successful for equations = 48. The 7 probable primes were 97 digits long. Their true primality was verified using the APRT-CLE program in UBASIC. The compositeness of all the numbers between these primes was verified using Fermat tests. The solution is shown below.

Number of modular equations = 48

Product of primes up to the 48th prime = m :

$m = 36700973182733191646503456555013673233980031295533178261946245703_$
 9988073311157667212930

Solution for the 48 modular equations = x :

$x = 11893061343242550473160091662536053989417322887015941546297601405_$
 6809082107460202605690

First prime = $Q1 = x + N^*m + 1$, where

$$N = 2968677222$$

$Q1$ is the 97 digit number,

TABLE 3. Estimated search time for 7 composite primes in AP

number of equations	max prime	comps left	approx digits	average number of prime-sets	average number of composite-sets	test time days
20	71	124	35	3969	5123727.81	14830.2
22	79	116	39	8466	394225.66	2711.9
24	89	109	43	16770	55405.79	832.4
26	101	102	47	31256	11049.37	338.2
28	107	96	51	55366	3119.92	183.5
30	113	91	55	93928	1152.62	124.1
32	131	86	59	153539	489.84	92.4
34	139	80	64	271342	199.64	72.2
36	151	74	68	414781	99.68	58.6
38	163	70	72	618849	60.59	56.2
40	173	65	77	990130	35.04	55.7
42	181	59	81	1411407	21.43	51.0
44	193	55	86	2146577	14.68	56.5
46	199	51	90	2950903	10.78	59.7
48	223	47	95	4308466	7.95	67.8
50	229	43	100	6169606	6.05	77.8
52	239	39	105	8681255	4.73	89.8
54	251	35	109	11278281	3.83	98.0
56	263	31	114	15438012	3.11	114.1
58	271	29	119	20849044	2.76	142.9
60	281	26	124	27810527	2.40	172.2
62	293	23	129	36676290	2.10	207.2
64	311	21	134	47862020	1.92	256.6
66	317	18	139	61853038	1.71	307.0
68	337	16	144	79212711	1.59	377.3
70	349	14	149	100591507	1.48	461.5
72	359	12	154	126736725	1.38	562.1
74	373	10	159	158502934	1.30	681.8
76	383	8	164	196863124	1.22	823.6
78	397	6	170	253162797	1.16	1038.1
80	409	4	175	310115716	1.10	1243.1
82	421	2	180	377715643	1.05	1483.2
84	433	0	185	457572099	1.00	1763.6

- Notes: 1. Computer system is PC486/66 with a Cruncher.
2. Approximate Fermat test time = .09·(digits/100) seconds.
3. 7-prime sets are sieved to 1,000,000.
4. Sieving time is included in table.
5. Column 5 is the expected number of prime sets that will have to be tested to find 7 primes in AP.
6. Column 6 is the expected number of composite sets that will have to be tested to find 1254 composites.

$Q1 = 1089533431247059310875780378922957732908036492993138195385213105_$
561742150447308967213141717486151

$$Q2 = Q1 + 210, Q3 = Q2 + 210, \dots, Q7 = Q6 + 210.$$

Finding 8 consecutive primes in AP would be expected to take about 20 times longer than our estimated search time for 7 primes. This is well within the capability of supercomputers or workstation and PC networks. It seems only a matter of time before 8 consecutive primes in AP will be found. Even 9 or 10 such primes seem within the realm of possibility using the above methods.

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449 BEVERLY ROAD, RIDGEWOOD, NEW JERSEY 07450
E-mail address: 70327.1170@compuserve.com

4259 ERORY WAY, LIVERMORE, CALIFORNIA 94550
E-mail address: hln@anduin.ocf.llnl.gov