

CONVERGENCE OF RELAXATION SCHEMES FOR HYPERBOLIC CONSERVATION LAWS WITH STIFF SOURCE TERMS

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ABSTRACT. We focus in this study on the convergence of a class of relaxation numerical schemes for hyperbolic scalar conservation laws including stiff source terms. Following Jin and Xin, we use as approximation of the scalar conservation law, a semi-linear hyperbolic system with a second stiff source term. This allows us to avoid the use of a Riemann solver in the construction of the numerical schemes. The convergence of the approximate solution toward a weak solution is established in the cases of first and second order accurate MUSCL relaxed methods.

1. INTRODUCTION

Hyperbolic conservation laws with stiff source terms could describe the effect of relaxation as in the kinetic theory of gases, elasticity with memory, water waves, traffic flows, etc.

These problems may be mathematically written in scalar case as a Cauchy problem of the form

$$(1.1) \quad u_t + f(u)_x = q(u), \quad (x, t) \in \mathbb{R} \times]0, T[; T > 0$$

$$(1.2) \quad u(x, 0) = u_0(x), \quad x \in \mathbb{R}.$$

Theoretical study of the relaxation phenomena may be found in Chen et al. [7], Liu [20], and Whitham [32].

Numerical methods have been derived for the approximation of the conservation laws including nonstiff source terms in [3], [4], [24], [26], and [29]. These methods are based on explicit difference schemes. It is well known that explicit schemes are not appropriate for the numerical treatment of the stiff source terms; this motivates the use of semi-implicit and fully implicit schemes.

The approximation of the stiff case was recently studied by several authors (see [2], [6], [8], [13], [15], [17], [19], [25], and [26]), where different methods like asymptotic or splitting methods are used.

The main difficulty when we deal with the numerical solution of the stiff problems is the wrong location of the discontinuities. This problem has been investigated in [2], [8], [11], [13], [15], [17], and [19].

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Error bounds related to the approximation of (1.1)–(1.2) were derived in [5], [25] and [29].

In a recent study, Jin and Xin ([16]) introduced explicit relaxing schemes for the approximation of systems of hyperbolic conservation laws without source terms. Our purpose in this paper is the study of the convergence of semi-implicit relaxed schemes for the approximation of scalar conservation laws, including stiff source terms.

We do not assume here that the source term q is a Lipschitzian function in u but we suppose that $q' \leq 0$. This last hypothesis is realistic since it does indicate the dissipativity of the source term q in the sense of [7]. This is the case in the models of combustion ([2], [11], [21]), gas dynamics with heat transfer ([13]), water waves in the presence of the friction force of the bottom ([7]), etc. We point out that in all these examples and many others the source term is not a Lipschitzian function. The nonpositivity of q' is also assumed by Liu in [20] and Chen et al. in [7].

Thanks to the implicit character and to the dissipativity of the source term q (nonpositivity of q'), some proposed schemes are TVD, entropy satisfying at the limit, and they are monotone in the first order accurate semi-implicit case. Then they possess almost all the same properties as in the homogeneous schemes ($q = 0$).

This paper is organized as follows. Section 2 is devoted to recalling some preliminaries related to the hyperbolic conservation laws, including source terms. At first we give results which summarize the properties of the exact solution of the Cauchy problem when q' is nonpositive. These properties are similar to those related to the homogeneous case. In Section 3, we give an error bound for the relaxation problem. Section 4 concerns the upwind semi-implicit relaxing schemes. Section 5 is devoted to the study of the first and second order accurate relaxed schemes. An extension to the two dimensional case is presented in Section 6. Finally, in Section 7, we give concluding remarks.

2. PRELIMINARIES

We seek a weak solution to the Cauchy problem (1.1)–(1.2); that is a function $u \in L^\infty(\mathbb{R} \times]0, T[)$ satisfying

$$\int_{\mathbb{R}} \int_{]0, T[} [u\varphi_t + f(u)\varphi_x] dxdt + \int_{\mathbb{R}} u_0(x)\varphi(x, 0)dx = - \int_{\mathbb{R}} \int_{]0, T[} q(u)\varphi(x, t) dxdt$$

for all $\varphi \in \mathcal{C}^1(\mathbb{R} \times [0, T])$, with compact support in $(\mathbb{R} \times [0, T])$.

Let $\eta \in \mathcal{C}^2(\mathbb{R})$ be a strictly convex function, whose entropy flux is F , that is

$$(2.1) \quad \eta'(u)f'(u) = F'(u) \quad \forall u \in \mathbb{R}.$$

The solution of (1.1)–(1.2) is not necessarily unique and the physical one is characterized by the following entropy condition:

$$(2.2) \quad \int_{\mathbb{R}} \int_{]0, T[} [\eta(u)\varphi_t + F(u)\varphi_x] dxdt + \int_{\mathbb{R}} \eta(u_0(x))\varphi(x, 0)dx \geq - \int_{\mathbb{R}} \int_{]0, T[} \eta'(u)q(u)\varphi(x, t) dxdt$$

for all test positive functions $\varphi \in \mathcal{C}^1(\mathbb{R} \times [0, T])$, with compact support in $\mathbb{R} \times [0, T]$.

Let $u(x, t) = S(t)u_0$ denote the unique weak solution of (1.1)–(1.2) which satisfies the entropy condition. Let us assume that

$$(2.3) \quad \sup_u q'(u) \leq \gamma, \quad \gamma = \text{constant.}$$

Using a result of Kruzkov ([18]), we can easily prove the following

Proposition 2.1. *If $u_0 \in BV(\mathbb{R}) \cap L^1(\mathbb{R}), f \in C^1(\mathbb{R}), q \in C^1(\mathbb{R})$ such that $q(0) = 0$ and $q' \leq 0$, then the problem (1.1)–(1.2) possesses a unique entropy solution $u(x, t) = S(t)u_0$ satisfying:*

- i) $\| S(t)u_0 \|_{L^\infty(\mathbb{R})} \leq \| u_0 \|_{L^\infty(\mathbb{R})}$;
- ii) *Moreover, for any $v_0 \in L^\infty(\mathbb{R})$ we have*

$$\| S(t)u_0 - S(t)v_0 \|_{L^1(\mathbb{R})} \leq \| u_0 - v_0 \|_{L^1(\mathbb{R})};$$

- iii) $TV(S(t)u_0) \leq TV(u_0)$;
- iv) *If $u_0(x) \leq v_0(x)$, then $S(t)u_0 \leq S(t)v_0$.*

Remark 2.1. If we suppose that $q \in C^1(\mathbb{R})$ such that $q(0) = 0$ and $q' \leq 0$, then q is dissipative in the sense of Chen et al. ([7]). Indeed if we take $\eta(u) = u^2/2$ as an entropy function and apply the mean value theorem to q , we get

$$\begin{aligned} \eta'(u)q(u) &= u[q(0) + uq'(\xi)] \quad (\min(0, u) < \xi < \max(0, u)) \\ &= q'(\xi)u^2 \leq 0. \end{aligned}$$

Let h be the spatial size and k be the time grid size related to h by the fixed positive number r through

$$(2.4) \quad r = k/h.$$

A weak solution of (2.1)–(2.3) is approximated by a function u_h defined on $\mathbb{R} \times]0, T[$ by

$$(2.5) \quad u_h(x, t) = u_j^n \quad \text{for } (x, t) \in I_j \times J_n$$

with

$$I_j \times J_n =](j - 1/2)h, (j + 1/2)h[\times](n - 1/2)k, (n + 1/2)k[$$

for all $j \in \mathbb{Z}$ and $n \leq N = E(T/k) + 1$, where E denotes the integer part function.

The initial condition (2.2) is projected into the space of piecewise constant functions by

$$(2.6) \quad u_j^0 = \frac{1}{h} \int_{I_j} u_0(x) dx \quad \forall j \in \mathbb{Z}.$$

In all the following, we assume that

$$(2.7) \quad u_0 \in BV(\mathbb{R}), \quad f \in C^1(\mathbb{R}),$$

$$(2.8) \quad q \in C^1(\mathbb{R}) \quad \text{such that} \quad q(0) = 0, \quad q' \leq 0.$$

3. THE RELAXATION PROBLEM

Now, as in [16], let us associate with the scalar Cauchy problem (1.1)–(1.2), the semi-linear relaxation system

$$(3.1) \quad \begin{cases} u_t + v_x = q(u) \\ v_t + au_x = -\frac{1}{\epsilon}(v - f(u)) \end{cases}$$

with the initial data

$$(3.2) \quad u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x) = f(u_0(x)),$$

where the small parameter ϵ is the relaxation rate and a is a positive constant. Let (u_ϵ, v_ϵ) be the solution of the system (3.1). If we consider a formal expansion of Hilbert type, we get

$$v^\epsilon = f(u^\epsilon) + \epsilon v_1^\epsilon + \epsilon^2 v_2^\epsilon + \dots$$

We obtain as a first order approximation of the system (3.1)

$$(3.3) \quad u_t + f(u)_x = q(u) + \epsilon(f'(u)q(u))_x + \epsilon((a - f'(u)^2)u_x)_x.$$

Thus, (3.3) is dissipative if the following subcharacteristic condition is satisfied:

$$(3.4) \quad -\sqrt{a} \leq f'(u) \leq \sqrt{a} \quad \text{for all } u.$$

Let $Q = \mathbb{R} \times]0, T[$, then by similar proof to that in [9], we prove

Theorem 3.1. *Suppose that the initial data is bounded and the subcharacteristic condition (3.4) is satisfied, then*

- i) *The relaxation problem (3.1)–(3.2) admits a unique solution.*
- ii) *If we suppose further that there is no interval on which the flux function f is affine, then the solution (u^ϵ, v^ϵ) of (3.1)–(3.2) converges in $L^p(Q) \times L^2(Q)$, $0 < p < \infty$ toward $(u, f(u))$ as ϵ tends to zero, where u is the entropy solution of (1.1)–(1.2).*
- iii) *We have the bound error:*

$$\| f(u^\epsilon) - v^\epsilon \|_{L^2(Q)} \leq C\sqrt{\epsilon},$$

where C is a positive constant.

Proof. The proof of i) is based on invariant regions (see [9]). Using the entropy inequality we obtain ii).

Let us prove property iii) as we do in the homogeneous case in [14]. Let $(\eta(u, v), F(u, v))$ be a convex entropy-flux entropy function associated with the system (3.1) such that

$$a\eta_v = F_u, \quad \eta_u = F_v$$

by multiplying the first equation of (3.1) by η_u and the second equation of (3.1) by η_v . Summing the obtained equations, we get

$$\frac{\partial}{\partial t} \eta(u, v) + \frac{\partial}{\partial x} F(u, v) = \frac{1}{\epsilon} \eta_v (f(u) - v) + q(u) \eta_u.$$

Hence,

$$\begin{aligned} & \frac{1}{\epsilon} \int_Q \eta_v(v - f(u)) dxdt - \int_Q q(u) \eta_u dxdt \\ &= \int_{\mathbb{R}} \eta(u(x, T), v(x, T)) dx - \int_{\mathbb{R}} \eta(u_0(x), v_0(x)) dx. \end{aligned}$$

Thus

$$\frac{1}{\epsilon} \int_Q \eta_v(v - f(u)) dxdt - \int_Q q'(\xi) u^2 \eta_{uu}(\alpha, \beta) \leq \int_{\mathbb{R}} \eta(u_0(x), v_0(x)) dxdt.$$

We have

$$\eta_v(u, v) = (v - f(u)) \eta_{vv}(\lambda, \nu).$$

Using the convexity and the nonpositivity of q' , we get

$$\frac{1}{\epsilon} \int_Q \eta_{vv}(\lambda, \nu) (v - f(u))^2 dxdt \leq \int_{\mathbb{R}} \eta(u_0(x), v_0(x)) dxdt.$$

Then

$$\|f(u^\epsilon) - v^\epsilon\|_{L^2(Q)} \leq C\sqrt{\epsilon}.$$

4. THE UPWIND SEMI-IMPLICIT RELAXING SCHEME

The linear hyperbolic part of the system (3.1) has two Riemann invariants

$$v + \sqrt{a}u \quad \text{and} \quad v - \sqrt{a}u$$

associated with the characteristic fields $\lambda_- = -\sqrt{a}$ and $\lambda_+ = +\sqrt{a}$.

The first-order upwind scheme applied to $v \pm \sqrt{a}u$ gives

$$(v + \sqrt{a}u)_{j+1/2} = (v + \sqrt{a}u)_j, \quad (v - \sqrt{a}u)_{j+1/2} = (v - \sqrt{a}u)_{j+1};$$

hence,

$$\begin{aligned} u_{j+1/2} &= \frac{1}{2}(u_j + u_{j+1}) - \frac{1}{2\sqrt{a}}(v_{j+1} - v_j), \\ v_{j+1/2} &= \frac{1}{2}(v_j + v_{j+1}) - \frac{1}{2}\sqrt{a}(u_{j+1} - u_j). \end{aligned}$$

We can then construct the following upwind scheme:

$$(4.1) \quad \begin{cases} \frac{1}{k}(u_j^{n+1} - u_j^n) + \frac{1}{2h}(v_{j+1}^n - v_{j-1}^n) - \frac{\sqrt{a}}{2h}(u_{j+1}^n - 2u_j^n + u_{j-1}^n) = q(u_j^{n+1}) \\ \frac{1}{k}(v_j^{n+1} - v_j^n) + \frac{a}{2h}(u_{j+1}^n - u_{j-1}^n) - \frac{\sqrt{a}}{2h}(v_{j+1}^n - 2v_j^n + v_{j-1}^n) \\ \quad = -\frac{1}{\epsilon}(v_j^{n+1} - f(u_j^{n+1})). \end{cases}$$

For the existence and uniqueness of the solution of (4.1), we prove the following result.

Proposition 4.1. *The system (4.1) admits a unique solution $(u_j^{n+1}, v_j^{n+1}), n \in \mathbb{N}, j \in \mathbb{Z}$.*

Proof. It suffices to prove that the first equation of (4.1) admits a unique solution (u_j^{n+1}) . Indeed the function

$$x \longmapsto x - kq(x)$$

is differentiable and strictly monotone since $q' \leq 0$. Then the first equation of (4.1) admits a unique solution u_j^{n+1} .

5. THE RELAXED SCHEMES

By making use of the Hilbert expansion again, applied to (4.1), we get as ϵ tends to zero, the following relaxed schemes.

5.1. The upwind semi-implicit relaxed scheme. Using a formal expansion of Hilbert type in (4.1), we get the following relaxed scheme:

$$(5.1) \quad \begin{cases} v_j^n = f(u_j^n) \\ u_j^{n+1} = u_j^n - \frac{\tau}{2}(f(u_{j+1}^n) - f(u_{j-1}^n)) + \frac{\sqrt{a}\tau}{2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n) + kq(u_j^{n+1}). \end{cases}$$

To deal with locally bounded variation functions, we set

$$\begin{aligned} \Omega_R &=] - R, R[\times] 0, T[, \quad \text{for } R > 0, \\ J &= E(R/h), \quad \text{where } E \text{ is the integer part function.} \end{aligned}$$

Throughout this section, we set

$$L = \sup_{u \in \mathcal{A}} |f'(u)|, \quad M = \sup_{u \in \mathcal{A}} |q'(u)|,$$

where

$$\mathcal{A} = \{v \in L^\infty(\mathbb{R}), \|v\|_{L^\infty(\mathbb{R})} \leq \|u_0\|_{L^\infty(\mathbb{R})}\}.$$

Proposition 5.1. *Under the subcharacteristic condition (3.4) and the CFL condition*

$$(5.2) \quad r\sqrt{a} \leq 1,$$

we have

- i) *The scheme (5.1) is monotone.*
- ii) $\|u^{n+1}\|_{L^\infty(\mathbb{Z})} \leq \|u^n\|_{L^\infty(\mathbb{Z})}$.
- iii) $TV(u^{n+1}) \leq TV(u^n)$; *that is, the scheme (5.1) is TVD.*
- iv) $\sum_{|j| \leq J} |u_j^{n+1} - u_j^n| \leq C_1 TV(u^n) + C_2 \|u^{n+1}\|_{L^\infty(\mathbb{Z})}$, C_1, C_2 *are positive constants.*

Proof. First we write the scheme (5.1) in the form

$$(5.3) \quad \begin{cases} u_j^{n+1} = \bar{u}_j^{n+1} + kq(u_j^{n+1}) \\ \bar{u}_j^{n+1} = u_j^n - \frac{\tau}{2}(f(u_{j+1}^n) - f(u_{j-1}^n)) + \frac{\sqrt{a}\tau}{2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n). \end{cases}$$

i) To show that the scheme (5.1) is monotone, let us suppose that

$$u_j^n \leq w_j^n \quad \forall j \in \mathbb{Z}.$$

Using (5.3), we have

$$[u_j^{n+1} - w_j^{n+1}][1 - kq'(\alpha_j^n)] = \bar{u}_j^{n+1} - \bar{w}_j^{n+1}.$$

The second equation of (5.3) is a monotone scheme under the subcharacteristic condition (3.4) and the CFL condition (5.2), then

$$u^n \leq w^n \Rightarrow \bar{u}^{n+1} \leq \bar{w}^{n+1}.$$

Using the nonpositivity of q' , we obtain

$$u_j^{n+1} \leq w_j^{n+1} \quad \forall j \in \mathbb{Z}.$$

ii) Using (5.3), we have

$$u_j^{n+1} - kq(u_j^{n+1}) = \bar{u}_j^{n+1}.$$

Thus

$$(1 - kq'(\xi_j^n))u_j^{n+1} = \bar{u}_j^{n+1},$$

where q' is nonpositive, then we obtain

$$\|u^{n+1}\|_{L^\infty(\mathbb{Z})} \leq \|\bar{u}^{n+1}\|_{L^\infty(\mathbb{Z})}.$$

Taking into account the subcharacteristic condition (3.4) and the CFL condition (5.2), we easily show that

$$\|\bar{u}^{n+1}\|_{L^\infty(\mathbb{Z})} \leq \|u^n\|_{L^\infty(\mathbb{Z})}.$$

Then

$$\|u^{n+1}\|_{L^\infty(\mathbb{Z})} \leq \|u^n\|_{L^\infty(\mathbb{Z})}.$$

iii) By making use of (5.3), we get

$$[u_j^{n+1} - u_{j-1}^{n+1}][1 - kq'(\alpha_j^n)] = \bar{u}_j^{n+1} - \bar{u}_{j-1}^{n+1},$$

and the nonpositivity of q' gives us

$$TV(u^{n+1}) \leq TV(u^n).$$

iv) From (5.1), we obtain

$$\begin{aligned} u_j^{n+1} - u_j^n &= -\frac{r}{2}[(f(u_{j+1}^n) - f(u_j^n)) + (f(u_j^n) - f(u_{j-1}^n))] \\ &\quad + \frac{\sqrt{ar}}{2}[(u_{j+1}^n - u_j^n) - (u_j^n - u_{j-1}^n)] + kq(u_j^{n+1}). \end{aligned}$$

Hence,

$$\begin{aligned} \sum_{|j| \leq J} |u_j^{n+1} - u_j^n| &\leq \frac{r}{2}L \left(\sum_{|j| \leq J} |u_{j+1}^n - u_j^n| + \sum_{|j| \leq J} |u_j^n - u_{j-1}^n| \right) \\ &\quad + \frac{\sqrt{ar}}{2} \left(\sum_{|j| \leq J} |u_{j+1}^n - u_j^n| + \sum_{|j| \leq J} |u_j^n - u_{j-1}^n| \right) + kM \sum_{|j| \leq J} |u_j^{n+1}|. \end{aligned}$$

Then

$$\sum_{|j| \leq J} |u_j^{n+1} - u_j^n| \leq C_1 TV(u^n) + C_2 \|u^{n+1}\|_{L^\infty(\mathbb{Z})},$$

with

$$C_1 = r(L + \sqrt{a}), \quad C_2 = rRM \quad \text{since} \quad Jh \leq R.$$

Theorem 5.1. *Under the subcharacteristic condition (3.4) and the CFL condition (5.2), the approximate solution u_h constructed by the semi-implicit scheme (5.1) converges in $L^1_{\text{loc}}(\mathbb{R} \times]0, T[)$ toward the entropy satisfying solution of (1.1)–(1.2), as h tends to zero.*

Proof. Using Proposition 5.1, we obtain that the sequence (u_h) is bounded in family in the space $L^\infty(\mathbb{R} \times]0, T[) \cap BV_{\text{loc}}(\mathbb{R} \times]0, T[)$. By a similar proof as in [6], we make use of Helly’s theorem to get a subsequence extracted from (u_h) which converges toward the entropy solution of (1.1)–(1.2), and the uniqueness of the entropy satisfying solution ensures that all the sequence (u_h) converges toward the entropy satisfying solution u of problem (1.1)–(1.2) as h tends toward zero.

5.2. The upwind fully implicit relaxed scheme. Here, we consider the following implicit relaxed scheme:

$$(5.4) \quad \begin{cases} v_j^n = f(u_j^n) \\ u_j^{n+1} = u_j^n - \frac{r}{2}(f(u_{j+1}^{n+1}) - f(u_{j-1}^{n+1})) + \frac{\sqrt{ar}}{2}(u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}) + kq(u_j^{n+1}). \end{cases}$$

First, we introduce, as in [31], the operator T_μ defined on piecewise constant functions by

$$(5.5) \quad T_\mu(u_h)_j = u_j - \frac{\mu r}{2}(f(u_{j+1}) - f(u_{j-1})) + \frac{\sqrt{a}\mu r}{2}(u_{j+1} - 2u_j + u_{j-1}), \quad j \in \mathbb{Z},$$

where μ is a parameter satisfying $\mu > 0$, and $u_h = (u_j)_{j \in \mathbb{Z}}$.

By a similar proof to that in [31] we can prove

Lemma 5.1. *Let $u_h = (u_j)_{j \in \mathbb{Z}}$ be a piecewise constant function. If the subcharacteristic condition (3.4) is satisfied and if the positive parameter μ is sufficiently small such that*

$$(5.6) \quad \mu r \sqrt{a} \leq 1,$$

then

- i) $\| T_\mu(u_h) \|_{L^\infty(\mathbb{Z})} \leq \| u_h \|_{L^\infty(\mathbb{Z})}$;
- ii) $TV(T_\mu(u_h)) \leq TV(u_h)$; (TVD character);
- iii) $\| T_\mu(u_h) - T_\mu(v_h) \|_{L^1(\mathbb{Z})} \leq \| u_h - v_h \|_{L^1(\mathbb{Z})}$.

Next, we prove

Proposition 5.2. *Under the subcharacteristic condition (3.4), the approximate solution given by (5.4) satisfies the following properties:*

- i) $\| u^{n+1} \|_{L^\infty(\mathbb{Z})} \leq \| u^n \|_{L^\infty(\mathbb{Z})}$;
- ii) $TV(u^{n+1}) \leq TV(u^n)$ (TVD character);
- iii) $\sum_{|j| \leq J} |u_j^{n+1} - u_j^n| \leq C_1 TV(u^{n+1}) + C_2 \| u^{n+1} \|_{L^\infty(\mathbb{Z})}$.

Proof. i) Using the operator T_μ defined in (5.5), the scheme (5.4) may be written

$$u^{n+1} = \frac{\mu}{1 + \mu} u^n + \frac{1}{1 + \mu} T_\mu(u^{n+1}) + \frac{\mu}{1 + \mu} kq(u^{n+1}).$$

Applying the mean value theorem to q , we get

$$\left[1 - \frac{\mu}{1 + \mu} kq'(\xi^{n+1})\right] u^{n+1} = \frac{\mu}{1 + \mu} u^n + \frac{1}{1 + \mu} T_\mu(u^{n+1}).$$

Taking into account the nonpositivity of q' , we obtain

$$\| u^{n+1} \|_{L^\infty(\mathbb{Z})} \leq \frac{\mu}{1 + \mu} \| u^n \|_{L^\infty(\mathbb{Z})} + \frac{1}{1 + \mu} \| T_\mu(u^{n+1}) \|_{L^\infty(\mathbb{Z})}.$$

By making use of Lemma 5.1, we have

$$\| T_\mu(u^{n+1}) \|_{L^\infty(\mathbb{Z})} \leq \| u^{n+1} \|_{L^\infty(\mathbb{Z})}.$$

Hence,

$$\| u^{n+1} \|_{L^\infty(\mathbb{Z})} \leq \frac{\mu}{1 + \mu} \| u^n \|_{L^\infty(\mathbb{Z})} + \frac{1}{1 + \mu} \| u^{n+1} \|_{L^\infty(\mathbb{Z})}.$$

Then

$$\| u^{n+1} \|_{L^\infty(\mathbb{Z})} \leq \| u^n \|_{L^\infty(\mathbb{Z})}.$$

The properties ii) and iii) may be proved as in the former semi-explicit case.

Proposition 5.3. *If the subcharacteristic condition (3.4) is satisfied and*

$$(5.7) \quad k \sup_{u \in \mathcal{A}} | q'(u) | < 1,$$

then the scheme (5.4) admits a unique solution $(u_j^{n+1})_{j \in \mathbb{Z}} \in L^1(\mathbb{Z})$.

Proof. The scheme (5.4) may be written

$$u^{n+1} = \frac{\mu}{1 + \mu} u^n + \frac{1}{1 + \mu} T_\mu(u^{n+1}) + \frac{\mu}{1 + \mu} kq(u^{n+1}).$$

Let us introduce the operator $T : L^1(\mathbb{Z}) \rightarrow L^1(\mathbb{Z})$ defined by

$$(5.8) \quad T(u)_j = \frac{1}{1 + \mu} T_\mu(u)_j + \frac{\mu}{1 + \mu} kq(u)_j \quad \forall j \in \mathbb{Z}.$$

We have

$$\begin{aligned} \| T(u) - T(v) \|_{L^1(\mathbb{Z})} &\leq \frac{1}{1 + \mu} \| T_\mu(u) - T_\mu(v) \|_{L^1(\mathbb{Z})} \\ &\quad + \frac{\mu}{1 + \mu} k \sup_{u \in \mathcal{A}} | q'(u) | \| u - v \|_{L^1(\mathbb{Z})}. \end{aligned}$$

Using Lemma 5.1, we get

$$\| T(u) - T(v) \|_{L^1(\mathbb{Z})} \leq C \| u - v \|_{L^1(\mathbb{Z})},$$

where

$$C = \frac{1}{1 + \mu} + \frac{\mu}{1 + \mu} k \sup_{u \in \mathcal{A}} | q'(u) |.$$

By condition (5.7), we have

$$0 < C < 1.$$

Then T is a contraction; therefore, there exists a unique solution of (5.4) in $L^1(\mathbb{Z})$.

Theorem 5.2. *Under the subcharacteristic condition (3.4) the approximate solution u_h constructed by the implicit scheme (5.4), converges in $L^1_{loc}(\mathbb{R} \times]0, T[)$ toward the entropy satisfying solution of (1.1)–(1.2), as h tends to zero.*

Proof. Using Proposition 5.2, the family of the approximate solution (u_h) is bounded in $L^\infty(\mathbb{R} \times]0, T[) \cap BV_{loc}(\mathbb{R} \times]0, T[)$. Then by Helly’s theorem, we can extract a subsequence still labeled (u_h) which converges toward u in $L^1_{loc}(\mathbb{R} \times]0, T[)$. To show that u satisfies the entropy condition (2.3), we proceed exactly as in [5].

Remark 5.1. We point out that the former result was proved without any CFL condition.

5.3. The MUSCL semi-implicit relaxed scheme. To construct a second order accurate (in space) scheme using the Van-Leer method ([30]), we will need the initial boundary values inside cell j :

$$\begin{aligned} (v^n + \sqrt{a}u^n)_{j+1/2} &= (v^n + \sqrt{a}u^n)_j + 1/2h\delta_j^{+n}, \\ (v^n - \sqrt{a}u^n)_{j+1/2} &= (v^n - \sqrt{a}u^n)_{j+1} - 1/2h\delta_{j+1}^{-n}, \end{aligned}$$

with

$$(5.9) \quad \delta_j^\pm = \frac{1}{h}(v_{j+1} \pm \sqrt{a}u_{j+1} - v_j \pm \sqrt{a}u_j)\phi(\theta_j^\pm)$$

and

$$\theta_j^\pm = \frac{v_j \pm \sqrt{a}u_j - v_{j-1} \pm \sqrt{a}u_{j-1}}{v_{j+1} \pm \sqrt{a}u_{j+1} - v_j \pm \sqrt{a}u_j},$$

where ϕ is a limiter function as defined by Sweby in [27] and satisfies

$$(5.10) \quad 0 \leq \frac{\phi(\theta)}{\theta} \leq 2, \quad 0 \leq \phi(\theta) \leq 2.$$

Then

$$u_{j+1/2}^n = \frac{1}{2}(u_j^n + u_{j+1}^n) - \frac{1}{2\sqrt{a}}(v_{j+1}^n - v_j^n) + \frac{h}{4\sqrt{a}}(\delta_j^{+n} + \delta_{j+1}^{-n}),$$

and

$$v_{j+1/2}^n = \frac{1}{2}(v_j^n + v_{j+1}^n) - \frac{\sqrt{a}}{2}(u_{j+1}^n - u_j^n) + \frac{h}{4}(\delta_j^{+n} - \delta_{j+1}^{-n}).$$

Then the second order (in space) accurate semi-implicit relaxed scheme is given by

$$(5.11) \quad \begin{cases} v_j^n = f(u_j^n) \\ u_j^{n+1} = u_j^n - \frac{r}{2}(f(u_{j+1}^n) - f(u_{j-1}^n)) + \frac{\sqrt{a}r}{2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n) \\ \quad \quad \quad + \frac{k}{4}(\delta_{j+1}^{-n} - \delta_j^n + \delta_{j-1}^{+n} - \delta_j^{+n}) + kq(u_j^{n+1}). \end{cases}$$

Let

$$C_j = \frac{1}{2}r \left(\sqrt{a} + \frac{f(u_{j+1}) - f(u_j)}{u_{j+1} - u_j} \right), \quad D_j = \frac{1}{2}r \left(\sqrt{a} - \frac{f(u_{j+1}) - f(u_j)}{u_{j+1} - u_j} \right).$$

By the subcharacteristic condition (3.4), we have

$$C_j \geq 0, \quad D_j \geq 0.$$

The scheme (5.11) may be written in the form

$$(5.12) \quad u_j^{n+1} = \bar{u}_j^{n+1} + kq(u_j^{n+1}),$$

with

$$(5.13) \quad \begin{aligned} \bar{u}_j^{n+1} &= u_j^n - C_{j-1}^n(u_j^n - u_{j-1}^n) + D_j^n(u_{j+1}^n - u_j^n) \\ &\quad + \frac{k}{4}(1 - r\sqrt{a})(\delta_{j+1}^{-n} - \delta_j^{-n} + \delta_{j-1}^{+n} - \delta_j^{+n}). \end{aligned}$$

By using (5.9) and the definition of C_j and D_j , we have

$$\delta_j^+ = \frac{2}{rh}C_j\phi_j^+(u_{j+1} - u_j), \quad \delta_j^- = -\frac{2}{rh}D_j\phi_j^-(u_{j+1} - u_j).$$

Dropping n in the right-hand side, (5.13) gives

$$(5.14) \quad \begin{aligned} \bar{u}_j^{n+1} = & u_j - C_{j-1}(u_j - u_{j-1}) + D_j(u_{j+1} - u_j) \\ & - \frac{1}{2}\alpha D_{j+1}\phi_{j+1}^-(u_{j+2} - u_{j+1}) + \frac{1}{2}\alpha D_j\phi_j^-(u_{j+1} - u_j) \\ & + \frac{1}{2}\alpha C_{j-1}\phi_{j-1}^+(u_j - u_{j-1}) - \frac{1}{2}\alpha C_j\phi_j^+(u_{j+1} - u_j), \end{aligned}$$

where

$$\alpha = (1 - r\sqrt{a}).$$

Taking into account the equalities

$$\theta_j^+ C_j(u_{j+1} - u_j) = C_{j-1}(u_j - u_{j-1}), \quad \theta_j^- D_j(u_{j+1} - u_j) = D_{j-1}(u_j - u_{j-1}),$$

this leads to

$$(5.15) \quad \begin{aligned} \bar{u}_j^{n+1} = & [1 + \frac{1}{2}\alpha(\frac{\phi_j^+}{\theta_j^+} - \phi_{j-1}^+)]C_{j-1}u_{j-1} \\ & + (1 - (1 - \frac{1}{2}\alpha\phi_{j-1}^+ + \frac{1}{2}\alpha\frac{\phi_j^+}{\theta_j^+})C_{j-1} - D_{j-1} - \frac{1}{2}\alpha(\phi_j^- - \frac{\phi_{j+1}^-}{\theta_{j+1}^-})D_j)u_j \\ & + (1 - \frac{1}{2}\alpha(\frac{\phi_{j+1}^-}{\theta_{j+1}^-} - \phi_j^-))D_ju_{j+1}. \end{aligned}$$

Now, we prove that the coefficients of u_{j-1} , u_j and u_{j+1} in (5.15) are positive.

Lemma 5.2. *Under the subcharacteristic condition (3.4) and the CFL condition*

$$(5.16) \quad \nu = r\sqrt{a} \leq \frac{1}{2},$$

the coefficients

$$\begin{aligned} A_{1,j} &= [1 + \frac{1}{2}\alpha(\frac{\phi_j^+}{\theta_j^+} - \phi_{j-1}^+)]C_{j-1}, \\ A_{2,j} &= 1 - (1 - \frac{1}{2}\alpha\phi_{j-1}^+ + \frac{1}{2}\alpha\frac{\phi_j^+}{\theta_j^+})C_{j-1} - D_{j-1} - \frac{1}{2}\alpha(\phi_j^- - \frac{\phi_{j+1}^-}{\theta_{j+1}^-})D_j, \\ A_{3,j} &= [1 - \frac{1}{2}\alpha(\frac{\phi_{j+1}^-}{\theta_{j+1}^-} - \phi_j^-)]D_j \end{aligned}$$

are positive.

Proof. From (5.10) we have that

$$|\frac{\phi_j^+}{\theta_j^+} - \phi_{j-1}^+| \leq 2.$$

Then

$$1 + \frac{1}{2}\alpha(\frac{\phi_j^+}{\theta_j^+} - \phi_{j-1}^+) \geq 1 - \alpha = 1 - (1 - \nu) = \nu \geq 0;$$

C_j being positive, then the coefficient $A_{1,j}$ is positive. By a proof similar to the former one, we get that $A_{3,j}$ is positive since D_j is positive.

Let us now prove that $A_{2,j}$ is positive.

$$A_{2,j} = 1 + \frac{1}{2}\alpha\phi_{j-1}^+C_{j-1} - (C_{j-1} + D_{j-1}) - \frac{1}{2}\alpha\left[\frac{\phi_j^+}{\theta_j^+}C_{j-1} + \left(\phi_j^- - \frac{\phi_{j+1}^-}{\theta_{j+1}^-}\right)D_j\right].$$

The definition of C_j and D_j gives

$$C_{j-1} + D_{j-1} = r\sqrt{a} = \nu \leq \frac{1}{2}.$$

Hence,

$$\begin{aligned} A_{2,j} &\geq 1 + \frac{1}{2}\alpha\phi_{j-1}^+C_{j-1} - \nu - \alpha(C_{j-1} + D_j) \\ &\geq 1 + \frac{1}{2}\alpha\phi_{j-1}^+C_{j-1} - \nu - \alpha\nu - r\alpha\sqrt{a} \\ &\geq \frac{1}{2}\alpha\phi_{j-1}^+C_{j-1} + 2\nu^2 - 3\nu + 1 \\ &\geq \frac{1}{2}\alpha\phi_{j-1}^+C_{j-1} \quad \text{since } \nu \leq \frac{1}{2} \\ &\geq 0. \end{aligned}$$

Proposition 5.4. *Under the subcharacteristic condition (3.4) and the CFL condition (5.16), the scheme (5.11) is $L^\infty D$. That is*

$$(5.17) \quad \|u^{n+1}\|_{L^\infty(\mathbb{Z})} \leq \|u^n\|_{L^\infty(\mathbb{Z})}.$$

Proof. The scheme (5.11) may be written

$$(5.18) \quad u_j^{n+1} = \bar{u}_j^{n+1} + kq(u_j^{n+1}),$$

with

$$\begin{aligned} (5.19) \quad \bar{u}_j^{n+1} &= \left[1 + \frac{1}{2}\alpha\left(\frac{\phi_j^+}{\theta_j^+} - \phi_{j-1}^+\right)\right]C_{j-1}u_{j-1} \\ &\quad + \left(1 - \frac{1}{2}\alpha\phi_{j-1}^+ + \frac{1}{2}\alpha\frac{\phi_j^+}{\theta_j^+}\right)C_{j-1} - D_{j-1} - \frac{1}{2}\alpha\left(\phi_j^- - \frac{\phi_{j+1}^-}{\theta_{j+1}^-}\right)D_j]u_j \\ &\quad + \left[1 - \frac{1}{2}\alpha\left(\frac{\phi_{j+1}^-}{\theta_{j+1}^-} - \phi_j^-\right)\right]D_ju_{j+1}. \end{aligned}$$

Let us first prove that

$$\|\bar{u}^{n+1}\|_{L^\infty(\mathbb{Z})} \leq \|u^n\|_{L^\infty(\mathbb{Z})};$$

indeed,

$$\bar{u}_j^{n+1} = A_{1,j}u_{j-1}^n + A_{2,j}u_j^n + A_{3,j}u_{j+1}^n.$$

By Lemma 5.2, $A_{1,j}, A_{2,j}$ and $A_{3,j}$ are positive under the CFL condition (5.16).

Hence

$$\begin{aligned} |\bar{u}_j^{n+1}| &\leq A_{1,j}|u_{j-1}^n| + A_{2,j}|u_j^n| + A_{3,j}|u_{j+1}^n| \\ &\leq A_{1,j}\|u^n\|_{L^\infty(\mathbb{Z})} + A_{2,j}\|u^n\|_{L^\infty(\mathbb{Z})} + A_{3,j}\|u^n\|_{L^\infty(\mathbb{Z})}. \end{aligned}$$

After simplification we get

$$|\bar{u}_j^{n+1}| \leq \|u^n\|_{L^\infty(\mathbb{Z})} \quad \forall j \in \mathbb{Z},$$

since

$$A_{1,j} + A_{2,j} + A_{3,j} = 1.$$

Then

$$(5.20) \quad \|\bar{u}^{n+1}\|_{L^\infty(\mathbb{Z})} \leq \|u^n\|_{L^\infty(\mathbb{Z})}.$$

Using the mean value formula in (5.18) and then as in the proof of Proposition 5.1, we obtain

$$\|u^{n+1}\|_{L^\infty(\mathbb{Z})} \leq \|u^n\|_{L^\infty(\mathbb{Z})}.$$

By a similar proof to that of Proposition 5.4, we may prove

Proposition 5.5. *Under the subcharacteristic condition (3.4) and the CFL condition (5.16), the scheme (5.20) satisfies*

- i) $TV(u^{n+1}) \leq TV(u^n)$;
- ii) $\sum_{|j| \leq J} |u_j^{n+1} - u_j^n| \leq C_1 TV(u^n) + C_2 \|u^{n+1}\|_{L^\infty(\mathbb{Z})}$.

Using Propositions 5.4 and 5.5, we get that the family of the approximate solutions (u_h) is bounded in $L^\infty(\mathbb{R} \times]0, T[) \cap BV_{loc}(\mathbb{R} \times]0, T[)$ and then we are able to prove the following result.

Theorem 5.3. *Under the subcharacteristic condition (3.4) and the CFL condition (5.16), there exists a subsequence from the approximate family (u_h) , which converges in $L^1_{loc}(\mathbb{R} \times]0, T[)$ toward a weak solution of (1.1)–(1.2) as h tends to 0.*

Remark 5.2. Time discretization may be achieved using the Runge-Kutta method as in [6] and [13].

6. EXTENSION TO THE TWO DIMENSIONAL CASE

We consider now the two dimensional Cauchy problem:

$$(6.1) \quad u_t + f(u)_x + g(u)_y = q(u); \quad (x, y, t) \in \mathbb{R}^2 \times]0, T[; T > 0$$

and

$$(6.2) \quad u(x, y, 0) = u_0(x, y); \quad (x, y) \in \mathbb{R}^2.$$

In this case, we assume that

$$u_0 \in BV(\mathbb{R}^2); \quad f, g \in C^1(\mathbb{R})$$

and

$$q \in C^1(\mathbb{R}), \quad q(0) = 0, \quad q' \leq 0.$$

The existence and uniqueness of the solution of (6.1)–(6.2) is established by Kruzkov in [18].

The associated relaxation system with (6.1) is

$$(6.3) \quad \begin{cases} u_t + v_x + w_y = q(u), \\ v_t + au_x = -\frac{1}{\epsilon}(v - f(u)), \\ w_t + bu_y = -\frac{1}{\epsilon}(w - g(u)). \end{cases}$$

As in the one dimensional case, the system (6.3) is dissipative under the following subcharacteristic condition:

$$(6.4) \quad \frac{(f'(u))^2}{a} + \frac{(g'(u))^2}{b} \leq 1 \quad \forall u.$$

Let

$$r = \frac{\Delta t}{\Delta x}, \quad s = \frac{\Delta t}{\Delta y}.$$

Using the same method as in Section 5, the relaxed scheme for (6.1)–(6.2) is

$$(6.5) \quad \begin{cases} v_{i,j}^n = f(u_{i,j}^n) \\ w_{i,j}^n = g(u_{i,j}^n) \\ u_{i,j}^{n+1} = u_{i,j}^n - \frac{r}{2}(f(u_{i+1,j}^n) - f(u_{i-1,j}^n)) + \frac{\sqrt{ar}}{2}(u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n) \\ \quad - \frac{s}{2}(g(u_{i,j+1}^n) - g(u_{i,j-1}^n)) + \frac{\sqrt{bs}}{2}(u_{i,j+1}^n - 2u_{i,j}^n + u_{i,j-1}^n) + \Delta t q(u_{i,j}^{n+1}) \end{cases}$$

with

$$u_{i,j}^0 = \frac{1}{\Delta x \Delta y} \int_{(j-1/2)\Delta x}^{(j+1/2)\Delta x} \int_{(j-1/2)\Delta y}^{(j+1/2)\Delta y} u_0(x, y) dx dy.$$

A solution of (6.1)–(6.2) is approximated by a function u_h defined on $\mathbb{R}^2 \times]0, T[$ by

$$(6.6) \quad u_h(x, y, t) = u_{i,j}^n \quad \text{for } (x, y, t) \in](i-1/2)\Delta x, (i+1/2)\Delta x[\\ \times](j-1/2)\Delta y, (j+1/2)\Delta y[\times [n\Delta t, (n+1)\Delta t[$$

with

$$(i, j) \in \mathbb{Z}^2 \text{ and } n \leq N = E(T/k) + 1.$$

Proposition 6.1. *Under the subcharacteristic condition (6.4) and the CFL condition*

$$(6.7) \quad \nu = r\sqrt{a} + s\sqrt{b} \leq 1,$$

the scheme (6.5) satisfies

- i) $\|u^{n+1}\|_{L^\infty(\mathbb{Z} \times \mathbb{Z})} \leq \|u^n\|_{L^\infty(\mathbb{Z} \times \mathbb{Z})}$,
- ii) $TV(u^{n+1}) \leq TV(u^n)$,
- iii) $\sum_{|i| \leq I, |j| \leq J} |u_{i,j}^{n+1} - u_{i,j}^n| \leq C_1 TV(u^n) + C_2 \|u^{n+1}\|_{L^\infty(\mathbb{Z} \times \mathbb{Z})}$,

where

$$\|u^n\|_{L^\infty(\mathbb{Z} \times \mathbb{Z})} = \sup_{i,j \in \mathbb{Z}} |u_{i,j}^n|$$

and

$$TV(u^n) = \sum_{i,j \in \mathbb{Z}} |u_{i+1,j}^n - u_{i,j}^n| \Delta y + \sum_{i,j \in \mathbb{Z}} |u_{i,j+1}^n - u_{i,j}^n| \Delta x.$$

The proof is similar to those of Propositions 5.1–5.5.

As in subsection 5.1, using Proposition 6.1 we prove the following theorem.

Theorem 6.1. *Under the subcharacteristic (6.4) and the CFL condition (6.7), the approximate solution u_h given by the relaxed scheme (6.5) converges toward the entropy solution of (6.1)–(6.2).*

Remark 6.1. Second order accurate schemes may be constructed as in subsection 5.3; however these schemes will not be TVD (see [12]).

7. CONCLUSION

We presented an analysis of a class of relaxation schemes for hyperbolic conservation laws including stiff source terms. This method was introduced for the first time by Jin and Xin in [16] for the approximation of hyperbolic conservation laws without source terms. The construction of these schemes is based on the approximation of an associated linear hyperbolic system with a stiff source term depending on a small parameter ϵ (relaxation time). This method allows us to avoid the use of a Riemann solver.

With an adequate and realistic hypothesis, ($q'(u) \leq 0$) on the source term $q(u)$, that is with a dissipative (in some sense) source $q(u)$, we proposed semi-implicit and fully implicit numerical schemes possessing good properties, such as monotony, TVD character, etc., exactly as in the case of conservation laws without a source term ($q = 0$).

Under reasonable CFL conditions, we established the convergence of the approximate solution toward a weak solution or to the entropy satisfying solution (in some cases).

An extension of this analysis to the multidimensional case is given at the end of this paper.

We think that a direct study (stability, bound error, convergence, etc.) of the relaxing schemes (4.1), which depend on the relaxation rate ϵ , will be of interest, especially as concerns the link between this small parameter ϵ and the discretization steps Δt and Δx .

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