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SQUARE PRODUCT OF THREE INTEGERS IN SHORT INTERVALS

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Abstract. In this paper we list all the integer triplets taken from an interval of length ≤ 12, whose products are perfect squares.

1. Introduction

Let f and k be positive integers with $f \leq k$. The sets of distinct integers $n_1, \ldots, n_f \in [n+1, \ldots, n+k]$ with the property that there is a nontrivial way to multiply them to obtain a perfect power was investigated by Erdős and Turk [ET]. This question is related to the Erdős-Selfridge theorem (see [ES]), which states that the product of two or more consecutive integers is never a perfect power—that is, if $f = k \geq 2$, then the equation

$$\prod_{i=1}^{f} n_i = x^m \qquad (x \in \mathbb{N},\ m \geq 2) \tag{1}$$

has no solutions. Moreover, Erdős and Turk conjectured (cf. [ET]) that (1) has no solutions with $(k, f, m) = (4, 3, 2)$. This conjecture was verified by Tzanakis [T].

In this paper we list all the integer triplets ($f = 3$) taken from a short interval ($k \leq 12$) whose products are perfect squares.

2. Result

Now we formulate our result.

Theorem. *Let $(a, b, c) \in \mathbb{Z}^3$ with $a < b < c$ such that $c - a = k - 1 < 12$. If $abc \neq 0$ is a perfect square, then the triplet (a, b, c) is one of the following:*

$k = 5$: $(-2, -1, 2)$, $(2, 3, 6)$;
$k = 6$: $(-4, -1, 1)$, $(3, 6, 8)$, $(5, 8, 10)$, $(240, 243, 245)$;
$k = 7$: $(-4, -2, 2)$, $(-3, -1, 3)$, $(2, 4, 8)$, $(6, 8, 12)$, $(48, 50, 54)$;
$k = 8$: $(-4, -3, 3)$, $(1, 2, 8)$, $(2, 8, 9)$, $(7, 8, 14)$, $(21, 27, 28)$;
$k = 9$: $(-6, -3, 2)$, $(-4, -1, 4)$, $(1, 4, 9)$, $(2, 5, 10)$, $(12, 15, 20)$, $(24, 27, 32)$, $(242, 245, 250)$;

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$k = 10$: $(-8,-2,1)$, $(-6,-2,3)$, $(-3,-2,6)$, $(3,4,12)$, $(3,9,12)$, $(6,10,15)$, $(18,24,27)$;

$k = 11$: $(-9,-4,1)$, $(-9,-1,1)$, $(-8,-4,2)$, $(-8,-1,2)$, $(-5,-4,5)$, $(-5,-1,5)$, $(-2,-1,8)$, $(2,6,12)$, $(5,12,15)$, $(8,9,18)$, $(8,16,18)$, $(10,18,20)$, $(14,21,24)$, $(20,24,30)$, $(40,45,50)$, $(2880,2888,2890)$, $(10082,10086,10092)$;

$k = 12$: $(-9,-8,2)$, $(-9,-2,2)$, $(-8,-6,3)$, $(1,3,12)$, $(7,14,18)$, $(11,18,22)$, $(22,24,33)$, $(44,45,55)$, $(88,98,99)$, $(693,700,704)$.

As a consequence of the theorem we obtain that the interval $[44,45,\ldots,55]$ is the smallest one which contains two disjoint triplets of positive integers with the relevant property: $\{44,45,55\}$ and $\{48,50,54\}$.

3. Proof

To prove our theorem, we will reduce equation (1) to several elliptic equations. Recently, Gebel, Pethő and Zimmer [GPZ], and independently Stroeker and Tzanakis [ST], have developed an algorithm for solving elliptic equations. Their method is based on the approach of Zagier [Z], and on the recent estimates of linear forms in elliptic logarithms due to David [D]. The algorithm outlined in [GPZ] has been implemented by Gebel in the program package SIMATH (cf. [SIM]), and we use this program package to solve our elliptic equations.

Proof of the Theorem. Let $(a,b,c)\in\mathbb{Z}^3$ be a triplet with the desired property, and put $x=a$, $u=b-a$ and $v=c-a$. To prove the theorem we have to solve the system of elliptic equations

$$x(x-u)(x-v)=y^2$$

with $0<u<v<12$ in integers x,y. Using the results of Erdős and Selfridge [ES], and Tzanakis [T], we may suppose that $v\geq 4$, and we obtain 52 equations. By a simple substitution we transform these elliptic equations into Weierstrass normal form, and we can solve them by SIMATH. We obtained just the solutions listed in our theorem. $\square$

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