

ON A CONJECTURE OF ERDŐS AND STEWART

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ABSTRACT. For any $k \geq 1$, let p_k be the k th prime number. In this paper, we confirm a conjecture of Erdős and Stewart concerning all the solutions of the diophantine equation $n! + 1 = p_k^a p_{k+1}^b$, when $p_{k-1} \leq n < p_k$.

1. INTRODUCTION

For any $k \geq 1$ let p_k be the k th prime number. From [3], we found out that Erdős and Stewart conjectured that the only solutions of the equation

$$(1) \quad n! + 1 = p_k^a p_{k+1}^b \quad \text{for some } a \geq 0, b \geq 0 \text{ and } p_{k-1} \leq n < p_k$$

are obtained for $n \leq 5$.

In this paper, we prove the following

Theorem. *Equation (1) has no solutions for $n \geq 6$.*

One can check that equation (1) has no solutions for $5 < n \leq 11$. From now on, we work with a potential solution of (1) with $n \geq 12$.

2. AN ELEMENTARY LEMMA

The following elementary result turns out to be helpful when searching for the values of n .

Lemma. *In equation (1), one has $ab \neq 0$.*

Proof of the Lemma. Assume that this is not so and write

$$(2) \quad n! + 1 = p^a \quad \text{for some } p \in \{p_k, p_{k+1}\}.$$

Let $a = 2^i a_1$ where $a_1 \geq 1$ is odd. Then,

$$(3) \quad \text{ord}_2(n!) = \text{ord}_2(p^a - 1) \leq \max(\text{ord}_2(p \pm 1)) + i \leq \log_2(p_{k+1} + 1) + \log_2(a).$$

From equation (2), we know that

$$(4) \quad n^a < p^a = n! + 1 < n^n,$$

therefore $a < n$. Since the interval $[n + 1, 2n]$ contains at least two primes for $n \geq 12$, we get $p_{k+1} + 1 \leq 2n$. Hence, inequality (3) implies

$$(5) \quad \text{ord}_2(n!) < \log_2(2n) + \log_2(n) = 2 \log_2(n) + 1.$$

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From Lemma 1 in [1], we know that

$$(6) \quad \text{ord}_2(n!) \geq n - \log_2(n + 1).$$

From inequalities (5) and (6), we get

$$(7) \quad n - \log_2(n + 1) < 2 \log_2(n) + 1,$$

which implies $n \leq 11$. This contradicts the assumption on $n \geq 12$.

3. A LINEAR FORM IN LOGARITHMS AND A BOUND ON n

Write

$$(8) \quad n! = p_k^a p_{k+1}^b - 1 = p_{k+1}^b \left(p_k^a - \left(\frac{1}{p_{k+1}} \right)^b \right).$$

We find an upper bound for $\text{ord}_2(n!)$. We apply Théorème 4 in [1] with the choices

$$\begin{aligned} p &= 2, \quad D = 1, \quad g = 1, \\ \alpha_1 &= p_k, \quad \alpha_2 = \frac{1}{p_{k+1}}, \quad b_1 = a, \quad b_2 = b, \\ A_1 &= p_k, \quad A_2 = p_{k+1} \end{aligned}$$

and

$$\mu = 15, \quad \nu = 10, \quad c(\mu, \nu) = 18.$$

From the result in [1], it follows that

$$(9) \quad \text{ord}_2(n!) \leq \frac{36}{(\log 2)^4} (\max\{\log b' + \log \log 2 + 0.4, 15 \log 2\})^2 \log p_k \log p_{k+1},$$

where

$$(10) \quad b' = \frac{a}{\log p_{k+1}} + \frac{b}{\log p_k}.$$

We now find a bound on b' in terms on n . Since

$$p_k^a p_{k+1}^b = n! + 1 < n^n,$$

it follows that

$$(11) \quad a \log p_k + b \log p_{k+1} = \log p_k^a p_{k+1}^b < \log n^n = n \log n.$$

Hence,

$$(12) \quad b' = \frac{a}{\log p_{k+1}} + \frac{b}{\log p_k} = \frac{a \log p_k + b \log p_{k+1}}{\log p_k \log p_{k+1}} < \frac{n \log n}{\log p_k \log p_{k+1}} < \frac{n}{\log n}.$$

Since the interval $[n + 1, 2n]$ contains at least two primes, it follows that $p_k < p_{k+1} < 2n$. Inequality (9) now implies

$$(13) \quad \text{ord}_2(n!) < \frac{36}{(\log 2)^4} \left(\max \left\{ \log \left(\frac{n}{\log n} \right) + \log \log 2 + 0.4, 15 \log 2 \right\} \right)^2 \log^2(2n).$$

When

$$\log \left(\frac{n}{\log n} \right) + \log \log 2 + 0.4 \leq 15 \log 2,$$

we get $n < 409\,506$. When

$$\log\left(\frac{n}{\log n}\right) + \log \log 2 + 0.4 > 15 \log 2,$$

we get, by inequalities (6) and (13), that

$$(14) \quad n - \log_2(n+1) < \frac{36}{(\log 2)^4} \left(\log\left(\frac{n}{\log n}\right) + \log \log 2 + 0.4 \right)^2 \log^2(2n),$$

which implies $n < 7\,242\,116$. The conclusion is that $n < p_k < p_{k+1} < 7.5 \cdot 10^6$.

4. THE REMAINING COMPUTATIONS

For the remaining computations, we used the following result due to Erdős and Obláth (see [2]).

Theorem EO. *The equation*

$$(15) \quad x^p \pm y^p = n!$$

has no solutions such that $p > 2$ is prime and $\gcd(x, y) = 1$.

Case 1. $n > 193$.

The idea here was to check, computationally, that if n leads to a solution of (1), then $a \equiv b \equiv 0 \pmod{3}$. Once we prove this, the impossibility of (1) follows from Theorem EO for $p = 3$.

Assume, for example, that (1) has a solution such that either $3 \nmid a$ or $3 \nmid b$. Write

$$(16) \quad n! + 1 = Ax^3 \quad \text{where } A = p_k^{\delta_1} p_{k+1}^{\delta_2} \text{ for some } \delta_1, \delta_2 \in \{0, 1, 2\} \text{ with } (\delta_1, \delta_2) \neq (0, 0).$$

Let $q \leq 193$ be a prime congruent to 1 modulo 3. Equation (1) implies that $Ax^3 \equiv 1 \pmod{q}$ for every such q . It now follows that A is a cubic residue modulo q for every $q \leq 193$ which is congruent to 1 modulo 3. Since a number y is a cubic residue modulo q if and only if y^2 is a cubic residue modulo q , it follows that we need to identify only those numbers A of the form

$$(17) \quad A = p_k \quad \text{or} \quad A = p_k p_{k+1} \quad \text{or} \quad A = p_k^2 p_{k+1}$$

in the range $193 < p_k < p_{k+1} < 7.5 \cdot 10^6$ which are cubic residues with respect to every prime $q \leq 193$ which is congruent to 1 modulo 3. Achim Flammenkamp wrote a computer program which checked in a few minutes that there are no such A 's. Hence, $n \leq 193$.

Case 2. $n \leq 193$.

By the Lemma, we know that if n leads to a solution of (1), then $ab > 0$. Achim Flammenkamp wrote another computer program which checked in less than a second that in this range $n! + 1 \not\equiv 0 \pmod{p_k p_{k+1}}$.

The Theorem is therefore proved.

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REFERENCES

- [1] Y. Bugeaud & M. Laurent, *Minoration effective de la distance p -adique entre puissances de nombres algébriques*, J. Number Theory **61** (1996), 311–342. MR **98b**:11086
- [2] P. Erdős & R. Obláth, *Über diophantische Gleichungen der Form $n! = x^p \pm y^p$ und $n! \pm m! = x^p$* , Acta Szeged **8** (1937), 241–255.
- [3] R. K. Guy, *Unsolved problems in number theory*, Springer-Verlag, New York, 1994, Problem **A2**. MR **96e**:11002

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