

CONVERGENCE OF NEWTON'S METHOD  
AND INVERSE FUNCTION THEOREM  
IN BANACH SPACE

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ABSTRACT. Under the hypothesis that the derivative satisfies some kind of weak Lipschitz condition, a proper condition which makes Newton's method converge, and an exact estimate for the radius of the ball of the inverse function theorem are given in a Banach space. Also, the relevant results on premises of Kantorovich and Smale types are improved in this paper.

We continue to discuss the problem of convergence in the Newton method

$$(0.1) \quad x_{n+1} = x_n - f'(x_n)^{-1}f(x_n), \quad n = 0, 1, \dots,$$

to solve an operator equation  $f$  which maps from some domain  $D$  in a real or complex Banach space  $\mathbf{X}$  to another Banach space  $\mathbf{Y}$ ,

$$(0.2) \quad f(x) = 0.$$

Now we come back to the problem which we bypassed in [1].

We always assume that  $f'(x_0)^{-1}$  exists and  $f'(x_0)^{-1}f'$  satisfies some kind of Lipschitz condition similar to that of [1] in some open ball  $B(x_0, r) \subset D$  with center  $x_0$  and radius  $r$  (or some closed ball  $\overline{B(x_0, r)} \subset D$ ) in order to study the convergence of Newton's method and the domain of the local inverse function of  $f$  at  $x_0$ .

1. THE DOMAIN OF THE INVERSE FUNCTION

The inverse function theorem asserts that there is an inverse function  $f_{x_0}^{-1}$  defined on some open ball  $B(f(x_0), \varepsilon) \subset \mathbf{Y}$  with the property that

$$\begin{aligned} f_{x_0}^{-1}(f(x_0)) &= x_0, \\ f(f_{x_0}^{-1}(y)) &= y, \quad \forall y \in B(f(x_0), \varepsilon), \end{aligned}$$

and  $f_{x_0}^{-1}$  is differentiable. Now we study the exact lower bound estimate of the radius of this ball.

For this reason, we assume that  $f$  has a continuous derivative in the ball  $B(x_0, r)$ ,  $f'(x_0)^{-1}$  exists and  $f'(x_0)^{-1}f'$  satisfies the center Lipschitz condition with the  $L$  average,

$$(1.1) \quad \|f'(x_0)^{-1}f'(x) - \mathbf{I}\| \leq \int_0^{\rho(x)} L(u)du, \quad \forall x \in B(x_0, r),$$

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where  $\rho(x) = \|x - x_0\|$  and  $L$  is a positive integrable function in the interval  $(0, r)$ . By Banach's theorem, when  $r_0 \leq r$ , for all  $x \in B(x_0, r_0)$ ,  $f'(x)^{-1}$  exists and

$$(1.2) \quad \|f'(x)^{-1}f'(x_0)\| \leq \frac{1}{1 - \int_0^{\rho(x)} L(u)du},$$

where  $r_0$  satisfies

$$(1.3) \quad \int_0^{r_0} L(u)du = 1.$$

**Theorem 1.1.** *Suppose that  $r \geq r_0$  and  $b = \int_0^{r_0} L(u)udu$ . Then under the hypothesis of condition (1.1), we have*

$$(1.4) \quad B(f(x_0), b/\|f'(x_0)^{-1}\|) \subset f(B(x_0, r_0)),$$

and in the open ball on the left,  $f_{x_0}^{-1}$  exists and is differentiable. Moreover, the radius of the ball is the best possible.

**Lemma 1.2.** *Let*

$$(1.5) \quad h(t) = \beta - t + \int_0^t L(u)(t-u)du, \quad 0 \leq t \leq R,$$

where  $R$  satisfies

$$(1.6) \quad \frac{1}{R} \int_0^R L(u)(R-u)du = 1.$$

Then when  $0 < \beta < b$ ,  $h$  is decreasing monotonically in  $[0, r_0]$ , while it is increasing monotonically in  $[r_0, R]$  and

$$h(\beta) > 0, \quad h(r_0) = \beta - b < 0, \quad h(R) = \beta > 0.$$

Moreover,  $h$  has a unique zero in each interval, denoted by  $r_1$  and  $r_2$ . They satisfy

$$(1.7) \quad \beta < r_1 < \frac{r_0}{b}\beta < r_0 < r_2 < R.$$

*Proof.* It is obvious by the sign of  $h'(t) = -1 + \int_0^t L(u)du$  that  $h(t)$  is piecewise monotone. By the positivity of  $L$ , we see that  $\varphi(t) := \frac{1}{t} \int_0^t L(u)(t-u)du$  is increasing monotonically with respect to  $t$ . In fact, for  $0 < t_1 < t_2$ ,

$$\begin{aligned} \varphi(t_2) - \varphi(t_1) &= \int_{t_1}^{t_2} L(u)du - \left( \frac{1}{t_2} \int_{t_1}^{t_2} + \left( \frac{1}{t_2} - \frac{1}{t_1} \right) \int_0^{t_1} \right) L(u)udu \\ &\geq \int_{t_1}^{t_2} L(u)du - \int_{t_1}^{t_2} L(u)du - \left( \frac{1}{t_2} - \frac{1}{t_1} \right) \int_0^{t_1} L(u)udu \\ &= \left( \frac{1}{t_1} - \frac{1}{t_2} \right) \int_0^{t_1} L(u)udu > 0. \end{aligned}$$

Thus we have

$$\begin{aligned} \beta < r_1 &= h(r_1) + r_1 = \beta + \varphi(r_1)r_1 < \beta + \varphi(r_0)r_1 \\ &= \beta + \frac{1}{r_0} \int_0^{r_0} L(u)(r_0-u)du \cdot r_1 = \beta + r_1 - \frac{b}{r_0}r_1. \quad \square \end{aligned}$$

By this lemma, Theorem 1.1 implies a more precise proposition, as follows. For this purpose, we assume the inequality (1.1) can be extended to the boundary, i.e.

$$(1.1') \quad \|f'(x_0)^{-1}f'(x) - \mathbf{I}\| \leq \int_0^{\rho(x)} L(u)du, \quad \forall x \in \overline{\mathbf{B}(x_0, r)}.$$

**Proposition 1.3.** *Suppose that  $r \geq r_1$  and  $0 < \beta < b = \int_0^{r_0} L(u)du$ , where  $r_1$  is determined by Lemma 1.2. Then, under the hypothesis of the condition (1.1'),*

$$(1.8) \quad \overline{\mathbf{B}(f(x_0), \beta/\|f'(x_0)^{-1}\|)} \subset f(\overline{\mathbf{B}(x_0, r_1)}),$$

and in the closed ball on the left,  $f_{x_0}^{-1}$  exists, is differentiable, and its derivative  $(f_{x_0}^{-1})'(y) = f'(x)^{-1}$  at  $y = f(x)$  satisfies (1.2). Moreover, as a closed ball of the image, the radius  $r_1$  is as small as possible.

*Proof.* Arbitrarily choosing

$$(1.9) \quad y \in \overline{\mathbf{B}(f(x_0), \beta/\|f'(x_0)^{-1}\|)},$$

we consider two sequences  $\{x_n\} \subset \mathbf{X}$  and  $\{t_n\} \subset \mathbf{R}$ , respectively given by

$$(1.10) \quad x_{n+1} = x_n - f'(x_0)^{-1}(f(x_n) - y), \quad n = 0, 1, \dots,$$

and

$$(1.11) \quad t_{n+1} = t_n + h(t_n), \quad t_0 = 0, \quad n = 0, 1, \dots.$$

First, by the fact that  $h(t) + t$  increases monotonically with respect to  $t$  and  $t_0 = 0 < t_1 = \beta < r_1$ , we inductively find that  $\{t_n\}$  increases monotonically and is less than  $r_1$ . Thus  $\{t_n\}$  converges to  $r_1$ .

Then, by induction, for all  $n$  we will prove that

$$(1.12) \quad \|x_{n+1} - x_n\| \leq t_{n+1} - t_n.$$

By (1.9) and (1.7),

$$\|x_1 - x_0\| \leq \|f'(x_0)^{-1}\| \cdot \|f(x_0) - y\| \leq \beta = t_1 - t_0.$$

This means (1.12) is true for  $n = 0$ . Suppose that (1.12) is valid until some  $n - 1$ . For  $0 \leq \tau \leq 1$ , let

$$(1.13) \quad \begin{aligned} x_{n-1+\tau} &= x_{n-1} + \tau(x_n - x_{n-1}), \\ t_{n-1+\tau} &= t_{n-1} + \tau(t_n - t_{n-1}). \end{aligned}$$

We have

$$\begin{aligned} \|x_{n-1+\tau} - x_0\| &\leq \|x_1 - x_0\| + \dots + \|x_{n-1} - x_{n-2}\| + \tau\|x_n - x_{n-1}\| \\ &\leq (t_1 - t_0) + \dots + (t_{n-1} - t_{n-2}) + \tau(t_n - t_{n-1}) \\ &= t_{n-1+\tau} < r_1 \leq r. \end{aligned}$$

Thus, by virtue of the equality

$$\begin{aligned} x_{n+1} - x_n &= -f'(x_0)^{-1}(f(x_n) - f(x_{n-1}) - f'(x_0)(x_n - x_{n-1})) \\ &= -\int_0^1 (f'(x_0)^{-1}f'(x_{n-1+\tau}) - \mathbf{I})(x_n - x_{n-1})d\tau \end{aligned}$$

and (1.1), we obtain

$$\begin{aligned}
\|x_{n+1} - x_n\| &\leq \int_0^1 \|f'(x_0)^{-1}f'(x_{n-1+\tau}) - I\| \|x_n - x_{n-1}\| d\tau \\
&\leq \int_0^1 \int_0^{\rho(x_{n-1+\tau})} L(u) du \|x_n - x_{n-1}\| d\tau \\
&\leq \int_0^1 \int_0^{t_{n-1+\tau}} L(u) du (t_n - t_{n-1}) d\tau \\
&= \int_0^{t_n} L(u)(t_n - u) du - \int_0^{t_{n-1}} L(u)(t_{n-1} - u) du \\
&= t_{n+1} - t_n.
\end{aligned}$$

This indicates that (1.12) is valid for all  $n$ .

The inequality (1.12) above shows that the sequence  $\{x_n\}$  is self-convergent and so is convergent. Taking the limit on both sides in (1.10), we see that  $x = \lim x_n$  satisfies

$$(1.14) \quad f(x) = y.$$

Also, since  $\|x_n - x_0\| \leq r_1$ , we have

$$(1.15) \quad x = f_{x_0}^{-1}(y) \in \overline{B(x_0, r_1)}.$$

For this reason we have to prove  $x$  satisfying (1.14) is unique in the closed ball. This will be given together with the proof of the next proposition. Finally, the differentiability of the inverse function follows by (1.2).  $\square$

*Remark.* Except for the differentiability of the inverse function, the proposition is also true for  $\beta = b$ .

Besides Proposition 1.3, we have the following proposition, which is called the branch separation theorem

**Proposition 1.4.** *Suppose that  $r_1 \leq r < r_2$  and  $0 < \beta < b$ , where  $r_1, r_2$  and  $b$  are determined by Lemma 1.2 and Theorem 1.1. Then, under the condition (1.1'),*

$$(1.16) \quad \overline{B(f(x_0), \beta/\|f'(x_0)^{-1}\|)} \cap f(B(x_0, r) \setminus \overline{B(x_0, r_1)}) = \emptyset.$$

*Proof.* Arbitrarily choose

$$(1.17) \quad y \in B(f(x_0), \beta/\|f'(x_0)^{-1}\|), \quad x'_0 \in B(x_0, r).$$

Let

$$(1.18) \quad x'_{n+1} = x'_n - f'(x_0)^{-1}(f(x'_n) - y), \quad n = 0, 1, \dots,$$

$$(1.19) \quad t'_{n+1} = t'_n + h(t'_n), \quad t'_0 = \|x'_0 - x_0\|, \quad n = 0, 1, \dots.$$

Since

$$(1.20) \quad x'_{n+1} - x_{n+1} = - \int_0^1 (f'(x_0)^{-1}f'(x_n + \tau(x'_n - x_n)) - I) (x'_n - x_n) d\tau,$$

we can prove that

$$(1.21) \quad \|x'_n - x_n\| \leq t'_n - t_n, \quad n = 0, 1, \dots.$$

Hence,  $\{x'_n\}$  is also convergent to  $x = \lim x_n$ . Therefore, there is only one  $x \in \overline{B(x_0, r_1)}$  in the open ball  $B(x_0, r)$  that satisfies (1.14).  $\square$

*Remark.* The proof of Propositions 1.3 and 1.4 may be viewed as the proof of the existence and uniqueness theorems about the solution of the equation  $f(x) = y$ , and the premise (1.9) and (1.17) can be replaced by

$$(1.22) \quad \|f'(x_0)^{-1}(f(x_0) - y)\| \leq \beta.$$

Hence, setting  $y = 0$  and  $\beta = \|f'(x_0)^{-1}f(x_0)\|$ , we have

**Theorem 1.5.** *Let  $\beta = \|f'(x_0)^{-1}f(x_0)\| \leq b$ . Assume that  $r_1 \leq r < r_2$  if  $\beta < b$ , or  $r = r_1$  if  $\beta = b$ , where  $r_1, r_2$  and  $b$  are determined by Lemma 1.2 and Theorem 1.1. Then, under the conditions (1.1'), the equation (0.2) has a unique solution*

$$(1.23) \quad x^* \in \overline{B(x_0 - f'(x_0)^{-1}f(x_0), r_1 - \beta)} \subset \overline{B(x_0, r_1)}$$

in the closed ball  $\overline{B(x_0, r)}$ .

## 2. FURTHER DISCUSSION OF LIPSCHITZ CONDITIONS

In the ball  $B(x_0, r)$ , the Lipschitz condition with the constant  $L$  is

$$(2.1) \quad \|f(x) - f(x')\| \leq L\|x - x'\|,$$

where  $x, x' \in B(x_0, r)$ . If (2.1) is only true for all  $x \in B(x_0, r)$  and  $x' = x_0$ , then it is called the center Lipschitz condition in [1]; if (2.1) is valid for all  $x' \in B(x_0, r)$  and for all  $x = x_0 + \tau(x' - x_0)$  ( $0 \leq \tau \leq 1$ ), then it is called the radius Lipschitz condition. Now, if (2.1) is valid for all  $x \in B(x_0, r)$  and for all  $x' \in B(x, r - \rho(x))$ , then we call it the center Lipschitz condition in the inscribed sphere. For a constant or positive integrable function  $L$ , among all Lipschitz conditions with the constant  $L$  or the average of  $L$ , there is an implication relation  $\succ$  as follows:

Lipschitz condition in a ball

- $\succ$  the center Lipschitz condition in the inscribed sphere
- $\succ$  the radius Lipschitz condition
- $\succ$  the center Lipschitz condition.

Custom is the only reason why we give the names of different Lipschitz conditions. It is not necessary in essence; see Theorems 6.3 and 6.4 in [1]. Sometimes, however, we have to pay attention to such accustomed thinking because it determines the development of the literature.

## 3. CONVERGENCE OF NEWTON'S METHOD

Suppose that  $f$  has a continuous derivative in the closed ball  $\overline{B(x_0, r)}$ ,  $f'(x_0)^{-1}$  exists and  $f'(x_0)^{-1}f'$  satisfies the center Lipschitz condition in the inscribed sphere with the average of  $L$ ,

$$(3.1) \quad \|f'(x_0)^{-1}(f'(x) - f'(x'))\| \leq \int_{\rho(x)}^{\rho(\overline{xx'})} L(u)du,$$

$$\forall x \in B(x_0, r), \quad \forall x' \in \overline{\rho m B(x, r - \rho(x))},$$

where  $\rho(x) = \|x - x_0\|$ ,  $\rho(\overline{xx'}) = \rho(x) + \|x' - x\| \leq r$ , and  $L$  is a positive nondecreasing function in  $[0, r]$ . Under this hypothesis, the conditions (1.1) and (1.1') are of course satisfied, and thus Theorems 1.1 and 1.5, Propositions 1.3 and 1.4 hold.

**Theorem 3.1.** Assume that  $\beta = \|f'(x_0)^{-1}f(x_0)\| \leq b$  and  $r \geq r_1$ , where  $b$  and  $r_1$  are determined by Theorem 1.1 and Lemma 1.2. Then, under the condition (3.1), Newton's method (0.1) is defined for all  $n$  and converges to a solution  $x^*$  of equation (0.2),

$$(3.2) \quad x^* \in \overline{\mathbb{B}(x_1, r_1 - \beta)} \subset \overline{\mathbb{B}(x_0, r_1)}.$$

Moreover, for all  $n \geq n_0 \geq 0$ , the best possible error bounds

$$(3.3) \quad \|x^* - x_n\| \leq (r_1 - t_n) \left( \frac{\|x^* - x_{n_0}\|}{r_1 - t_{n_0}} \right)^{2^{n-n_0}}$$

and

$$(3.4) \quad \frac{2\|x_{n+1} - x_n\|}{1 + \sqrt{1 + 4\frac{r_1 - t_{n+1}}{(r_1 - t_n)^2}(t_{n+1} - t_n)}} \leq \|x^* - x_n\| \leq (r_1 - t_n) \left( \frac{\|x_{n_0+1} - x_{n_0}\|}{t_{n_0+1} - t_{n_0}} \right)^{2^{n-n_0}}$$

are valid with

$$(3.5) \quad t_{n+1} = t_n - \frac{h(t_n)}{h'(t_n)}, \quad t_0 = 0, \quad n = 0, 1, \dots$$

In order to prove Theorem 3.1, we need

**Proposition 3.2.** Under the assumptions of Theorem 3.1, for any natural number  $n \geq 1$ , we have

$$(3.6) \quad \|x_n - x_{n-1}\| \leq t_n - t_{n-1},$$

$$(3.7) \quad \|f'(x_0)^{-1}f(x_n)\| \leq h(t_n) \left( \frac{\|x_n - x_{n-1}\|}{t_n - t_{n-1}} \right)^2,$$

$$(3.8) \quad \frac{\|f'(x_0)^{-1}f(x_n)\|}{\|f'(x_0)^{-1}f(x_{n-1})\|} \leq \frac{h(t_n)}{h(t_{n-1})} \cdot \frac{\|x_n - x_{n-1}\|}{t_n - t_{n-1}},$$

and

$$(3.9) \quad \|x_{n+1} - x_n\| \leq (t_{n+1} - t_n) \left( \frac{\|x_n - x_{n-1}\|}{t_n - t_{n-1}} \right)^2.$$

*Proof.* By the hypotheses, (3.6) is true for  $n = 1$ . Now assume that it holds for some  $n \geq 1$ . Then

$$(3.10) \quad x_n \in \mathbb{B}(x^*, t_n) \subset \mathbb{B}(x^*, r_1).$$

Since

$$\begin{aligned} f(x_n) &= f(x_n) - f(x_{n-1}) - f'(x_{n-1})(x_n - x_{n-1}) \\ &= \int_0^1 (f'(x_{n-1+\tau}) - f'(x_{n-1}))(x_n - x_{n-1})d\tau, \end{aligned}$$

where

$$x_{n-1+\tau} = x_{n-1} + \tau(x_n - x_{n-1}), \quad 0 \leq \tau \leq 1,$$

we obtain

$$\|f'(x_0)^{-1}f(x_n)\| \leq \int_0^1 \|f'(x_0)^{-1}(f'(x_{n-1+\tau}) - f'(x_{n-1}))\| \cdot \|x_n - x_{n-1}\|d\tau.$$

By the hypothesis (3.1), we have

$$\begin{aligned} \|f'(x_0)^{-1}f(x_n)\| &\leq \int_0^1 \int_{\rho(x_{n-1})}^{\rho(\overline{x_{n-1}x_{n-1+\tau}})} L(u)du \|x_n - x_{n-1}\| d\tau \\ &= \int_0^{\|x_n - x_{n-1}\|} L(\|x_{n-1} - x_0\| + u)(\|x_n - x_{n-1}\| - u)du. \end{aligned}$$

Since  $L$  is a nondecreasing function,  $\varphi(t) := \frac{1}{t^2} \int_0^t L(\rho+u)(t-u)du$  is nondecreasing with respect to  $t$  in  $[0, r - \rho]$ . In fact, when  $0 < t_1 < t_2 \leq r - \rho$ , we have

$$\begin{aligned} \varphi(t_2) - \varphi(t_1) &= \left(\frac{1}{t_1^2} - \frac{1}{t_2^2}\right) \int_0^{\frac{t_1}{2}} \left(L\left(\rho + \frac{t_1}{2} + u\right) - L\left(\rho + \frac{t_1}{2} - u\right)\right) udu \\ &\quad + \frac{(t_2 - t_1)^2}{2t_2^2 t_1} \int_0^{t_1} (L(\rho + t_1) - L(\rho + u))du \\ &\quad + \frac{1}{t_2^2} \int_{t_1}^{t_2} (L(\rho + u) - L(\rho + t_1))(t_2 - u)du \geq 0. \end{aligned}$$

Hence

$$\begin{aligned} \|f'(x_0)^{-1}f(x_n)\| &\leq \frac{1}{\|x_n - x_{n-1}\|^2} \int_0^{\|x_n - x_{n-1}\|} L(\|x_{n-1} - x_0\| + u)(\|x_n - x_{n-1}\| - u)du \\ &\quad \cdot \|x_n - x_{n-1}\|^2 \\ &\leq \frac{1}{(t_n - t_{n-1})^2} \int_0^{t_n - t_{n-1}} L(t_{n-1} + u)(t_n - t_{n-1} - u)du \|x_n - x_{n-1}\|^2 \\ &= h(t_n) \left(\frac{\|x_n - x_{n-1}\|}{t_n - t_{n-1}}\right)^2, \end{aligned}$$

where we have used the inductive hypothesis (3.6). Therefore, (3.7) holds for all  $n$ , which makes (3.6) hold.

Since

$$(3.11) \quad \|x_n - x_{n-1}\| \leq \|f'(x_{n-1})^{-1}f'(x_0)\| \cdot \|f'(x_0)^{-1}f(x_{n-1})\|,$$

we obtain

$$(3.12) \quad \frac{\|f'(x_0)^{-1}f(x_n)\|}{\|f'(x_0)^{-1}f(x_{n-1})\|} \leq h(t_n) \frac{\|x_n - x_{n-1}\|}{(t_n - t_{n-1})^2} \|f'(x_{n-1})^{-1}f'(x_0)\|.$$

By (1.2) and (1.5) we have

$$(3.13) \quad \|f'(x_{n-1})^{-1}f'(x_0)\| \leq \frac{1}{1 - \int_0^{\|x_{n-1} - x_0\|} L(u)du} \leq \frac{1}{h'(t_{n-1})}.$$

Combining (3.12) and (3.13) and using (3.5), we get that (3.8) is also true if (3.6) is true for some  $n$ .

Increasing  $n$  to  $n+1$  in (3.11) and (3.13) and applying (3.7) and (3.13) to (3.11), we get (3.9).

So (3.6) can be continued, and (3.6)-(3.9) hold for all  $n \geq 1$ .  $\square$

*Proof of Theorem 3.1.* Obviously,  $\{t_n\}$  is convergent to  $r_1$  monotonically. Therefore, the sequence  $\{x_n\} \subset B(x_0, r_1)$  converges. Also by (1.1),  $\|f'(x_n)\|$  is bounded uniformly. So from

$$(3.14) \quad f(x_n) + f'(x_n)(x_{n+1} - x_n) = 0$$

we get  $\lim x_n = x^*$ .

Finally, by (3.1) and

$$\begin{aligned} e_n &:= f'(x_0)^{-1}(f(x^*) - f(x_n) - f'(x_n)(x^* - x_n)) \\ &= \int_0^1 f'(x_0)^{-1}(f'(z_\tau) - f'(x_n))(x^* - x_n)d\tau, \\ z_\tau &:= x_n + \tau(x^* - x_n) \end{aligned}$$

we obtain

$$\begin{aligned} \|e_n\| &\leq \int_0^1 \int_{\rho(x_n)}^{\rho(\overline{x_n z_\tau})} L(u)du \|x^* - x_n\| d\tau \\ &= \int_0^{\|x^* - x_n\|} L(\|x_n - x_0\| + u)(\|x^* - x_n\| - u)du. \end{aligned}$$

Since  $\frac{1}{t^2} \int_0^t L(\rho + u)(t - u)du$  is nondecreasing with respect to  $t$ , we have

$$\begin{aligned} \|e_n\| &\leq \frac{1}{\|x^* - x_n\|^2} \int_0^{\|x^* - x_n\|} L(\|x_n - x_0\| + u)(\|x^* - x_n\| - u)du \\ &\quad \cdot \|x^* - x_n\|^2 \\ &\leq \frac{1}{(r_1 - t_n)^2} \int_0^{r_1 - t_n} L(t_n + u)(r_1 - t_n - u)du \|x^* - x_n\|^2 \\ &= \int_{t_n}^{r_1} L(u)(r_1 - u)du \left( \frac{\|x^* - x_n\|}{r_1 - t_n} \right)^2. \end{aligned}$$

Therefore,

$$\begin{aligned} \|x^* - x_{n+1}\| &\leq \|f'(x_n)^{-1}f'(x_0)\| \cdot \|e_n\| \\ &\leq \frac{\int_{t_n}^{r_1} L(u)(r_1 - u)du}{1 - \int_0^{t_n} L(u)du} \left( \frac{\|x^* - x_n\|}{r_1 - t_n} \right)^2. \end{aligned}$$

By the induction method, (3.3) follows.

By (3.9), for all  $i \geq 0$  and  $n \geq n_0 \geq 0$ , we have

$$\|x_{n+i+1} - x_{n+i}\| \leq (t_{n+i+1} - t_{n+i}) \left( \frac{\|x_{n_0+1} - x_{n_0}\|}{t_{n_0+1} - t_{n_0}} \right)^{2^{n-n_0}}.$$

Summing for all  $i \geq 0$  results in the upper bound (3.4). It follows from (3.3) that

$$\|x_{n+1} - x_n\| \leq \|x^* - x_n\| + \|x^* - x_{n+1}\| \leq \|x^* - x_n\| + \frac{r_1 - t_{n+1}}{(r_1 - t_n)^2} \|x^* - x_n\|^2.$$

Then, using Gragg and Tapia [3], we obtain the proof of the lower bound (3.4).  $\square$

4. UNDER THE PREMISE OF A KANTOROVICH TYPE

About the convergence of Newton's method, the main point of Kantorovich [2] type premise is to make

$$(4.1) \quad h(t) = \beta - t + \frac{1}{2}Lt^2, \quad 0 \leq t \leq R,$$

become a majorizing function. For this reason, as  $\|x - x_0\| + \|x' - x\| \leq r$ , it is sufficient to assume that

$$(4.2) \quad \|f'(x_0)^{-1}(f'(x) - f'(x'))\| \leq L\|x - x'\|,$$

for a positive constant  $L$ . As

$$(4.3) \quad \lambda = L\beta \leq \frac{1}{2},$$

corresponding to (1.7), the zeros of  $h$

$$(4.4) \quad \left. \begin{array}{l} r_1 \\ r_2 \end{array} \right\} = \frac{1 \mp \sqrt{1 - 2\lambda}}{L}$$

satisfy

$$(4.5) \quad \beta \leq r_1 \leq 2\beta \leq \frac{1}{L} \leq r_2 \leq \frac{2}{L},$$

because  $r_0 = 1/L, R = 2/L, b = 1/(2L)$  in this case. So Theorems 1.1 and 1.5, Propositions 1.3 and 1.4 all have concrete forms. The concretization of Theorem 3.1 requires that the solution of the sequence (3.4) has a closed form

$$(4.6) \quad t_n = \frac{1 - q^{2^n - 1}}{1 - q^{2^n}} r_1, \quad n = 0, 1, \dots,$$

where

$$(4.7) \quad q = \frac{1 - \sqrt{1 - 2\lambda}}{1 + \sqrt{1 - 2\lambda}}.$$

(4.6) is independently given by [3]-[5].

For instance, the concrete forms of Theorems 1.1, 1.5 and 3.1 are, respectively,

**Theorem 4.1.** *Let  $L$  be a positive constant. Assume that  $f$  satisfies the condition*

$$(4.8) \quad \|f'(x_0)^{-1}f'(x) - I\| \leq L\|x - x_0\|, \quad \forall x \in B(x_0, 1/L).$$

*Then  $f_{x_0}^{-1}$  exists and is differentiable in the open ball*

$$(4.9) \quad B(f(x_0), 1/(2L\|f'(x_0)^{-1}\|)) \subset f(B(x_0, 1/L)).$$

*Moreover, the radius of this ball (the left in (4.9)) is the best possible.*

**Theorem 4.2.** *Let  $L$  be a positive constant,  $\beta = \|f'(x_0)^{-1}f(x_0)\|$  and  $\lambda = L\beta \leq \frac{1}{2}$ . Assume that  $f$  satisfies the condition*

$$(4.10) \quad \|f'(x_0)^{-1}f'(x) - I\| \leq L\|x - x_0\|, \quad \forall x \in \overline{B(x_0, r)},$$

*where  $r_1 \leq r < r_2$  if  $\lambda < \frac{1}{2}$ , or  $r = r_1$  if  $\lambda = \frac{1}{2}$ , while  $r_1$  and  $r_2$  are determined by (4.4). Then the equation (0.2) has a unique solution  $x^*$  satisfying (1.23) in the closed ball  $\overline{B(x_0, r)}$ .*

**Theorem 4.3.** *Let  $L$  be a positive constant,  $\beta = \|f'(x_0)^{-1}f(x_0)\|$  and  $\lambda = L\beta \leq \frac{1}{2}$ . Assume that  $f$  satisfies the condition (4.2). Then Newton's method (0.1) is well defined for all  $n$  and converges to the solution  $x^*$  satisfying (3.2) of the equation (0.2). Moreover, for all  $n \geq 0$ , the best possible error bounds*

$$(4.11) \quad \|x^* - x_n\| \leq \frac{q^{2^n-1}}{\sum_{i=0}^{2^n-1} q^i} \|x^* - x_0\| \leq \frac{q^{2^n-1}}{\sum_{i=0}^{2^{n-1}-1} q^{2i}} \|x_1 - x_0\|$$

and

$$(4.12) \quad \frac{2\|x_{n+1} - x_n\|}{1 + \sqrt{1 + 4q^{2^n}/(1 + q^{2^n})^2}} \leq \|x^* - x_n\| \leq \frac{q}{\beta} \sum_{i=0}^{2^{n-1}-1} q^{2i} \|x_n - x_{n-1}\|^2 \leq q^{2^{n-1}} \|x_n - x_{n-1}\|$$

are valid with (4.7).

*Remark.* It is a posterior estimation to use  $\|x_n - x_{n-1}\|$  to estimate  $\|x^* - x_n\|$ . The posterior estimation in (4.12) can be obtained by setting  $n_0 = n - 1$  in (3.4). In the hypothesis of Kantorovich's type, more precise posterior estimations were studied by Potra [6] and Potra & Ptak [7].

## 5. UNDER A PREMISE OF SMALE TYPE

Under the hypotheses that  $f$  is analytic and satisfies

$$(5.1) \quad \left\| f'(x_0)^{-1} f^{(n)}(x_0) \right\| \leq n! \gamma^{n-1}, \quad n \geq 2,$$

Smale [8] studied the convergence and error estimation of Newton's iteration. Wang and Han [9] (also see [10], [11]) completely improved Smale's results by introducing a majorizing function

$$(5.2) \quad h(t) = \beta - t + \frac{\gamma t^2}{1 - \gamma t}, \quad 0 \leq t \leq R.$$

When  $\gamma\|x - x_0\| < 1$ , it is easy to derive from (5.1) that

$$\left\| f'(x_0)^{-1} f''(x) \right\| \leq h''(\|x - x_0\|) = \frac{2\gamma}{(1 - \gamma\|x - x_0\|)^3}$$

(see Lemma 3 in [12] or Lemma 3.5 in [13]). Hence, conditions (1.1) and (3.1) are satisfied for the function  $L$  defined by

$$(5.3) \quad L(u) = \frac{2\gamma}{(1 - \gamma u)^3}.$$

Furthermore, for this  $L$ , the function  $h$  given in (1.5) coincides with the one in (5.2).

As  $\alpha = \gamma\beta \leq 3 - 2\sqrt{2}$ , corresponding to (1.7), the zeros of  $h$

$$(5.4) \quad \left. \begin{array}{l} r_1 \\ r_2 \end{array} \right\} = \frac{1 + \alpha \mp \sqrt{(1 + \alpha)^2 - 8\alpha}}{4\gamma}$$

satisfy

$$(5.5) \quad \beta \leq r_1 \leq \left(1 + \frac{1}{\sqrt{2}}\right)\beta \leq \left(1 - \frac{1}{\sqrt{2}}\right)\frac{1}{\gamma} \leq r_2 \leq \frac{1}{2\gamma},$$

because  $r_0 = (1 - \frac{1}{\sqrt{2}})\frac{1}{\gamma}$ ,  $R = \frac{1}{2\gamma}$ ,  $b = (3 - 2\sqrt{2})\frac{1}{\gamma}$  in this case. So Theorems 1.1 and 1.5, Propositions 1.3 and 1.4 all have concrete forms. The concretization of Theorem 3.1 requires that the solution of the sequence (3.4) has a closed form

$$(5.6) \quad t_n = \frac{1 - q^{2^n - 1}}{1 - q^{2^n - 1}\eta} r_1, \quad n = 0, 1, \dots,$$

where

$$(5.7) \quad q = \frac{1 - \alpha - \sqrt{(1 + \alpha)^2 - 8\alpha}}{1 - \alpha + \sqrt{(1 + \alpha)^2 - 8\alpha}}, \quad \eta = \frac{1 + \alpha - \sqrt{(1 + \alpha)^2 - 8\alpha}}{1 + \alpha + \sqrt{(1 + \alpha)^2 - 8\alpha}}.$$

For instance, the concrete forms of Theorems 1.1, 1.5 and 3.1 are, respectively,

**Theorem 5.1.** *Let  $\gamma$  be a positive constant. Assume that  $f$  satisfies the condition*

$$(5.8) \quad \|f'(x_0)^{-1}f'(x) - I\| \leq \frac{1}{(1 - \gamma\|x - x_0\|)^2} - 1, \\ \forall x \in B(x_0, (1 - \frac{1}{\sqrt{2}})/\gamma).$$

*Then  $f_{x_0}^{-1}$  exists and is differentiable in the open ball*

$$(5.9) \quad B(f(x_0), (3 - 2\sqrt{2})/(\gamma\|f'(x_0)^{-1}\|)) \subset f(B(x_0, (1 - \frac{1}{\sqrt{2}})/\gamma)).$$

*Moreover, the radius of this ball (the left in (5.9)) is the best possible.*

**Theorem 5.2.** *Let  $\gamma$  be a positive constant,  $\beta = \|f'(x_0)^{-1}f(x_0)\|$  and  $\alpha = \beta\gamma \leq 3 - 2\sqrt{2}$ . Assume that  $f$  satisfies the condition*

$$(5.10) \quad \|f'(x_0)^{-1}f'(x) - I\| \leq \frac{1}{(1 - \gamma\|x - x_0\|)^2} - 1, \quad \forall x \in \overline{B(x_0, r)},$$

*where  $r_1 \leq r < r_2$  if  $\alpha < 3 - 2\sqrt{2}$ , or  $r = r_1$  if  $\alpha = 3 - 2\sqrt{2}$ , while  $r_1$  and  $r_2$  are determined by (5.4). Then the equation (0.2) has a unique solution  $x^*$  satisfying (1.23) in the closed ball  $\overline{B(x_0, r)}$ .*

**Theorem 5.3.** *Let  $\gamma$  be a positive constant,  $\beta = \|f'(x_0)^{-1}f(x_0)\|$  and  $\alpha = \beta\gamma \leq 3 - 2\sqrt{2}$ . Assume that  $f$  satisfies the condition*

$$(5.11) \quad \|f'(x_0)^{-1}(f'(x) - f'(x'))\| \leq \frac{1}{(1 - \gamma\|x - x_0\| - \gamma\|x' - x_0\|)^2} - \frac{1}{(1 - \gamma\|x - x_0\|)^2}, \\ \|x - x_0\| + \|x' - x_0\| \leq r.$$

*Then Newton's method (0.1) is well defined for all  $n$  and converges to the solution  $x^*$  satisfying (3.2) of the equation (0.2). Moreover, for all  $n \geq 0$ , the best possible error bounds*

$$(5.12) \quad \|x^* - x_n\| \leq \frac{1 - \eta}{1 - q^{2^n - 1}\eta} q^{2^n - 1} \|x^* - x_0\| \\ \leq \frac{1 - \eta^2}{(1 + \alpha)(1 - q^{2^n - 1}\eta)} q^{2^n - 1} \|x_1 - x_0\|$$

and

(5.13)

$$\begin{aligned} & \frac{2\|x_{n+1} - x_n\|}{1 + \sqrt{1 + 4q^{2^n} \frac{(1 - q^{2^n})(1 - q^{2^n-1}\eta)}{(1 - q^{2^{n+1}-1}\eta)^2}}} \\ & \leq \|x^* - x_n\| \leq \frac{q(1 - q^{2^n-1}\eta)}{r_1(1 - \eta)} \left( \frac{1 - q^{2^{n-1}-1}\eta}{1 - q^{2^{n-1}}} \right)^2 \|x_n - x_{n-1}\|^2 \\ & \leq q^{2^{n-1}} \frac{1 - q^{2^{n-1}-1}\eta}{1 - q^{2^{n-1}}} \|x_n - x_{n-1}\| \end{aligned}$$

are valid with (5.7).

The results above can be made more general by replacing (5.3). Now we take

$$(5.3') \quad L(u) = \frac{2c\gamma}{(1 - \gamma u)^3},$$

where  $c$  is a positive number. In this case the majorizing function is

$$(5.2') \quad h(t) = \beta - t + \frac{c\gamma t^2}{1 - \gamma t},$$

and its zeros are

$$(5.4') \quad \left. \begin{array}{l} r_1 \\ r_2 \end{array} \right\} = \frac{1 + \alpha \mp \sqrt{(1 + \alpha)^2 - 4(1 + c)\alpha}}{2(1 + c)\gamma}.$$

They satisfy

$$(5.5') \quad \beta \leq r_1 \leq (1 + \sqrt{\frac{c}{c+1}})\beta \leq (1 - \sqrt{\frac{c}{c+1}})\frac{1}{\gamma} \leq r_2 \leq \frac{1}{(c+1)\gamma},$$

because  $r_0 = (1 - \sqrt{\frac{c}{c+1}})\frac{1}{\gamma}$ ,  $R = \frac{1}{(c+1)\gamma}$ ,  $b = (1 + 2c - 2\sqrt{c(c+1)})\frac{1}{\gamma}$ . Hence, we have

**Theorem 5.3'.** *Let  $\gamma$  and  $c$  be positive constants,  $\beta = \|f'(x_0)^{-1}f(x_0)\|$  and  $\alpha = \beta\gamma \leq 1 + 2c - 2\sqrt{c(c+1)}$ . Assume that  $f$  satisfies the condition*

(5.11')

$$\|f'(x_0)^{-1}(f'(x) - f'(x'))\| \leq \frac{c}{(1 - \gamma\|x - x_0\| - \gamma\|x' - x\|)^2} - \frac{c}{(1 - \gamma\|x - x_0\|)^2},$$

$$\|x - x_0\| + \|x' - x\| \leq r.$$

Then Newton's method (0.1) is well defined for all  $n$  and converges to the solution  $x^*$  satisfying (3.2) of the equation (0.2). Moreover, for all  $n \geq 0$ , the best possible error bounds (5.12) and (5.13) are valid with

(5.7')

$$q = \frac{1 - \alpha - \sqrt{(1 + \alpha)^2 - 4(1 + c)\alpha}}{1 - \alpha + \sqrt{(1 + \alpha)^2 - 4(1 + c)\alpha}}, \quad \eta = \frac{1 + \alpha - \sqrt{(1 + \alpha)^2 - 4(1 + c)\alpha}}{1 + \alpha + \sqrt{(1 + \alpha)^2 - 4(1 + c)\alpha}}.$$

6. UNDER THE PREMISE OF ANALYTICITY

We come back to the analytic premise about  $f$ , to see what stronger conclusion can be obtained. When  $f$  is assumed to be analytic in the ball  $B(x_0, r)$ ,  $f$  can be expanded to a convergent power series

$$(6.1) \quad f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x_0)(x - x_0)^n.$$

If we suppose

$$(6.2) \quad \left\| f'(x_0)^{-1} f^{(n)}(x_0) \right\| \leq \gamma_n, \quad n \geq 2,$$

and write

$$(6.3) \quad g(t) = \sum_{n=2}^{\infty} \frac{\gamma_n}{n!} t^n,$$

where the sequence  $\gamma_n$  satisfies

$$(6.4) \quad \limsup \sqrt[n]{\frac{\gamma_n}{n!}} \leq \frac{1}{r},$$

then  $f'(x_0)^{-1} f'$  satisfies the Lipschitz condition about  $g''$  in  $B(x_0, r)$ . Thus, Theorem 1.1 asserts that  $f_{x_0}^{-1}$  exists in  $B(f(x_0), b/\|f'(x_0)^{-1}\|)$  and is analytic, where

$$(6.5) \quad b = \int_0^{r_0} g''(u)u du = r_0 - g(r_0),$$

and  $r_0$  satisfies

$$(6.6) \quad \int_0^{r_0} g''(u)du = g'(r_0) = 1.$$

So, we have

**Theorem 6.1.** *Assume that  $f$  is analytic in the ball  $B(x_0, r)$  and  $r \geq r_0$ . If*

$$(6.7) \quad \|y - f(x_0)\| < \frac{b}{\|f'(x_0)^{-1}\|},$$

*then the Euler series*

$$(6.8) \quad x = x_0 + \sum_{n=1}^{\infty} \frac{1}{n!} \left( \frac{d}{dy} \right)^n f_{x_0}^{-1}(y)_{y=f(x_0)} (y - f(x_0))^n$$

*converges, and the constant  $b$  in the right of (6.7), which is determined by (6.5), is the best possible.*

When  $\mathbf{X} = \mathbf{Y} = \mathbf{C}$ , we have

**Theorem 6.2.** *Assume that  $f$  and  $F$  are analytic in the open ball  $B(x_0, r) \subset \mathbf{C}$  and  $r \geq r_0$ . Then the convergence radius,  $R(F \circ f_{x_0}^{-1})$ , of the Lagrange series*

$$(6.9) \quad \begin{aligned} & F(f_{x_0}^{-1}(y)) \\ &= F(x_0) + \sum_{n=1}^{\infty} \frac{1}{n!} \left( \frac{d}{dx} \right)^{n-1} \left( F'(x) \left( \frac{x - x_0}{f(x) - f(x_0)} \right)^n \right)_{x=x_0} (y - f(x_0))^n \end{aligned}$$

has an exact lower bound

$$(6.10) \quad R(F \circ f_{x_0}^{-1}) \geq b|f'(x_0)|.$$

It is a very technical thing to choose the sequence  $\{\gamma_n\}$  or the function  $g$  such that it can give a bound of the different Taylor coefficients of  $f$  and be convenient to give the values of the parameters  $b$  and  $r_0$  in Theorems 6.1 and 6.2. In this paper we propose to choose a different generating function  $G$  of the unit sequence  $\{1, 1, \dots\}$  with the positive constants  $\gamma$  and  $c$ , and then the function  $g$  can be obtained by

$$g(t) = \frac{c}{G'(0)\gamma} \{G(\gamma t) - G'(0)\gamma t - G(0)\}.$$

**Example 1** (Exponential type). Taking  $G(t) = e^t$  as the exponential generating function of the unit sequence, we have

$$g(t) = \frac{c}{\gamma}(e^{\gamma t} - \gamma t - 1).$$

Under the condition

$$\|f'(x_0)^{-1}f^{(n)}(x_0)\| \leq c\gamma^{n-1}, \quad n \geq 2,$$

we obtain that

$$\begin{aligned} \gamma r_0 &= \ln \frac{c+1}{c}, \\ \gamma b &= (c+1) \ln \frac{c+1}{c} - 1. \end{aligned}$$

Especially, as  $c = 1$ , we have

$$\begin{aligned} \gamma r_0 &= \ln 2 = 0.69314\dots, \\ \gamma b &= \ln 4 - 1 = 0.38629\dots. \end{aligned}$$

Theorem 6.1 with the values above has been obtained in [14] by the method of taking the exponential generating function of the number of Schröder system as the majorizing sequence.

**Example 2** (Binomial type). Taking  $G(t) = 1 + \text{sign}(m)\{(1-t)^{-m} - 1\}$  as the binomial generating function of the unit sequence, where  $m > -1$  and  $m \neq 0$  is a real number, we have

$$g(t) = \frac{c}{m\gamma} \{(1-\gamma t)^{-m} - m\gamma t - 1\}.$$

Under the condition

$$\|f'(x_0)^{-1}f^{(n)}(x_0)\| \leq c(m+1)(m+2)\dots(m+n-1)\gamma^{n-1}, \quad n \geq 2,$$

we obtain that

$$\begin{aligned} \gamma r_0 &= 1 - \left(\frac{c}{c+1}\right)^{\frac{1}{m+1}}, \\ \gamma b &= 1 + c \frac{m+1}{m} \left(1 - \left(\frac{c+1}{c}\right)^{\frac{m}{m+1}}\right). \end{aligned}$$

Especially, as  $m = 1$  and  $c = 1$ , we have

$$\begin{aligned} \gamma r_0 &= 1 - \frac{1}{\sqrt{2}} = 0.29289\dots, \\ \gamma b &= 3 - 2\sqrt{2} = 0.17157\dots. \end{aligned}$$

Theorem 6.1 with the values above has been obtained in [13] by the method of taking the normal generating function of the number of blankets added to  $n$  letters as the majorizing sequence.

Also, as  $m = \frac{1}{2}, c = 1$ , we have

$$\begin{aligned}\gamma r_0 &= 1 - \sqrt[3]{\frac{1}{4}} = 0.37003\dots, \\ \gamma b &= 4 - 3\sqrt[3]{2} = 0.22023\dots.\end{aligned}$$

As  $m = -\frac{1}{2}, c = 1$ , we have

$$\begin{aligned}\gamma r_0 &= \frac{3}{4}, \\ \gamma b &= \frac{1}{2}.\end{aligned}$$

The required condition of these simple numbers is not complicated, i.e.

$$\|f'(x_0)^{-1}f^{(n)}(x_0)\| \leq \frac{(2n-3)!!}{2^{n-1}}\gamma^{n-1}, \quad n \geq 2,$$

**Example 3** (The first logarithmic type). Taking  $G(t) = 1 - \ln(1-t)$  as the first logarithmic generating function of the unit sequence, we have

$$g(t) = \frac{c}{\gamma} \ln \frac{1}{1-\gamma t} - ct.$$

Under the condition

$$\|f'(x_0)^{-1}f^{(n)}(x_0)\| \leq c(n-1)!\gamma^{n-1}, \quad n \geq 2,$$

we obtain that

$$\begin{aligned}\gamma r_0 &= \frac{1}{c+1}, \\ \gamma b &= 1 - c \ln \frac{c+1}{c}.\end{aligned}$$

Especially, as  $c = 1$ , we have

$$\begin{aligned}\gamma r_0 &= \frac{1}{2}, \\ \gamma b &= 1 - \ln 2 = 0.30685\dots.\end{aligned}$$

**Example 4** (The second logarithmic type). Taking  $G(t) = 1 + 2t + (1-t)\ln(1-t)$  as the second logarithmic generating function of the unit sequence, we have

$$g(t) = \frac{c}{\gamma}(1-\gamma t)\ln(1-\gamma t) + ct.$$

Under the condition

$$\|f'(x_0)^{-1}f^{(n)}(x_0)\| \leq c(n-2)!\gamma^{n-1}, \quad n \geq 2,$$

we obtain that

$$\begin{aligned}\gamma r_0 &= 1 - e^{-\frac{1}{c}}, \\ \gamma b &= 1 - c + ce^{-\frac{1}{c}}.\end{aligned}$$

Especially, as  $c = 1$ , we have

$$\begin{aligned}\gamma r_0 &= 1 - \frac{1}{e} = 0.63212\dots, \\ \gamma b &= \frac{1}{e} = 0.36787\dots.\end{aligned}$$

7. APPLICATIONS TO SMALE'S  $\alpha$ -THEORY

We continue the discussion of Chapter 7 in [1]. It is well known that Smale [8] first used the criterion

$$(7.1) \quad \alpha(f, x_0) = \gamma \|f'(x_0)^{-1}f(x_0)\|$$

to judge  $x_0$  is an approximate zero of Newton's iteration of  $f$ , where

$$(7.2) \quad \gamma = \sup_{n \geq 2} \left\| \frac{1}{n!} f'(x_0)^{-1} f^{(n)}(x_0) \right\|^{\frac{1}{n-1}}.$$

**Definition 7.1.** Suppose  $x_0 \in D$  is such that Newton's iteration (0.1) is well defined for  $f : D \subset X \rightarrow Y$  and satisfies

$$e(x_n) \leq \left(\frac{1}{2}\right)^{2^{n-1}} e(x_{n-1}),$$

for all positive integers  $n$ , where  $e(x_n)$  denotes some measurement of the approximation degree between  $x_n$  and  $x^*$ . Then  $x_0$  is said to be an approximate zero of  $f$  in the sense of  $e(x_n)$ .

The approximate zero defined in [8] was introduced in the sense of  $\|x_{n+1} - x_n\|$ , while the second kind of approximate zero is defined in the sense of  $\|x^* - x_n\|$ . Now a more reasonable definition for the second kind was introduced in [15]. We find that it is not necessary to introduce the definition of an approximate zero in the sense of  $\|f'(x_0)^{-1}f(x_n)\|$ .

In fact, similarly to Theorem 7.2 in [1], by Theorem 5.3' we have

**Theorem 7.2.** Let  $\gamma, c$  and  $q$  be positive numbers,  $0 < q < 1$ . Assume that  $f$  satisfies the condition

$$(7.3) \quad \begin{aligned} & \|f'(x_0)^{-1}(f'(x) - f'(x'))\| \\ & \leq \frac{c}{(1 - \gamma\|x - x_0\| - \gamma\|x' - x\|)^2} - \frac{c}{(1 - \gamma\|x - x_0\|)^2}, \\ & \gamma\|x - x_0\| + \gamma\|x' - x\| \leq 1 - \sqrt{\frac{c}{c+1}}. \end{aligned}$$

Then, as

$$(7.4) \quad \alpha \leq \frac{2q + c(1+q)^2 - (1+q)\sqrt{c^2(1+q)^2 + 4cq}}{2q}$$

for all natural numbers  $n \geq 1$ , it follows that

$$(7.5) \quad \|x_n - x^*\| \leq q^{2^{n-1}} \|x_{n-1} - x^*\|,$$

$$(7.6) \quad \|x_{n+1} - x_n\| \leq q^{2^{n-1}} \|x_n - x_{n-1}\|,$$

and

$$(7.7) \quad \|f'(x_0)^{-1}f(x_n)\| \leq q^{2^{n-1}} \|f'(x_0)^{-1}f(x_{n-1})\|,$$

where  $x^*$  satisfies  $f(x^*) = 0$ .

*Epecially, as*

$$(7.4a) \quad \alpha(f, x_0) \leq \frac{4 + 9c - 3\sqrt{c(9c+8)}}{4},$$

$x_0$  is an approximate zero of  $f$  in any sense of  $\|x^* - x_n\|, \|x_{n+1} - x_n\|$  or  $\|f'(x_0)^{-1}f(x_n)\|$ .

*Proof.* The representation in the inequality (7.4) at the right side can be obtained from (5.7') by representing  $q$  by  $\alpha$ . Hence, under the hypothesis of (7.4), by Theorem 5.3' and Proposition 3.2, we have

$$\begin{aligned} \|x_n - x^*\| &\leq \frac{r_1 - t_n}{r_1 - t_{n-1}} \|x_{n-1} - x^*\|, \\ \|x_{n+1} - x_n\| &\leq \frac{t_{n+1} - t_n}{t_n - t_{n-1}} \|x_n - x_{n-1}\| \end{aligned}$$

and

$$\|f'(x_0)^{-1}f(x_n)\| \leq \frac{h(t_n)}{h(t_{n-1})} \|f'(x_0)^{-1}f(x_{n-1})\|.$$

Thus, Theorem 7.2 follows from the following lemma. □

**Lemma 7.3.** For (5.2'), (5.4') and (5.7'), if  $\alpha = \beta\gamma \leq 1 = 2c - 2\sqrt{c(c+1)}$ , then

$$(7.8) \quad \frac{r_1 - t_n}{r_1 - t_{n-1}} = \frac{1 - q^{2^{n-1}-1}\eta}{1 - q^{2^n-1}\eta} q^{2^{n-1}} \leq q^{2^{n-1}},$$

$$(7.9) \quad \frac{t_{n+1} - t_n}{t_n - t_{n-1}} = \frac{1 - q^{2^n}}{1 - q^{2^{n-1}}} \cdot \frac{1 - q^{2^{n-1}-1}\eta}{1 - q^{2^{n+1}-1}\eta} q^{2^{n-1}} \leq q^{2^{n-1}}$$

and

$$(7.10) \quad \frac{h(t_n)}{h(t_{n-1})} = \frac{1 - c \left( \frac{1}{(1-\gamma t_n)^2} - 1 \right)}{1 - c \left( \frac{1}{(1-\gamma t_{n-1})^2} - 1 \right)} \cdot \frac{t_{n+1} - t_n}{t_n - t_{n-1}} \leq q^{2^{n-1}}.$$

*Proof.* As (5.2) becomes (5.2'), the representation (5.6) about  $r_1 - t_n$  remains true provided that  $r_1$  and  $r_2$ ,  $q$  and  $\eta$  are determined by (5.4') and (5.7'). Hence, (7.8)-(7.10) follow immediately. □

Finally, similarly to Corollary 7.3 in [1], we have

**Corollary 7.4.** Let  $\gamma$  be a positive number. Assume that  $f'(x_0)^{-1}$  exists,  $f$  is analytic in  $B(x_0, 1/\gamma)$ , and for some  $q \in (0, 1)$

$$(7.11) \quad \left\| \frac{1}{n!} f'(x_0)^{-1} f^{(n)}(x_0) \right\| \leq \left( \frac{1-q}{1+q} \right)^2 \gamma^{n-1}, \quad n \geq 2.$$

Then, as

$$(7.12) \quad \alpha(f, x_0) \leq q,$$

(7.5) holds.

Especially, as

$$(7.11a) \quad \left\| \frac{1}{n!} f'(x_0)^{-1} f^{(n)}(x_0) \right\| \leq \frac{\gamma^{n-1}}{9}, \quad n \geq 2,$$

and

$$(7.12a) \quad \alpha(f, x_0) \leq \frac{1}{2}$$

$x_0$  is an approximate zero of Newton's iteration of  $f$ .

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