

## GAPS BETWEEN INTEGERS WITH THE SAME PRIME FACTORS

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ABSTRACT. We give numerical and theoretical evidence in support of the conjecture of Dressler that between any two positive integers having the same prime factors there is a prime. In particular, it is shown that the abc conjecture implies that the gap between two consecutive such numbers  $a < c$  is  $\gg a^{1/2-\epsilon}$ , and it is shown that this lower bound is best possible. Dressler's conjecture is verified for values of  $a$  and  $c$  up to  $7 \cdot 10^{13}$ .

### 1. INTRODUCTION

We start with the following conjecture of Dressler.

**Conjecture 1.** *Between any two positive integers having the same prime factors there is a prime.*

If the two integers have just one prime factor then the conjecture is a trivial consequence of Bertrand's Postulate. On the other hand, the validity of the conjecture for numbers composed of 2's and 3's implies Bertrand's Postulate. Indeed, for  $n \geq 5$  one can always find positive integers  $i$  and  $j$  such that  $n \leq 2^i 3, 2^j 3^2 < 2n$ . The primary reason for believing the conjecture is evidence, both numerical and theoretical, indicating that the gap between two integers with the same prime factors is relatively large.

**Conjecture 2.** *For any  $\epsilon > 0$  there exists a constant  $C(\epsilon)$  such that if  $a < c$  are positive integers having the same prime factors, then*

$$(1) \quad c - a \geq C(\epsilon) a^{\frac{1}{2} - \epsilon}.$$

It is clear that Conjecture 1 is an easy consequence of Conjecture 2 modulo good information on  $C(\epsilon)$  and on the maximal gap between consecutive primes. In this paper we shall prove that Conjecture 2 in turn is an easy consequence of the abc conjecture.

**Theorem 1.** *The abc conjecture implies Conjecture 2.*

We shall also deduce the following unconditional result as a consequence of a weaker version of the abc conjecture due to Stewart and Yu [9].

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**Theorem 2.** *If  $a < c$  are positive integers having the same prime factors, then*

$$c - a \geq C(\epsilon)(\log c)^{\frac{3}{4}-\epsilon}.$$

If the prime factors of  $a$  and  $c$  are restricted to a fixed finite set  $\mathcal{S}$  of primes, then we have the much stronger lower bound of Tijdeman [10],

$$c - a > \frac{a}{(\log a)^C},$$

with the drawback being that the constant  $C$  depends on the set  $\mathcal{S}$ .

Cramér [4] conjectured that the gap between consecutive primes  $p_n$  and  $p_{n+1}$  is  $O(\log^2 p_n)$ , in fact he made the stronger conjecture that  $\limsup_{n \rightarrow \infty} (p_{n+1} - p_n) / \log^2(p_n) = 1$ . Computer searches have shown that  $p_{n+1} - p_n < \log^2 p_n$  for values of  $p_n$  up to  $7 \times 10^{13}$ ; see Shanks [8], Lander and Parkin [6], Brent [1], and Young and Potler [12]. On the assumption of the Riemann Hypothesis, Cramér proved that there always exists a prime between  $n$  and  $n + O(n^{\frac{1}{2}} \log n)$ . In order to deduce Conjecture 1 from Conjecture 2 one needs gaps of size  $O(n^{\frac{1}{2}-\epsilon})$ , which is somewhere between what one obtains from the Riemann Hypothesis and what Cramér has conjectured. On the other hand, with just a “modest” improvement in Theorem 2, specifically obtaining  $c - a \geq (\log c)^2$ , Conjecture 1 is essentially a consequence of Cramér’s conjecture.

The following example shows that the exponent in (1) cannot be taken to be equal to  $1/2$ . Indeed, we obtain an infinite family of pairs of positive integers  $a < c$  having the same prime factors and satisfying

$$(2) \quad c - a \leq \frac{2\sqrt{2\log 2} a^{1/2}}{(\log a)^{\frac{1}{2}}}.$$

**Example.** Let  $k$  be any positive integer and define  $a_1, c_1$  by

$$a_1 = 2(2^k - 1)^2, \quad c_1 = 2^{k+1}(2^k - 1).$$

Then  $c_1, a_1$  have the same prime divisors and  $c_1 - a_1 = \sqrt{2}a_1^{1/2}$ . Suppose now that  $k = 2 \cdot 3^{j-1}$ , where  $j \geq 2$  is a positive integer. Then we have  $3^j | (2^k - 1)$  and so we can divide  $a_1$  and  $c_1$  by  $3^{j-1}$  and end up with two smaller numbers

$$a = \frac{2(2^k - 1)^2}{3^{j-1}}, \quad c = \frac{2^{k+1}(2^k - 1)}{3^{j-1}}$$

having the same prime factors and satisfying

$$c - a = \frac{\sqrt{2}}{3^{(j-1)/2}} a^{1/2} = \frac{2}{\sqrt{k}} a^{1/2}.$$

Now,

$$\log a = \log 2 + 2 \log(2^k - 1) - (j - 1) \log 3 < 2k \log 2,$$

that is,  $k > \log a / (2 \log 2)$ , and thus we obtain (2). Similar examples may be obtained by dividing out other prime powers or by replacing  $(2^k - 1)$  with  $(2^k + 1)$  or by replacing 2 with any other positive integer  $m > 1$ , but we know of no example where the order of magnitude is less than what we obtain in (2).

If  $a$  and  $c$  have just two prime divisors then we show that the exponent in (1) can be taken to be equal to  $1/2$  on the assumption of the abc conjecture.

**Theorem 3.** *Suppose that  $a < c$  are positive integers having the same two prime divisors. Then, on the assumption of the abc conjecture,  $c - a \gg a^{1/2}$ .*

In Section 3 we use results of de Weger [11] to prove (Theorem 4 in this paper) that the only positive integers  $a < c$  composed of the same two primes  $p, q$  with  $p < q < 200$  and

$$(3) \quad c - a < \sqrt{a}$$

are  $(a, c) = (48, 54) = (2^4 \cdot 3, 2 \cdot 3^3)$ ,  $(a, c) = (1250, 1280) = (2 \cdot 5^4, 2^8 \cdot 5)$  and  $(a, c) = (11859482, 11862016) = (2 \cdot 181^3, 2^{16} \cdot 181)$ . The following is an open question.

*Question 1.* Are there infinitely many pairs  $a < c$  having the same two prime factors satisfying (3)?

Using the table of Young and Potler [12] on first occurrences of prime gaps, we have been able to verify with a computer search that Conjecture 1 is valid for  $a < c < 7 \cdot 10^{13}$ . The only example in this range with  $c - a$  less than the maximal gap between primes up to  $c$  is  $(a, c) = (2400, 2430)$ . The largest gap between consecutive primes up to  $7 \cdot 10^{13}$  is just 778, substantially smaller than the cube root of  $7 \cdot 10^{13}$ . Thus for  $n > 7 \cdot 10^{13}$  it is reasonable to believe that there is always a prime between  $n$  and  $n + n^{1/3}$ . In this case, Conjecture 1 follows if one can establish that for any  $a < c$  having the same prime factors,

$$(4) \quad c - a > a^{1/3}.$$

We know of no example for which (4) fails, and so we ask

*Question 2.* Is there any pair  $a < c$  composed of the same prime factors with  $c - a < a^{1/3}$ ?

From de Weger's work in [11] we can obtain (Theorem 5) all solutions of (3) with  $a$  and  $c$  composed of the primes 2, 3, 5, 7, 11 and 13, and having the same prime factors. All of these solutions satisfy (4) as well. Thus (4) holds for all  $a, c$  composed of the same primes from the set 2, 3, 5, 7, 11, and 13. Further examples satisfying (3) may be gleaned from the tables of Nitaj [7] and Browkin and Brzezinski [2] on extremal examples for the abc conjecture. All of these examples satisfy (4) as well.

## 2. PROOFS OF THEOREMS 1 AND 2

For any positive integer  $n$  let  $N_0(n) = \prod_{p|n} p$ , the product being over the distinct prime factors of  $n$ .

**The abc conjecture.** For any  $\epsilon > 0$  there exists a constant  $C(\epsilon)$  such that for any nonzero relatively prime integers  $a, b$  and  $c$  with  $a + b = c$  we have

$$(5) \quad \max(|a|, |b|, |c|) \leq C(\epsilon) N_0(abc)^{1+\epsilon}.$$

Suppose now that  $a < c$  are positive integers having the same prime factors. Let  $b = c - a$ . Put  $P = N_0(a) = N_0(c)$  and  $d = (a, b) = (a, c) = (b, c)$ . Then  $\frac{a}{d} + \frac{b}{d} = \frac{c}{d}$  and the integers  $\frac{a}{d}$ ,  $\frac{b}{d}$  and  $\frac{c}{d}$  are relatively prime. Now

$$(6) \quad N_0\left(\frac{a}{d} \frac{b}{d} \frac{c}{d}\right) \leq N_0(ac) N_0\left(\frac{b}{d}\right) \leq P \frac{b}{d} \leq \frac{b^2}{d},$$

the last inequality following since  $P|b$ . It follows from (5) that  $\frac{c}{d} \leq C(\epsilon) \left(\frac{b^2}{d}\right)^{1+\epsilon}$ , and so  $c \leq C(\epsilon) b^{2(1+\epsilon)}$ , that is  $b \geq C'(\epsilon) c^{\frac{1}{2}-\epsilon}$ . This establishes Theorem 1.

For the proof of Theorem 2 we proceed as above but instead of applying the abc conjecture we apply the following weaker, but proven, result of Stewart and Yu [9]. Under the same assumptions as in the abc conjecture above we have

$$\max(\log |a|, \log |b|, \log |c|) \leq C(\epsilon)N_0(abc)^{\frac{2}{3}+\epsilon}.$$

In our application we obtain

$$\log(c/d) \ll (b^2/d)^{\frac{2}{3}+\epsilon},$$

from which we deduce

$$b^2 \gg d(\log(c/d))^{\frac{3}{2}-\epsilon} \gg (\log c)^{\frac{3}{2}-\epsilon},$$

which completes the proof of Theorem 2. The latter inequality follows from the claim, for  $2 \leq d \leq c/2$  and  $0 < \epsilon < 3/2$  we have

$$d(\log(c/d))^{\frac{3}{2}-\epsilon} \geq .7(\log c)^{\frac{3}{2}-\epsilon}.$$

The claim follows from observing that

$$d\left(1 - \frac{\log d}{\log c}\right)^{\frac{3}{2}-\epsilon} \geq d\left(1 - \frac{\log d}{\log 2d}\right)^{3/2} = d\left(\frac{\log 2}{\log 2d}\right)^{3/2} \geq 2\left(\frac{\log 2}{\log 4}\right)^{3/2} > .7.$$

### 3. THE CASE OF TWO PRIME FACTORS: PROOF OF THEOREM 3

Suppose that  $a < c$  are positive integers composed of the same two prime divisors  $p, q$ . Let  $(a, c) = p^e q^f$  and write

$$(7) \quad c = p^{e+g} q^f, \quad a = p^e q^{f+h}, \quad b = c - a = p^e q^f (p^g - q^h).$$

We start by observing that in this case a large gap between  $a$  and  $c$  is tantamount to a large gap between the prime powers  $p^g$  and  $q^h$ . To be precise, the inequality

$$(8) \quad c - a \gg a^{1/2}$$

is equivalent to the inequality

$$(9) \quad p^g - q^h \gg p^{\frac{g}{2}(1-\frac{f}{h}-\frac{e}{g})}.$$

To see this we consider two cases. If  $q^h < \frac{1}{2}p^g$ , then (8) and (9) are both trivially true, and so we may assume that  $\frac{1}{2}p^g \leq q^h < p^g$ . Now (8) is equivalent to

$$p^g - q^h \gg p^{\frac{-e}{2}} q^{\frac{-f+h}{2}}.$$

Substituting  $q \approx p^{g/h}$  into the right-hand side yields (9).

We conclude the proof of Theorem 3 by showing that (9) holds true under the assumption of the abc conjecture.

It suffices to consider the case  $e = f = 1$  whence (9) becomes

$$(10) \quad p^g - q^h \gg p^{\frac{g}{2}(1-\frac{1}{h}-\frac{1}{g})}.$$

If  $h = 1$  or  $g = 1$  or  $(h, g) = (2, 2)$ , then (10) is trivial. Thus we may assume that  $h \geq 2, g \geq 2$ , and that either  $h$  or  $g$  is  $\geq 3$ . Now, the abc conjecture, applied to the sum  $p^g - q^h = (p^g - q^h)$ , implies that

$$p^g \ll (pq|p^g - q^h|)^{1+\epsilon},$$

or equivalently

$$(11) \quad |p^g - q^h| \gg p^{g(1 - \frac{1}{g} - \frac{1}{h} - \epsilon)},$$

the constants depending on  $\epsilon$ . Since  $1/h + 1/g \leq 5/6$ , one obtains (10) from (11) on taking  $\epsilon < 1/12$ . This completes the proof of Theorem 3.  $\square$

*Remark.* The argument above applies just as well to any relatively prime integers  $p$  and  $q$  (not necessarily primes). Thus Theorem 3 is valid for any  $a, c$  as in (7) with  $p, q$  relatively prime positive integers.

As one can see by the equivalence of (8) and (9), finding pairs  $a, c$  with  $c - a$  small amounts to finding two prime powers close together. Cijssouw, Korlaar and Tijdeman [3] found all solutions of the inequality

$$(12) \quad |p^g - q^h| < p^{g/2},$$

in positive integers  $g, h$  and primes  $p < q < 20$ . Their work was extended by de Weger ([11], Theorem 4.3) to the range  $p < q < 200$ ; see also Deze and Tijdeman ([5], Lemma 1). Now any solution of (3) with  $a, b, c$  as in (7) satisfies

$$p^g - q^h < p^{\frac{-\epsilon}{2}} q^{\frac{h-f}{2}} < p^{\frac{-1}{2}} q^{\frac{h-1}{2}}.$$

If  $p < q$ , then using the fact that  $q < p^{g/h}$  we obtain

$$(13) \quad p^g - q^h < p^{\frac{g}{2}(1 - \frac{1}{h} - \frac{1}{g})},$$

which is a stronger inequality than (12). If  $q < p$ , then  $p^{-1/2} < q^{-1/2}$  and so we obtain

$$p^g - q^h < q^{\frac{h}{2}-1},$$

which again is stronger than (12) with the roles of  $p$  and  $q$  reversed. Thus all solutions of (3) with  $p, q < 200$  may be found by testing the solutions of (12) given by de Weger in [11]. By doing so we obtain

**Theorem 4.** *Suppose that  $a < c$  are positive integers as in (7) with  $p, q < 200$  and  $c - a < a^{1/2}$ . Then  $(a, c) = (48, 54) = (2^4 \cdot 3, 2 \cdot 3^3)$ ,  $(1250, 1280) = (2 \cdot 5^4, 2^8 \cdot 5)$  or  $(11859482, 11862016) = (2 \cdot 181^3, 2^{16} \cdot 181)$ .*

#### 4. $a, c$ RESTRICTED TO THE PRIMES 2, 3, 5, 7, 11 AND 13

In [11, Theorem 4.6], de Weger solved the diophantine inequality

$$(14) \quad 0 < c - a < a^{1/2}$$

with

$$a, c \in \{2^{x_1} \dots 13^{x_6} : x_i \in \mathbb{Z}, x_i \geq 0, (1 \leq i \leq 6)\},$$

and  $(a, c) = 1$ . He found exactly 605 solutions, and all of them satisfy  $\nu_2(ac) \leq 26$ ,  $\nu_3(ac) \leq 19$ ,  $\nu_5(ac) \leq 13$ ,  $\nu_7(ac) \leq 13$ ,  $\nu_{11}(ac) \leq 7$ , and  $\nu_{13}(ac) \leq 8$ . Here,  $\nu_p(n)$  denotes the multiplicity of  $p$  dividing  $n$ . We ran a program in UBASIC to test which of these satisfy the stronger inequality

$$0 < P(c - a) < (Pa)^{1/2},$$

where  $P$  is the product of the primes appearing in  $ac$ . In this manner we were able to establish

**Theorem 5.** *There are 58 pairs of positive integers  $a < c$  having the same prime factors, with the primes selected from the set  $\{2, 3, 5, 7, 11, 13\}$ , such that  $c - a < a^{1/2}$ . In every such pair we have  $c < 15 \cdot 10^9$ , and  $c - a > a^{1/3}$ . Of these pairs, 19 are primitive,  $(a, c) = 1$ .*

A complete listing of the pairs in Theorem 5 is available upon request.

#### 5. SMALL GAPS WITH $a < c < 7 \cdot 10^{13}$

In the chart below we list all pairs  $0 < a < c < 7 \cdot 10^{13}$ , having the same prime factors, with  $c - a$  less than twice the maximal gap between primes up to  $c$ .

$a$	$c$	$c - a$	max prime gap
$48 = 2^4 \cdot 3$	$54 = 2 \cdot 3^3$	6	6
$1250 = 2 \cdot 5^4$	$1280 = 2^8 \cdot 5$	30	22
$2016 = 2^5 \cdot 3^2 \cdot 7$	$2058 = 2 \cdot 3 \cdot 7^3$	42	34
$2400 = 2^5 \cdot 3 \cdot 5^2$	$2430 = 2 \cdot 3^5 \cdot 5$	30	34
$2646 = 2 \cdot 3^3 \cdot 7^2$	$2688 = 2^7 \cdot 3 \cdot 7$	42	34
$15972 = 2^2 \cdot 3 \cdot 11^3$	$16038 = 2 \cdot 3^6 \cdot 11$	66	44
$29376 = 2^6 \cdot 3^3 \cdot 17$	$29478 = 2 \cdot 3 \cdot 17^2$	102	52
$58368 = 2^{10} \cdot 3 \cdot 19$	$58482 = 2 \cdot 3^4 \cdot 19^2$	114	72
$504000 = 2^6 \cdot 3^2 \cdot 5^3 \cdot 7$	$504210 = 2 \cdot 3 \cdot 5 \cdot 7^5$	210	114
$918540 = 2^2 \cdot 3^8 \cdot 5 \cdot 7$	$918750 = 2 \cdot 3 \cdot 5^5 \cdot 7^2$	210	114

The table above was obtained by a direct search on a PC using UBASIC. The idea of the program is very simple, and it runs extremely fast. For example if  $a, c$  have three odd primes in common, say  $p_1, p_2, p_3$ , then we know  $p_1 p_2 p_3 < 778/2$ , half the maximal gap between consecutive primes up to  $7 \cdot 10^{13}$ , and so the choices for  $p_1, p_2$  and  $p_3$  are very restricted, etc.

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