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BIVARIATE COMPOSITE VECTOR VALUED RATIONAL INTERPOLATION

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ABSTRACT. In this paper we point out that bivariate vector valued rational interpolants (BVRI) have much to do with the vector-grid to be interpolated. When a vector-grid is well-defined, one can directly design an algorithm to compute the BVRI. However, the algorithm no longer works if a vector-grid is ill-defined. Taking the policy of "divide and conquer", we define a kind of bivariate composite vector valued rational interpolant and establish the corresponding algorithm. A numerical example shows our algorithm still works even if a vector-grid is ill-defined.

1. MOTIVATION

Let $\{(x_i, y_j)|i, j = 0, 1, \ldots, n\}$ be a set of points in $\mathbb{R}^2$ and let these points be reordered into the following array

$$\begin{matrix} (x_0, y_0) & \cdots & (x_0, y_n) \\ \vdots & \ddots & \vdots \\ (x_n, y_0) & \cdots & (x_n, y_n) \end{matrix} \tag{1.1}$$

where $x_i > x_{i+1}$ and $y_i < y_{i+1}$ for $i = 0, 1, \cdots, n-1$. We call this array a *square point-grid*, and denote it by Π^n. Let $\vec{v}_{i,j} \in \mathbb{C}^d$ be a d-dimensional vector associated with the point $(x_i, y_j) \in \Pi^n$. Similarly we arrange these $\vec{v}_{i,j}$ into the following array

$$\begin{matrix} \vec{v}_{0,0} & \vec{v}_{0,1} & \cdots & \vec{v}_{0,n} \\ \vec{v}_{1,0} & \vec{v}_{1,1} & \cdots & \vec{v}_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ \vec{v}_{n,0} & \vec{v}_{n,1} & \cdots & \vec{v}_{n,n} \end{matrix} \tag{1.2}$$

and call it a vector-grid, denoted by $\vec{V}^n$.

For a d-dimensional vector $\vec{v} = (v_1, v_2, \cdots, v_d) \in \mathbb{C}^d$, its generalized inverse (or the Samelson inverse) is defined as (see [3])

$$\vec{v}^{-1} = \frac{(v_1^*, v_2^*, \cdots, v_d^*)}{\sum_{i=1}^d v_i v_i^*}, \tag{1.3}$$

where v_i^* denotes the complex conjugate of v_i.

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Definition 1.1. A d-dimensional vector valued polynomial

$$\vec{N}(x,y) = (N_1(x,y), N_2(x,y), \cdots, N_d(x,y))$$

is said to be *of degree* n and we write $\partial\vec{N} = n$, if $\partial N_i(x,y) \leq n$ for $i = 1,2,\cdots,d$ and $\partial N_j(x,y) = n$ for some j $(1 \leq j \leq d)$.

Definition 1.2. Denote by H_n the collection of all bivariate polynomials with total degree not exceeding n and by $\vec{H}_n$ the collection of d dimensional bivariate vector valued polynomials of degree n. Then the set

$$\vec{H}_{n,m} = \{\vec{N}(x,y)/M(x,y) | \vec{N}(x,y) \in \vec{H}_n, M(x,y) \in H_m\}$$

is called *the collection of bivariate vector valued rational functions of type* (n/m).

All vectors in this paper are d-dimensional unless otherwise specified.

Making use of the Samelson inverse and reciprocal difference, one of the authors constructed the following Thiele-type branched continued fraction (see [5]):

$$\vec{r}_n(x,y) = \vec{t}_0(y) + \frac{x-x_0}{\vec{t}_1(y)} + \cdots + \frac{x-x_{n-1}}{\vec{t}_n(y)}, \tag{1.4}$$

where

$$\begin{aligned} \vec{t}_l(y) &= \vec{c}_{l,0}(\overline{x_0,\cdots,x_l};y_0) \\ &\quad + \frac{y-y_0}{\vec{c}_{l,1}(\overline{x_0,\cdots,x_l};y_0,y_1)} + \cdots + \frac{y-y_{n-1}}{\vec{c}_{l,n}(\overline{x_0,\cdots,x_l};y_0,\cdots,y_n)}, \end{aligned} \tag{1.5}$$

and $\vec{c}_{i,j}(\overline{x_0,\cdots,x_i};y_0,\cdots,y_j)$ are computed through the following recursive process

$$\vec{c}_{0,0}(\bar{x}_i,y_j) = \vec{v}_{i,j} \qquad (i=0,1,\cdots,n,\ j=0,1,\cdots,n), \tag{1.6}$$

$$\vec{c}_{0,j}(\bar{x}_i;y_0,\cdots,y_j) = \frac{y_j-y_{j-1}}{\vec{c}_{0,j-1}(\bar{x}_i;y_0,\cdots,y_{j-2},y_j) - \vec{c}_{0,j-1}(\bar{x}_i;y_0,\cdots,y_{j-2},y_{j-1})}, \tag{1.7}$$

$$\vec{c}_{i,0}(\overline{x_0,\cdots,x_i};y_j) = \frac{x_i-x_{i-1}}{\vec{c}_{i-1,0}(\overline{x_0,\cdots,x_{i-2},x_i};y_j) - \vec{c}_{i-1,0}(\overline{x_0,\cdots,x_{i-2},x_{i-1}};y_j)}, \tag{1.8}$$

$$\begin{aligned} &\vec{c}_{i,j}(\overline{x_0,\cdots,x_i};y_0,\cdots,y_j) \\ &\quad = \frac{y_j-y_{j-1}}{\vec{c}_{i,j-1}(\overline{x_0,\cdots,x_i};y_0,\cdots,y_{j-2},y_j) - \vec{c}_{i,j-1}(\overline{x_0,\cdots,x_i};y_0,\cdots,y_{j-2},y_{j-1})}. \end{aligned} \tag{1.9}$$

It is not difficult to prove (see [5]) that $\vec{r}_n(x,y) \in \vec{H}_{n^2+2n,2[(n^2+2n)/2]}$ (here $[x]$ denotes the greatest integer not exceeding x) and

$$\vec{r}_n(x_i,y_j) = \vec{v}_{i,j} \qquad (i=0,1,\cdots,n,\ j=0,1,\cdots,n). \tag{1.10}$$

If the roles of x and y are interchanged, one will obtain a so-called dual Thiele-type branched continued fraction (see [5])

$$\vec{r}^*_n(x,y) = \vec{t}^*_0(x) + \frac{y-y_0}{\vec{t}^*_1(x)} + \cdots + \frac{y-y_{n-1}}{\vec{t}^*_n(x)}, \tag{1.11}$$

where

$$\vec{t}_l^*(x) = \vec{c}_{0,l}^*(x_0;\overline{y_0,\cdots,y_l}) + \frac{x-x_0}{\vec{c}_{1,l}^*(x_0,x_1;\overline{y_0,\cdots,y_l})} + \cdots + \frac{x-x_{n-1}}{\vec{c}_{n,l}^*(x_0,\cdots,x_n;\overline{y_0,\cdots,y_l})} \tag{1.12}$$

and $\vec{c}_{i,j}^*(x_0,\cdots,x_i;\overline{y_0,\cdots,y_j})$ are computed according to the following recursive process

$$\vec{c}_{0,0}^*(x_i,\bar{y}_j) = \vec{v}_{i,j} \qquad (i=0,1,\cdots,n,\ j=0,1,\cdots,n), \tag{1.13}$$

(1.14)

$$\vec{c}_{0,j}^*(x_i;\overline{y_0,\cdots,y_j}) = \frac{y_j - y_{j-1}}{\vec{c}_{0,j-1}^*(x_i;\overline{y_0,\cdots,y_{j-2},y_j}) - \vec{c}_{0,j-1}^*(x_i;\overline{y_0,\cdots,y_{j-2},y_{j-1}})},$$

(1.15)

$$\vec{c}_{i,0}^*(x_0,\cdots,x_i;\bar{y}_j) = \frac{x_i - x_{i-1}}{\vec{c}_{i-1,0}^*(x_0,\cdots,x_{i-2},x_i;\bar{y}_j) - \vec{c}_{i-1,0}^*(x_0,\cdots,x_{i-2},x_{i-1};\bar{y}_j)},$$

(1.16)

$$\vec{c}_{i,j}^*(x_0,\cdots,x_i;\overline{y_0,\cdots,y_j}) = \frac{x_i - x_{i-1}}{\vec{c}_{i-1,j}^*(x_0,\cdots,x_{i-2},x_i;\overline{y_0,\cdots,y_j}) - \vec{c}_{i-1,j}(x_0,\cdots,x_{i-2},x_{i-1};\overline{y_0,\cdots,y_j})}.$$

To distinguish $\vec{r}_n(x,y)$ from $\vec{r}_n^*(x,y)$, we might as well call $\vec{r}_n(x,y)$ defined in (1.4)–(1.9) an *x/y-type* and $\vec{r}_n^*(x,y)$ defined in (1.11)–(1.16) a *y/x-type*. It can be proved that $\vec{r}_n^*(x,y) \in \vec{H}_{n^2+2n,2[(n^2+2n)/2]}$ and

$$\vec{r}_n^*(x_i,y_j) = \vec{v}_{i,j} \qquad (i=0,1,\ldots,n,\ j=0,1,\ldots,n). \tag{1.17}$$

Although both $\vec{r}_n(x,y)$ and $\vec{r}_n^*(x,y)$ are of the same rational type and have the same interpolation properties, one can by no means assert that $\vec{r}_n(x,y) \equiv \vec{r}_n^*(x,y)$, as is shown by a numerical example in [5]. However, if the square point-grid Π^n is symmetric, by which we mean $x_i = y_i$ for $i=0,1,\ldots,n$, and the vector-grid $\vec{V}^n$ is symmetric, by which we mean $\vec{v}_{i,j} = \vec{v}_{j,i}$ for $i,j=0,1,\ldots,n$, then we can conclude $\vec{r}_n(x,y) \equiv \vec{r}_n^*(y,x)$ (see [5]).

In what follows, we only restrict ourselves to the discussion of x/y-type bivariate vector valued rational interpolants (BVRI), because the results in x/y-type BVRI can easily be transplanted into y/x-type.

For convenience, let us simply set

$$\vec{c}_{i,j}^{(i,j)} = \vec{c}_{i,j}(\overline{x_0,\ldots,x_i};y_0,\ldots,y_j) \qquad (i=0,1,\ldots,n,\ j=0,1,\ldots,n). \tag{1.18}$$

Then we have the following algorithm to compute $\vec{r}_n(x,y)$.

Algorithm 1.1. *This algorithm is carried out according to the following three steps.*

a) *For $i=0,1,\cdots,n$ and $j=0,1,\cdots,n$, let*

$$\vec{c}_{i,j}^{(0,0)} = \vec{v}_{i,j}.$$

b) *For* $j = 0, 1, \cdots, n,\ p = 1, 2, \cdots, n,$ *and* $i = p, p+1, \cdots, n,$ *let*

$$\vec{c}_{i,j}^{(p,0)} = \frac{x_i - x_{p-1}}{\vec{c}_{i,j}^{(p-1,0)} - \vec{c}_{p-1,j}^{(p-1,0)}}.$$

c) *For* $i = 0, 1, \cdots, n,\ q = 1, 2, \cdots, n,$ *and* $j = q, q+1, \cdots, n,$ *let*

$$\vec{c}_{i,j}^{(i,q)} = \frac{y_j - y_{q-1}}{\vec{c}_{i,j}^{(i,q-1)} - \vec{c}_{i,q-1}^{(i,q-1)}}.$$

It is easy to verify that

$$\vec{r}_n(x_i, y_j) = \vec{v}_{i,j}, \quad \forall (x_i, y_j) \in \Pi^n.$$

Definition 1.3. A vector-grid $\vec{V}^n$ is said to be *well-defined* if the $\vec{c}_{i,j}^{(p,q)}$ as defined in Algorithm 1.1 satisfy $\vec{c}_{i,j}^{(p-1,0)} \neq \vec{c}_{p-1,j}^{(p-1,0)}$ for $j = 0, 1, \ldots, n,\ p = 1, 2, \ldots, n,$ and $i = p, p+1, \ldots, n,$ and $\vec{c}_{i,j}^{(i,q-1)} \neq \vec{c}_{i,q-1}^{(i,q-1)}$ for $i = 0, 1, \ldots, n,\ q = 1, 2, \ldots, n,$ and $j = q, q+1, \ldots, n$. Otherwise the grid $\vec{V}^n$ is said to be *ill-defined*.

It is clear that if a vector-grid $\vec{V}^n$ is ill-defined, then Algorithm 1.1 does not work any more.

Example 1.1. Let Π^1 and a two-dimensional vector-grid $\vec{V}^1$ be given as follows:

$$\Pi^1: \quad \begin{array}{ll} (1,0) & (1,1) \\ (0,0) & (0,1), \end{array}$$

$$\vec{V}^1: \quad \begin{array}{ll} (1,0) & (1,1) \\ (0,0) & (1,0). \end{array}$$

Proceeding by Algorithm 1.1, we obtain

$$\begin{array}{ll} \vec{c}_{0,0}^{(0,0)} = (1,0) & \vec{c}_{0,1}^{(0,0)} = (1,1) \\ \vec{c}_{1,0}^{(0,0)} = (0,0) & \vec{c}_{1,1}^{(0,0)} = (1,0) \\ & \downarrow \\ \vec{c}_{0,0}^{(0,0)} = (1,0) & \vec{c}_{0,1}^{(0,0)} = (1,1) \\ \vec{c}_{1,0}^{(1,0)} = (1,0) & \vec{c}_{1,1}^{(1,0)} = (0,1) \\ & \downarrow \\ \vec{c}_{0,0}^{(0,0)} = (1,0) & \vec{c}_{0,1}^{(0,1)} = (0,1) \\ \vec{c}_{1,0}^{(1,0)} = (1,0) & \vec{c}_{1,1}^{(1,1)} = (-\frac{1}{2}, \frac{1}{2}). \end{array}$$

Obviously $\vec{V}^1$ is well-defined. As a result,

$$\begin{aligned} \vec{t}_0(y) &= \vec{c}_{0,0} + \frac{y - y_0}{\vec{c}_{0,1}} = (1,0) + \frac{y}{(0,1)} = (1, y), \\ \vec{t}_1(y) &= \vec{c}_{1,0} + \frac{y - y_0}{\vec{c}_{1,1}} = (1,0) + \frac{y}{(-\frac{1}{2}, \frac{1}{2})} = (1 - y, y). \end{aligned}$$

Consequently we get

$$\begin{aligned} \vec{r}_1(x,y) &= \vec{t}_0(y) + \frac{x - x_0}{\vec{t}_1(y)} = (1, y) + \frac{x - 1}{(1 - y, y)} \\ &= \frac{(y^2 + (1-y)^2 + (x-1)(1-y),\ y(y^2 + (1-y)^2 + x - 1))}{(1-y)^2 + y^2}. \end{aligned}$$

Example 1.2. Let Π^2 and the two-dimensional vector-grid $\vec{V}^2$ be given as follows:

$$\Pi^2: \quad \begin{matrix} (0,0) & (0,1) & (0,2) \\ (-1,0) & (-1,1) & (-1,2) \\ (-2,0) & (-2,1) & (-2,2), \end{matrix}$$

$$\vec{V}^2: \quad \begin{matrix} (2,2) & (6,0) & (24,24) \\ (12,6) & (6,0) & (12,6) \\ (0,0) & (6,0) & (-2,2). \end{matrix}$$

We see $\vec{v}_{0,1} = \vec{v}_{1,1} = \vec{v}_{2,1} = (6,0)$, which leads to $\vec{c}_{0,1}^{(0,0)} = \vec{c}_{1,1}^{(0,0)} = \vec{c}_{2,1}^{(0,0)}$; therefore $\vec{V}^2$ is ill-defined and we cannot use Algorithm 1.1 to construct a vector-valued rational function $\vec{r}_2(x,y)$ that interpolates $\vec{V}^2$ over Π^2.

In the next section, we define a new interpolant with a corresponding algorithm more reliable than Algorithm 1.1.

2. The definition and computation of BCVRI

Let us decompose the grid Π^n into the following two triangular grids:

$$\begin{matrix} (x_0,y_0) & & & \\ (x_1,y_0) & (x_1,y_1) & & \\ \vdots & \vdots & \ddots & \\ (x_n,y_0) & (x_n,y_1) & \cdots & (x_n,y_n) \end{matrix} \tag{2.1}$$

and

$$\begin{matrix} (x_0,y_1) & (x_0,y_2) & \cdots & (x_0,y_n) \\ & (x_1,y_2) & \cdots & (x_1,y_n) \\ & & \ddots & \vdots \\ & & & (x_{n-1},y_n), \end{matrix} \tag{2.2}$$

denoted by LB and RU, respectively. We hope to use the policy of "divide and conquer" to construct a kind of composite vector valued rational interpolant. In what follows we abbreviate the term *bivariate composite vector valued rational interpolant* as BCVRI.

Let

$$\vec{R}_n(LB;x,y) = \vec{S}_0(LB;y) + \frac{x-x_n}{\vec{S}_1(LB;y)}\Big| + \cdots + \frac{x-x_1}{\vec{S}_n(LB;y)}\Big|, \tag{2.3}$$

$$\vec{R}_n(RU;x,y) = \vec{S}_0(RU;y) + \frac{x-x_0}{\vec{S}_1(RU;y)}\Big| + \cdots + \frac{x-x_{n-2}}{\vec{S}_{n-1}(RU;y)}\Big|, \tag{2.4}$$

where

$$\vec{S}_k(LB;y) = \vec{a}_{k,0} + \frac{y-y_0}{\vec{a}_{k,1}}\Big| + \cdots + \frac{y-y_{n-k-1}}{\vec{a}_{k,n-k}}\Big|, \qquad k = 0,1,\cdots,n, \tag{2.5}$$

(2.6)

$$\vec{S}_k(RU;y) = \vec{b}_{k,k+1} + \frac{y-y_{k+1}}{\vec{b}_{k,k+2}}\Big| + \cdots + \frac{y-y_{n-1}}{\vec{b}_{k,n}}\Big|, \qquad k = 0,1,\cdots,n-1.$$

Suppose Π^n is uniform, i.e.,

$$x_{i-1} - x_i = x_i - x_{i+1} = y_{i+1} - y_i = y_i - y_{i-1}, \qquad i = 1, 2, \cdots, n-1, \tag{2.7}$$

and let

$$P(x, y) = \prod_{i=0}^{n} (x + y - x_n - y_i), \tag{2.8}$$

$$Q(x, y) = \prod_{i=0}^{n-1} (x + y - x_i - y_n). \tag{2.9}$$

It is clear that $P(x, y)$ and $Q(x, y)$ are polynomials of degree $n + 1$ and n, respectively, and

$$\begin{array}{lll} P(x_i, y_j) = 0, & Q(x_i, y_j) \neq 0 & \text{if } (x_i, y_j) \in \text{LB}, \\ P(x_i, y_j) \neq 0, & Q(x_i, y_j) = 0 & \text{if } (x_i, y_j) \in \text{RU}. \end{array} \tag{2.10}$$

When Π^n is not uniform, by which we mean that the conditions (2.7) are not satisfied, one can also construct polynomials $P(x, y)$ and $Q(x, y)$ such that (2.10) holds. In general, however, the degrees of the polynomials will be much higher. For example,

$$\begin{aligned} P(x, y) &= \prod_{(x_i, y_j) \in \text{LB}} [(x - x_i)^2 + (y - y_j)^2] \\ Q(x, y) &= \prod_{(x_i, y_j) \in \text{RU}} [(x - x_i)^2 + (y - y_j)^2] \end{aligned}$$

are the polynomials satisfying (2.10) with degree $(n + 1)(n + 2)$ and $n(n + 1)$, respectively.

Now we define a BCVRI over Π^n as follows:

$$\vec{R}_n(x, y) = Q(x, y)\vec{R}_n(\text{LB}; x, y) + P(x, y)\vec{R}_n(\text{RU}; x, y). \tag{2.11}$$

The following algorithm aims at computing the coefficients $\vec{a}_{k,l}$ and $\vec{b}_{k,l}$ in branched continued fractions $\vec{R}_n(\text{LB}; x, y)$ and $\vec{R}_n(\text{RU}; x, y)$ simultaneously.

Algorithm 2.1. *This algorithm proceeds as follows.*

a) *For* $i = 0, 1, \cdots, n$ *and* $j = 0, 1, \cdots, i$, *let*

$$\vec{A}_{i,j}^{(0,0)} = \vec{v}_{i,j} / Q(x_i, y_j). \tag{2.12}$$

For $i = 0, 1, \cdots, n-1$ *and* $j = i+1, i+2, \cdots, n$, *let*

$$\vec{B}_{i,j}^{(0,0)} = \vec{v}_{i,j} / P(x_i, y_j). \tag{2.13}$$

b) *For* $j = 0, 1, \cdots, n$, $p = 1, 2, \cdots, n-j$, *and* $i = j, j+1, \cdots, n-p$, *let*

$$\vec{A}_{i,j}^{(p,0)} = \frac{x_i - x_{n-p+1}}{\vec{A}_{i,j}^{(p-1,0)} - \vec{A}_{n-p+1,j}^{(p-1,0)}}. \tag{2.14}$$

c) *For* $i = 0, 1, \cdots, n$, $q = 1, 2, \cdots, i$, *and* $j = q, q+1, \cdots, i$, *let*

$$\vec{A}_{i,j}^{(n-i,q)} = \frac{y_j - y_{q-1}}{\vec{A}_{i,j}^{(n-i,q-1)} - \vec{A}_{i,q-1}^{(n-i,q-1)}}. \tag{2.15}$$

d) *For* $j=1,2,\cdots,n,\ p=1,2,\cdots,j-1,$ *and* $i=p,p+1,\cdots,j-1,$ *let*

$$\vec{B}_{i,j}^{(p,0)}=\frac{x_i-x_{p-1}}{\vec{B}_{i,j}^{(p-1,0)}-\vec{B}_{p-1,j}^{(p-1,0)}}. \tag{2.16}$$

e) *For* $i=0,1,\cdots,n-1$ *and* $j=i+1,i+2,\cdots,n,$ *let*

$$\vec{B}_{i,j}^{(i,i+1)}=\vec{B}_{i,j}^{(i,0)}. \tag{2.17}$$

f) *For* $i=0,1,\cdots,n-2,\ j=i+2,i+3,\cdots,n,$ *and* $q=i+2,i+3,\cdots,j,$ *let*

$$\vec{B}_{i,j}^{(i,q)}=\frac{y_j-y_{q-1}}{\vec{B}_{i,j}^{(i,q-1)}-\vec{B}_{i,q-1}^{(i,q-1)}}. \tag{2.18}$$

Theorem 2.1. *Let*

$$\vec{a}_{k,l}=\vec{A}_{n-k,l}^{(k,l)} \qquad (k=0,1,\cdots,n,\ l=0,1,\cdots,n-k), \tag{2.19}$$

$$\vec{b}_{k,l}=\vec{B}_{k,l}^{(k,l)} \qquad (k=0,1,\cdots,n-1,l=k+1,\cdots,n). \tag{2.20}$$

Then

$$\vec{R}_n(x,y)\in\vec{H}_{\max(\partial Q+n^2+n-2,\partial P+n^2+n-1),n^2+n-2} \quad \textit{for even } n,$$
$$\vec{R}_n(x,y)\in\vec{H}_{\max(\partial Q+n^2+n-1,\partial P+n^2+n-2),n^2+n-2} \quad \textit{for odd } n,$$

where ∂Q *and* ∂P *denote the total degrees of polynomials* $Q(x,y)$ *and* $P(x,y)$*, respectively. (In particular, if* Π^n *is uniform, then* $\partial Q=n$ *and* $\partial P=n+1$*. In this case,* $\vec{R}_n(x,y)\in\vec{H}_{n^2+2n,n^2+n-2}$ *for even* n *and* $\vec{R}_n(x,y)\in\vec{H}_{n^2+2n-1,n^2+n-2}$ *for odd* n*.) Moreover,*

$$\vec{R}_n(x_i,y_j)=\vec{v}_{i,j} \qquad \forall(x_i,y_j)\in\Pi^n.$$

Proof. It is not difficult to show by induction that

$$\vec{R}_n(LB;x,y)\in\vec{H}_{(n^2+2n)/2,(n^2+2n)/2} \quad \text{for even } n,$$
$$\vec{R}_n(LB;x,y)\in\vec{H}_{(n^2+2n-1)/2,(n^2+2n-3)/2} \quad \text{for odd } n,$$
$$\vec{R}_n(RU;x,y)\in\vec{H}_{(n^2-2)/2,(n^2-4)/2} \quad \text{for even } n,$$
$$\vec{R}_n(RU;x,y)\in\vec{H}_{(n^2-1)/2,(n^2-1)/2} \quad \text{for odd } n.$$

Therefore

$$\vec{R}_n(x,y)\in\vec{H}_{\max(\partial Q+n^2+n-2,\partial P+n^2+n-1),n^2+n-2} \quad \text{for even } n$$
$$\vec{R}_n(x,y)\in\vec{H}_{\max(\partial Q+n^2+n-1,\partial P+n^2+n-2),n^2+n-2} \quad \text{for odd } n.$$

Since $\Pi^n=\mathrm{LB}\cup\mathrm{RU}$, $(x_i,y_j)\in\Pi^n$ implies $(x_i,y_j)\in\mathrm{LB}$ or $(x_i,y_j)\in\mathrm{RU}$. If $(x_i,y_j)\in\mathrm{LB}$, then from (2.5), (2.19) and (2.15) it follows that

$$\begin{aligned}\vec{S}_k(\mathrm{LB};y_j)&=\vec{A}_{n-k,0}^{(k,0)}+\frac{y_j-y_0|}{|\vec{A}_{n-k,1}^{(k,1)}}+\cdots+\frac{y_j-y_{j-1}|}{|\vec{A}_{n-k,j}^{(k,j)}}\\&=\vec{A}_{n-k,0}^{(k,0)}+\frac{y_j-y_0|}{|\vec{A}_{n-k,1}^{(k,1)}}+\cdots+\frac{y_j-y_{j-2}|}{|\vec{A}_{n-k,j}^{(k,j-1)}}\\&=\cdots=\vec{A}_{n-k,j}^{(k,0)}.\end{aligned}$$

By (2.3), (2.14) and (2.12) one has

$$
\begin{aligned}
\vec{R}_n(\mathrm{LB}; x_i, y_j) &= \vec{A}_{n,j}^{(0,0)} + \frac{x_i - x_n}{\vec{A}_{n-1,j}^{(1,0)}} + \cdots + \frac{x_i - x_{i+1}}{\vec{A}_{i,j}^{(n-i,0)}} \\
&= \vec{A}_{n,j}^{(0,0)} + \frac{x_i - x_n}{\vec{A}_{n-1,j}^{(1,0)}} + \cdots + \frac{x_i - x_{i+2}}{\vec{A}_{i,j}^{(n-i-1,0)}} \\
&= \cdots = \vec{A}_{i,j}^{(0,0)} = \vec{v}_{i,j}/Q(x_i, y_j).
\end{aligned}
$$

Therefore, by (2.11) one finally gets

$$
\vec{R}_n(x_i, y_j) = Q(x_i, y_j)\vec{R}_n(\mathrm{LB}; x_i, y_j) = \vec{v}_{i,j}.
$$

If $(x_i, y_j) \in \mathrm{RU}$, then from (2.6), (2.20) and (2.18) it follows that

$$
\begin{aligned}
\vec{S}_k(\mathrm{RU}; y_j) &= \vec{B}_{k,k+1}^{(k,k+1)} + \frac{y_j - y_{k+1}}{\vec{B}_{k,k+2}^{(k,k+2)}} + \cdots + \frac{y_j - y_{j-1}}{\vec{B}_{k,j}^{(k,j)}} \\
&= \vec{B}_{k,k+1}^{(k,k+1)} + \frac{y_j - y_{k+1}}{\vec{B}_{k,k+2}^{(k,k+2)}} + \cdots + \frac{y_j - y_{j-2}}{\vec{B}_{k,j}^{(k,j-1)}} \\
&= \cdots = \vec{B}_{k,j}^{(k,k+1)}.
\end{aligned}
$$

Thus, from (2.4), (2.17), (2.16) and (2.13) we get

$$
\begin{aligned}
\vec{R}_n(\mathrm{RU}; x_i, y_j) &= \vec{B}_{0,j}^{(0,1)} + \frac{x_i - x_0}{\vec{B}_{1,j}^{(1,2)}} + \cdots + \frac{x_i - x_{i-1}}{\vec{B}_{i,j}^{(i,i+1)}} \\
&= \vec{B}_{0,j}^{(0,0)} + \frac{x_i - x_0}{\vec{B}_{1,j}^{(1,0)}} + \cdots + \frac{x_i - x_{i-1}}{\vec{B}_{i,j}^{(i,0)}} \\
&= \vec{B}_{0,j}^{(0,0)} + \frac{x_i - x_0}{\vec{B}_{1,j}^{(1,0)}} + \cdots + \frac{x_i - x_{i-2}}{\vec{B}_{i,j}^{(i-1,0)}} \\
&= \cdots = \vec{B}_{i,j}^{(0,0)} = \vec{v}_{i,j}/P(x_i, y_j).
\end{aligned}
$$

Hence, by (2.11) we have

$$
\vec{R}_n(x_i, y_j) = P(x_i, y_j)\vec{R}_n(\mathrm{RU}; x_i, y_j) = \vec{v}_{i,j}.
$$

The proof is completed. □

3. The complexity of algorithms

Instead of (2.1) and (2.2) one can also carry out other triangular decompositions of the square grid, for instance, the decomposition

$$
\begin{array}{llll}
(x_1, y_0) & & & \\
(x_2, y_0) & (x_2, y_1) & & \\
\vdots & \vdots & \ddots & \\
(x_n, y_0) & (x_n, y_1) & \cdots & (x_n, y_{n-1})
\end{array}
\qquad
\begin{array}{llll}
(x_0, y_0) & (x_0, y_1) & \cdots & (x_0, y_n) \\
 & (x_1, y_1) & \cdots & (x_1, y_n) \\
 & & \ddots & \vdots \\
 & & & (x_n, y_n)
\end{array}
$$

and the decomposition, along another diagonal,

$$\begin{array}{cccccccc} (x_0,y_0) & \cdots & (x_0,y_{n-1}) & (x_0,y_n) & & & & (x_1,y_n) \\ (x_1,y_0) & \cdots & (x_1,y_{n-1}) & & & & (x_2,y_{n-1}) & (x_2,y_n) \\ \vdots & & & & & & \vdots & \vdots \\ (x_n,y_0) & & & & (x_n,y_1) & \cdots & (x_n,y_{n-1}) & (x_n,y_n). \end{array}$$

It is not difficult to define the corresponding BCVRIs based on the above decompositions which interpolate $\vec{V}^n$ over Π^n.

For a vector valued continued fraction, the complexity is obviously related to the computation of the Samelson inverses. From (1.3) we know that carrying out a Samelson inversion for a d-dimensional vector demands at least $2d$ operations of multiplications or divisions. Therefore we take the number of Samelson inverses in an algorithm as the criterion for judging whether the algorithm is complicated or not.

Suppose N_1 and N_2 are the total numbers of Samelson inverses to be computed for the vector valued rational interpolants of form (2.11) and (1.4)–(1.5), respectively. Then

$$\begin{aligned} N_1 &= \sum_{j=0}^{n} \frac{(n-j)(n-j+1)}{2} + \sum_{i=0}^{n} \frac{i(i+1)}{2} + \sum_{j=1}^{n} \frac{j(j-1)}{2} + \sum_{i=0}^{n-2} \frac{(n-i+1)(n-i)}{2} \\ &= \frac{n(n+1)(n+2)}{6} + \frac{n(n+1)(n+2)}{6} + \frac{(n-1)n(n+1)}{6} + \frac{(n-1)n(n+1)}{6} \\ &= \frac{n(n+1)(2n+1)}{3}, \end{aligned}$$

$$N_2 = n(n+1)^2,$$

which shows that it is $n(n+1)(n+2)/3$ times more economical to compute the BCVRI $\vec{R}_n(x,y)$ in (2.11) yielded by the decomposition (2.1) and (2.2) of Π^n than to compute (1.4) and (1.5) directly. Therefore at least $2n(n+1)(n+2)d/3$ multiplications are saved through our decomposition method.

4. Numerical example

Let us consider again the grid Π^2 and the corresponding vector-grid $\vec{V}^2$ in Example 1.2, i.e.,

$$\Pi^2: \quad \begin{array}{ccc} (0,0) & (0,1) & (0,2) \\ (-1,0) & (-1,1) & (-1,2) \\ (-2,0) & (-2,1) & (-2,2), \end{array}$$

$$\vec{V}^2: \quad \begin{array}{ccc} (2,2) & (6,0) & (24,24) \\ (12,6) & (6,0) & (12,6) \\ (0,0) & (6,0) & (-2,2). \end{array}$$

In this case, Π^2 is uniform and $\vec{V}^2$ is ill-defined. We mentioned in Example 1.2 that the computational procedure in Algorithm 1.1 breaks down. In fact, $\vec{r}_2(x,y)$ does not exist at all in this case. Otherwise, $\vec{r}_2(x,y)$ can be written as

$$\vec{r}_2(x,y) = \vec{t}_0(y) + \cfrac{x-x_0}{\vec{t}_1(y) + \cfrac{x-x_1}{\vec{t}_2(y)}}.$$

Whatever a reordering of the square point-grid Π^2 and vector-grid $\vec{V}^2$ is made, we always have a whole column in $\vec{V}^2$, entries of which are all equal to $(6,0)$, i.e., $\vec{v}_{0,j} = \vec{v}_{1,j} = \vec{v}_{2,j} = (6,0)$ with some j in $\{0,1,2\}$. Therefore we have

$$\vec{r}_2(x_0, y_j) = \vec{r}_2(x_1, y_j) = \vec{r}_2(x_2, y_j),$$

which leads to

$$\vec{t}_0(y_j) + \frac{x_1 - x_0}{\vec{t}_1(y_j)} = \vec{t}_0(y_j).$$

The above relations imply $(x_1 - x_0)/\vec{t}_1(y_j) = 0$ which is impossible because $x_1 \neq x_0$.

Next, we turn to the construction of a BCVRI defined in (2.11). By (2.8) and (2.9),

$$\begin{aligned} P(x,y) &= (x+y+2)(x+y+1)(x+y), \\ Q(x,y) &= (x+y-2)(x+y-1). \end{aligned}$$

By (2.12) and (2.13), one gets

$$\begin{aligned} &\vec{A}^{(0,0)}_{0,0} = (1,1), \\ &\vec{A}^{(0,0)}_{1,0} = (2,1), \quad \vec{A}^{(0,0)}_{1,1} = (3,0), \\ &\vec{A}^{(0,0)}_{2,0} = (0,0), \quad \vec{A}^{(0,0)}_{2,1} = (1,0), \quad \vec{A}^{(0,0)}_{2,2} = (-1,1) \end{aligned}$$

and

$$\begin{aligned} \vec{B}^{(0,0)}_{0,1} = (1,0), \quad &\vec{B}^{(0,0)}_{0,2} = (1,1), \\ &\vec{B}^{(0,0)}_{1,2} = (2,1). \end{aligned}$$

According to (2.14) and (2.15), one obtains in order

$$\begin{aligned} &\vec{A}^{(1,0)}_{0,0} = (1,1), \\ &\vec{A}^{(1,0)}_{1,0} = (2/5,1/5), \quad \vec{A}^{(1,0)}_{1,1} = (1/2,0), \\ &\vec{A}^{(0,0)}_{2,0} = (0,0), \qquad \vec{A}^{(0,0)}_{2,1} = (1,0), \qquad \vec{A}^{(0,0)}_{2,2} = (-1,1), \end{aligned}$$

$$\begin{aligned} &\vec{A}^{(2,0)}_{0,0} = (3/5,4/5), \\ &\vec{A}^{(1,0)}_{1,0} = (2/5,1/5), \quad \vec{A}^{(1,0)}_{1,1} = (1/2,0), \\ &\vec{A}^{(0,0)}_{2,0} = (0,0), \qquad \vec{A}^{(0,0)}_{2,1} = (1,0), \qquad \vec{A}^{(0,0)}_{2,2} = (-1,1), \end{aligned}$$

$$\begin{aligned} &\vec{A}^{(2,0)}_{0,0} = (3/5,4/5), \\ &\vec{A}^{(1,0)}_{1,0} = (2/5,1/5), \quad \vec{A}^{(1,1)}_{1,1} = (2,-4), \\ &\vec{A}^{(0,0)}_{2,0} = (0,0), \qquad \vec{A}^{(0,1)}_{2,1} = (1,0), \qquad \vec{A}^{(0,1)}_{2,2} = (-1,1), \end{aligned}$$

$$\begin{aligned} &\vec{A}^{(2,0)}_{0,0} = (3/5,4/5), \\ &\vec{A}^{(1,0)}_{1,0} = (2/5,1/5), \quad \vec{A}^{(1,1)}_{1,1} = (2,-4), \\ &\vec{A}^{(0,0)}_{2,0} = (0,0), \qquad \vec{A}^{(0,1)}_{2,1} = (1,0), \qquad \vec{A}^{(0,2)}_{2,2} = (-2/5,1/5). \end{aligned}$$

By Theorem 2.1

$$\begin{aligned}\vec{S}_0(LB;y) &= \vec{A}_{2,0}^{(0,0)} + \frac{y-y_0}{\lceil \vec{A}_{2,1}^{(0,1)}} + \frac{y-y_1}{\lceil \vec{A}_{2,2}^{(0,2)}}\\ &= (0,0) + \frac{y}{\lceil (1,0)} + \frac{y-1}{\lceil (-2/5,1/5)}\\ &= \frac{(3y-2y^2, y^2-y)}{(3-2y)^2+(y-1)^2},\\ \vec{S}_1(LB;y) &= \vec{A}_{1,0}^{(1,0)} + \frac{y-y_0}{\lceil \vec{A}_{1,1}^{(1,1)}} = (2/5,1/5) + \frac{y}{\lceil (2,-4)}\\ &= (\frac{4+y}{10}, \frac{1-y}{5}),\\ \vec{S}_2(LB;y) &= \vec{A}_{0,0}^{(2,0)} = (3/5,4/5).\end{aligned}$$

This leads to

$$\begin{aligned}\vec{R}_2(LB;x,y) &= \vec{S}_0(LB;y) + \frac{x-x_2}{\lceil \vec{S}_1(LB;y)} + \frac{x-x_1}{\lceil \vec{S}_2(LB;y)}\\ &= \frac{(3y-2y^2, y^2-y)}{(3-2y)^2+(y-1)^2} + \frac{x+2}{\lceil (\frac{4+y}{10}, \frac{1-y}{5})} + \frac{x+1}{\lceil (3/5,4/5)}\\ &= \frac{(3y-2y^2, y^2-y)}{(3-2y)^2+(y-1)^2} + \frac{(10x+20)(6x+y+10, 8x-2y+10)}{(6x+y+10)^2+(8x-2y+10)^2}.\end{aligned}$$

According to (2.16)–(2.18), one derives

$$\begin{aligned}\vec{B}_{0,1}^{(0,0)} = (1,0),\quad &\vec{B}_{0,2}^{(0,0)} = (1,1),\\ &\vec{B}_{1,2}^{(1,0)} = (-1,0),\end{aligned}$$

$$\begin{aligned}\vec{B}_{0,1}^{(0,1)} = (1,0),\quad &\vec{B}_{0,2}^{(0,1)} = (1,1),\\ &\vec{B}_{1,2}^{(1,2)} = (-1,0),\end{aligned}$$

$$\begin{aligned}\vec{B}_{0,1}^{(0,1)} = (1,0),\quad &\vec{B}_{0,2}^{(0,2)} = (0,1),\\ &\vec{B}_{1,2}^{(1,2)} = (-1,0).\end{aligned}$$

By Theorem 2.1

$$\vec{S}_0(RU;y) = \vec{B}_{0,1}^{(0,1)} + \frac{y-y_1}{\lceil \vec{B}_{0,2}^{(0,2)}} = (1,0) + \frac{y-1}{\lceil (0,1)} = (1, y-1),$$

$$\vec{S}_1(RU;y) = \vec{B}_{1,2}^{(1,2)} = (-1,0),$$

which results in

$$\vec{R}_2(RU;x,y) = \vec{S}_0(RU;y) + \frac{x-x_0}{\vec{S}_1(RU;y)}$$

$$= (1, y-1) + \frac{x}{(-1,0)} = (1-x, y-1).$$

Hence we finally obtain

$$\begin{aligned}\vec{R}_2(x,y) &= Q(x,y)\vec{R}_2(LB;x,y) + P(x,y)\vec{R}_2(RU;x,y)\\ &= (x+y-2)(x+y-1)\left[\frac{(3y-2y^2, y^2-y)}{(3-2y)^2+(y-1)^2}\right.\\ &\qquad\left. + \frac{(10x+20)(6x+y+10, 8x-2y+10)}{(6x+y+10)^2+(8x-2y+10)^2}\right]\\ &\quad + (x+y+2)(x+y+1)(x+y)(1-x, y-1).\end{aligned}$$

It is easy to verify that $\vec{R}_2(x,y)$ interpolates $\vec{V}^2$ over Π^2. In our example, the vector-grid $\vec{V}^2$ is ill-defined, and, what is more, as mentioned at the beginning of this section, in this case one fails to find a rational interpolant $\vec{r}_2(x,y)$ of the form (1.4). However, $\vec{R}_2(x,y)$, as a BCVRI defined in (2.11), still exists. Hence, compared with Algorithm 1.1, our new algorithm for BCVRI is more reliable in the sense that it can overcome the nonexistence of some $\vec{r}_n(x,y)$, and more economical in the sense that it involves fewer Samelson inverses.

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