

## SQUARE PRODUCT OF THREE INTEGERS IN SHORT INTERVALS

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ABSTRACT. In this paper we list all the integer triplets taken from an interval of length  $\leq 12$ , whose products are perfect squares.

### 1. INTRODUCTION

Let  $f$  and  $k$  be positive integers with  $f \leq k$ . The sets of distinct integers  $n_1, \dots, n_f \in [n+1, \dots, n+k]$  with the property that there is a nontrivial way to multiply them to obtain a perfect power was investigated by Erdős and Turk [ET]. This question is related to the Erdős-Selfridge theorem (see [ES]), which states that the product of two or more consecutive integers is never a perfect power—that is, if  $f = k \geq 2$ , then the equation

$$(1) \quad \prod_{i=1}^f n_i = x^m \quad (x \in \mathbb{N}, m \geq 2)$$

has no solutions. Moreover, Erdős and Turk conjectured (cf. [ET]) that (1) has no solutions with  $(k, f, m) = (4, 3, 2)$ . This conjecture was verified by Tzanakis [T].

In this paper we list all the integer triplets ( $f = 3$ ) taken from a short interval ( $k \leq 12$ ) whose products are perfect squares.

### 2. RESULT

Now we formulate our result.

**Theorem.** *Let  $(a, b, c) \in \mathbb{Z}^3$  with  $a < b < c$  such that  $c - a = k - 1 < 12$ . If  $abc \neq 0$  is a perfect square, then the triplet  $(a, b, c)$  is one of the following:*

- $k = 5 : (-2, -1, 2), (2, 3, 6);$
- $k = 6 : (-4, -1, 1), (3, 6, 8), (5, 8, 10), (240, 243, 245);$
- $k = 7 : (-4, -2, 2), (-3, -1, 3), (2, 4, 8), (6, 8, 12), (48, 50, 54);$
- $k = 8 : (-4, -3, 3), (1, 2, 8), (2, 8, 9), (7, 8, 14), (21, 27, 28);$
- $k = 9 : (-6, -3, 2), (-4, -1, 4), (1, 4, 9), (2, 5, 10), (12, 15, 20), (24, 27, 32), (242, 245, 250);$

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$k = 10 : (-8, -2, 1), (-6, -2, 3), (-3, -2, 6), (3, 4, 12), (3, 9, 12), (6, 10, 15), (18, 24, 27);$

$k = 11 : (-9, -4, 1), (-9, -1, 1), (-8, -4, 2), (-8, -1, 2), (-5, -4, 5), (-5, -1, 5), (-2, -1, 8), (2, 6, 12), (5, 12, 15), (8, 9, 18), (8, 16, 18), (10, 18, 20), (14, 21, 24), (20, 24, 30), (40, 45, 50), (2880, 2888, 2890), (10082, 10086, 10092);$

$k = 12 : (-9, -8, 2), (-9, -2, 2), (-8, -6, 3), (1, 3, 12), (7, 14, 18), (11, 18, 22), (22, 24, 33), (44, 45, 55), (88, 98, 99), (693, 700, 704).$

As a consequence of the theorem we obtain that the interval  $[44, 45, \dots, 55]$  is the smallest one which contains two disjoint triplets of positive integers with the relevant property:  $\{44, 45, 55\}$  and  $\{48, 50, 54\}$ .

### 3. PROOF

To prove our theorem, we will reduce equation (1) to several elliptic equations. Recently, Gebel, Pethő and Zimmer [GPZ], and independently Stroeker and Tzanakis [ST], have developed an algorithm for solving elliptic equations. Their method is based on the approach of Zagier [Z], and on the recent estimates of linear forms in elliptic logarithms due to David [D]. The algorithm outlined in [GPZ] has been implemented by Gebel in the program package SIMATH (cf. [SIM]), and we use this program package to solve our elliptic equations.

*Proof of the Theorem.* Let  $(a, b, c) \in \mathbb{Z}^3$  be a triplet with the desired property, and put  $x = a$ ,  $u = b - a$  and  $v = c - a$ . To prove the theorem we have to solve the system of elliptic equations

$$x(x - u)(x - v) = y^2$$

with  $0 < u < v < 12$  in integers  $x, y$ . Using the results of Erdős and Selfridge [ES], and Tzanakis [T], we may suppose that  $v \geq 4$ , and we obtain 52 equations. By a simple substitution we transform these elliptic equations into Weierstrass normal form, and we can solve them by SIMATH. We obtained just the solutions listed in our theorem.  $\square$

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