DENSITY OF CUBIC FIELD DISCRIMINANTS

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ABSTRACT. In this paper we give a conjectural refinement of the Davenport-Heilbronn theorem on the density of cubic field discriminants. Our refinement is plausible theoretically and agrees very well with computational data.

1. INTRODUCTION

Let a_n be the number of isomorphism classes of abelian cubic fields with discriminant n. Let b_n be the number of isomorphism classes of nonabelian cubic fields with discriminant n. The numbers a_n are very well understood. The numbers b_n have been the subject of extensive theoretical and computational study for at least sixty years, but are less well understood. The object of this note is to contribute to the study of these b_n by bringing together the theoretical and computational literature.

For $\alpha \in \{-, +\}$, define

$$(1.1) \qquad g_\alpha(x) = \sum_{n=1}^{x} b_{\alpha n}.$$

Put $C_- = 3$ and $C_+ = 1$. The main theoretical result is due to Davenport and Heilbronn [8]:

$$(1.2) \qquad g_\alpha(x) \sim C_\alpha \frac{1}{12\zeta(3)} x.$$

The numbers $b_{\alpha n}$ have been computed for larger and larger n by many authors. The most recent results are due to Belabas [3], who introduced a vastly improved method and worked through the cutoff $x = 10^{11}$.

It has been discussed by several authors that the "match" between computational evaluation of $g_\alpha(x)$ and the asymptotic formula (1.2) is not very good, as illustrated by Tables 1, 2, and 3 below. For example, the paper [9] dealt with the case $\alpha = -$; here is an excerpt from pages 322–324, with some trivial notational changes:

> If, however, the reader were to plot the density versus x, he would be somewhat astonished to see that this density is increasing so slowly that his first impression would be that it will not make it to the Davenport-Heilbronn limit. Thus it remains a challenging problem, assuming that the D-H limit is not in error, to explain the origin of this very slow convergence. This problem was indicated by Shanks in [14], [13], and on the real side in [14], [10], where the problem is further aggravated

by even slower convergence. To date, and to our knowledge, no good
quantitative explanation of this phenomenon has been given.

In this paper we conjecture a quantitative explanation of this slow-convergence
phenomenon.

2. The abelian case

It is best to briefly review the situation in the abelian case. One has the formula

$$(2.1) \qquad \sum_{n=1}^{\infty} \frac{a_n}{n^s} = -1 + \left(1 + \frac{2}{81^s}\right) \prod_{p \equiv 1 \ (6)} \left(1 + \frac{2}{p^{2s}}\right).$$

Put

$$(2.2) \qquad f(x) = \sum_{n=1}^{x} a_n.$$

Cohn [4] showed that the function (2.1) continues analytically to $\mathrm{Re}(s) > 1/4$,
except for a simple pole at $s = 1/2$. He computed the residue of this pole. This
gave him more than enough information to conclude that

$$(2.3) \qquad f(x) \sim \left(\frac{11\sqrt{3}}{36\pi} \prod_{p \equiv 1 \ (6)} \frac{(p+2)(p-1)}{p(p+1)}\right) x^{1/2}.$$

Define $h_-(x) = g_-(x)$ and

$$(2.4) \qquad h_+(x) = g_+(x) + \frac{1}{3}f(x).$$

From (1.2) and (2.3) one sees that (1.2) would remain true with g_α replaced by h_α.
We henceforth focus more on h_α, as it appears naturally in (3.2) and (3.3).

3. The basic conjecture

Belabas proved [3, (8) and (20)] that, for either α, the difference between the
sides of (1.2) is $o(x/(\log x)^2)$. From his extensive computational data he speculated
[3, page 1232] that these differences are $o(x/(\log x)^\beta)$ for arbitrarily large β. We
believe a stronger statement. Recall that $C_- = 3$ and $C_+ = 1$. Put $K_- = \sqrt{3}$ and
$K_+ = 1$.

Conjecture 3.1. *For each α,*

$$(3.1) \qquad h_\alpha(x) = C_\alpha \frac{1}{12\zeta(3)}x + K_\alpha \frac{4\zeta(1/3)}{5\Gamma(2/3)^3\zeta(5/3)}x^{5/6} + o(x^{5/6}).$$

Numerically, (3.1) works out to

$$\begin{aligned}
h_-(x) &= (0.2079768431\ldots)x - (0.2557983756\ldots)x^{5/6} + o(x^{5/6}), \\
h_+(x) &= (0.0693256143\ldots)x - (0.1476852610\ldots)x^{5/6} + o(x^{5/6}).
\end{aligned}$$

Note that the coefficient of $x^{5/6}$ is negative in each case, the negative sign coming
from $\zeta(1/3) < 0$. This negativity corresponds to the fact that computationally
one has always found fewer fields than the Davenport-Heilbronn asymptotic would
suggest.

The heuristic we are about to sketch is too crude to suggest any particular
error term in (3.1). The $o(x^{5/6})$ we have written there corresponds to the weakest

conjecture in which our conjectured new term actually plays a role. However, Tables 1 and 2 below suggest that the error term should be reducible at least most of the way to $o(x^{1/2})$.

We are led to our conjecture theoretically as follows. Define a Dirichlet series

$$(3.2) \qquad \xi_\alpha(s) \quad = \quad \sum_{k \in \mathcal{K}_\alpha} \frac{1}{|\operatorname{Aut}(k)||D_k|^s}.$$

Here $\mathcal{K}_\alpha$ is the set of isomorphism classes of cubic fields k with discriminant of sign α. Also, D_k is the discriminant of k while $\operatorname{Aut}(k)$ is the automorphism group of k; so $|\operatorname{Aut}(k)|$ is 3 or 1 according to whether k is abelian or not, the former case occuring only for $\alpha = +$.

For S a finite set of primes, define a weighted version of (3.2):

$$(3.3) \qquad \xi_{\alpha,S}(s) = \sum_{k \in \mathcal{K}_\alpha} \frac{\eta_{k,S}(s)}{|\operatorname{Aut}(k)||D_K|^s}.$$

Here the weights $\eta_{k,S}(s)$ are given in [6, page 122]. These weights are themselves Dirichlet series $\sum c_m(k,S)m^{-s}$ with integral coefficients $c_m(k,S)$ decreasing to the Kronecker symbol δ_{m1} as S grows. Thus $\xi_{\alpha,S} \to \xi_\alpha$ in the sense of formal Dirichlet series.

The point of introducing the approximations $\xi_{\alpha,S}$ is that one understands them analytically better than one understands ξ_α itself. From Davenport's earlier work [7] one knows that the right-most pole of $\xi_{\alpha,\emptyset}$ is at $s = 1$. The analytic behavior of $\xi_{\alpha,\emptyset}$ was investigated more explicitly by Shintani [15], who found the next pole at $s = 5/6$. More generally, it is proved in [5, Prop. 6.2], that each $\xi_{\alpha,S}$ continues to the whole s-plane, meromorphically with only poles at $s = 1$ and $s = 5/6$. These poles are simple, their residues $C_\alpha r_{1,S}$ and $K_\alpha r_{5/6,S}$ being deducible from the first two equations in [5, Thm. 6.2i]. In the case $s = 1$, one has to set b_α equal to zero in the corresponding equation; this b_α corresponds to quadratic fields, included in $\xi_{\alpha,S}$ in [5] and [6], but not included here.

From the formulas in [5, Thm. 6.2i], one sees that the numbers $r_{1,S}$ are positive, and decreasing as S grows. Similarly, the numbers $r_{5/6,S}$ are negative and increasing as S grows. Let r_1 and $r_{5/6}$ be their respective limits. One does not have enough control at present to rigorously apply any "poles-control-growth" theorems. However the general formalism suggests that $h_\alpha(x)$ grows as $C_\alpha r_1 x + K_\alpha(6/5)r_{5/6}x^{5/6}$. This is the source of our Conjecture 3.1. The constant $r_{5/6}$ is actually nicer than we had initially expected from its source as the limit of the $r_{5/6,S}$; special zeta values arise because of the factorization of the right side of (5.3) below.

To follow our heuristic in a more detailed way, one should first understand it in the more refined context of Section 5 below. We recommend looking next at Prop. 2.1 and the discussion around it in [6]; this treats in more classical language the special case of [5, Thm 6.2i] we need, except it does not repeat the formula for $r_{5/6,S}$.

4. Numerical evidence for the basic conjecture

Our conjecture for $\alpha = -$ compares with Belabas's data [3, page 1232] for negative-discriminant cubic fields as in Table 1. Here and below, $x = 10^j$. Also $H_\alpha(x)$ denotes the right side of (1.2) while $H_\alpha^*(x)$ denotes the first two terms of the right side of (3.1). Clearly H_-^* matches the data h_- substantially better than H_-

TABLE 1. Imaginary cubic fields: Computation h_- and theory H_-^*

j	$h_-(x)$	$\dfrac{h_-(x)}{H_-(x)}$	$\dfrac{h_-(x)}{H_-^*(x)}$	$\dfrac{h_-(x) - H_-^*(x)}{x^{1/2}}$
2	7	0.337	0.7843510	-0.1925
3	127	0.611	0.9993210	-0.0027
4	1,520	0.731	0.9943300	-0.0867
5	17,041	0.819	0.9998781	-0.0066
6	182,417	0.877	1.0001096	0.0200
7	1,905,514	0.916	1.0000100	0.0060
8	19,609,185	0.943	0.9999394	-0.1188
9	199,884,780	0.961	0.9999850	-0.0951
10	2,024,660,098	0.974	1.0000009	0.0176
11	20,422,230,540	0.982	1.0000003	0.0218

TABLE 2. Real cubic fields: Computation h_+ and theory H_+^*

j	$g_+(x)$	$f(x)$	$\dfrac{h_+(x)}{H_+(x)}$	$\dfrac{h_+(x)}{H_+^*(x)}$	$\dfrac{h_+(x) - H_+^*(x)}{x^{1/2}}$	$\dfrac{g_+(x) - H_+^*(x)}{x^{1/2}}$
2	0	2	0.096	8.5889786	0.0589	-0.0078
3	22	5	0.341	1.0461129	0.0330	-0.0197
4	366	16	0.536	0.9900166	-0.0374	-0.0908
5	4,753	51	0.688	1.0010833	0.0163	-0.0374
6	54,441	159	0.786	0.9988436	-0.0631	-0.1161
7	592,421	501	0.855	0.9999134	-0.0162	-0.0690
8	6,246,698	1,592	0.901	1.0000259	0.0161	-0.0369
9	64,654,353	5,008	0.933	1.0000097	0.0198	-0.0330
10	661,432,230	15,851	0.954	0.9999988	-0.0081	-0.0609
11	6,715,773,873	50,152	0.969	1.0000002	0.0046	-0.0482

does. The poor match of H_- to h_- in the range $10^5 \leq x \leq 10^6$ formed the basis of the quote in Section 1.

For $\alpha = +$, we start from the values of $g_+(x) + f(x)$ listed in [3, page 1232]. We use (2.1) to compute $f(x)$ and thereby deduce $h_+(x) = g_+(x) + (1/3)f(x)$. Our hope that the error term in Conjecture 3.1 can be reduced to $o(x^{1/2})$ would require that the numbers in the last column of Table 1 and the next to last column of Table 2 tend to zero. The data seems rather borderline in support of $o(x^{1/2})$, but strongly indicative that our conjecture as stated is true, and moreover that the $o(x^{5/6})$ can be reduced at least somewhat.

From Cohn's result (2.3), at most one linear combination $g_+(x) + uf(x)$ can agree with $H_+^*(x)$ to within $o(x^{1/2})$. The last two columns of Table 2 suggest that $u = 1/3$ is more likely than $u = 0$. In fact $u = 0.31\ldots$ would give the best fit, in a least squares sense, and this agrees well with the theoretically favored $u = 1/3$.

5. The refined conjecture

The Davenport-Heilbronn theorem comes in a more refined version and so does our conjecture. Let A be the set of symbols $\{111, 21, 3, 1^2 1, 1^3\}$. Then for K a cubic field and p a prime, the splitting behavior of p in K determines an element

of A. If p is unramified, then it either splits completely (111), splits partially (21), or remains inert (3). If p is ramified, then it is either partially ramified ($1^2 1$) or totally ramified (1^3).

Until now, we have been using α to denote a sign, $+$ or $-$. Henceforth, we use α to denote the following data. First, a finite set $\text{support}(\alpha) \subset \{\infty, 2, 3, 5, \dots\}$ of places of $\mathbf{Q}$, containing ∞. Second, a sign $\alpha_\infty \in \{+, -\}$, as before. Third, for each prime $p \in \text{support}(\alpha)$, an element $\alpha_p \in A$. Let $\mathcal{K}_\alpha$ be the set of isomorphism classes of cubic fields meeting the given local condition α_v for each $v \in \text{support}(\alpha)$. One naturally has f_α, g_α, h_α, ξ_α, and $\xi_{\alpha,S}$, all direct generalizations of the notions introduced above in the special case $\text{support}(\alpha) = \{\infty\}$.

Define $C_{\infty,-} = 3$ and $C_{\infty,+} = 1$, as in Section 1 and $K_{\infty,-} = \sqrt{3}$ and $K_{\infty,+} = 1$, as in Section 3. Define p-adic analogs as follows.

(5.1)

α_p	C_{p,α_p}	K_{p,α_p}
111	$1/6C_p$	$(1+p^{-1/3})^3/6K_p$
21	$1/2C_p$	$(1+p^{-1/3})(1+p^{-2/3})/2K_p$
3	$1/3C_p$	$(1+p^{-1})/3K_p$
$1^2 1$	$1/pC_p$	$(1+p^{-1/3})^2/pK_p$
1^3	$1/p^2 C_p$	$(1+p^{-1/3})/p^2 K_p$

Here

$$(5.2) \qquad C_p \;=\; 1 + p^{-1} + p^{-2},$$

$$(5.3) \qquad K_p \;=\; \frac{(1-p^{-5/3})(1+p^{-1})}{1-p^{-1/3}}$$

are normalizing factors, chosen so that the column-sums $\sum C_{p,\alpha_p}$ and $\sum K_{p,\alpha_p}$ are both 1. The values of C_{p,α_p} are taken from, e.g., [10, page 589] while the K_{p,α_p} are taken from [5, Table 5.1]. Note that both the C_{p,α_p} and the K_{p,α_p} are actually given by procedures uniform in α_p. For example, the ratios $(K_{p,\alpha_p} K_p)/(C_{p,\alpha_p} C_p)$ are all given by replacing a factor f^e in α_p by a factor $(1+p^{-f/3})$, e.g., $1^2 1 \mapsto (1+p^{-1/3})(1+p^{-1/3})$.

We expect that Conjecture 3.1 holds verbatim for these more general α, with

$$(5.4) \qquad C_\alpha \;=\; \prod_{v \in \text{support}(\alpha)} C_{v,\alpha_v},$$

$$(5.5) \qquad K_\alpha \;=\; \prod_{v \in \text{support}(\alpha)} K_{v,\alpha_v}.$$

Our heuristic continues to make sense in this more localized context. In fact the basic references [5] and [6] are in this context.

6. Numerical evidence for the refined conjecture

In this section, we work with twenty different α. All of them have support $\{\infty, p\}$ and $\alpha_\infty = +$. We let p run over $\{2, 3, 5, 7\}$ and let α_p run over the five-element set A. For simplicity, we compare computation with theory only at $x = 10^7$.

We get the numbers $g_\alpha(10^7)$ by renormalizing the first four lines of [10, Table 11]. We get the numbers $f_\alpha(10^7)$ by a simple procedure based on (2.1). Including these extra 501 fields, with total mass $501/3 = 167$, improves the fit slightly, as would be expected from line $j = 7$ of Table 2.

TABLE 3. Real cubic fields and their p-adic decomposition: Computation h_α and theory H_α^* (x fixed at 10^7)

α_p	$p=2$		$p=3$		$p=5$		$p=7$	
	$\dfrac{h_\alpha(x)}{H_\alpha(x)}$	$\dfrac{h_\alpha(x)}{H_\alpha^*(x)}$	$\dfrac{h_\alpha(x)}{H_\alpha(x)}$	$\dfrac{h_\alpha(x)}{H_\alpha^*(x)}$	$\dfrac{h_\alpha(x)}{H_\alpha(x)}$	$\dfrac{h_\alpha(x)}{H_\alpha^*(x)}$	$\dfrac{h_\alpha(x)}{H_\alpha(x)}$	$\dfrac{h_\alpha(x)}{H_\alpha^*(x)}$
111	0.706	0.9999	0.721	0.9998	0.734	0.9994	0.740	0.9994
21	0.851	0.9999	0.856	1.0001	0.858	0.9996	0.858	0.9997
3	0.924	1.0001	0.923	0.9999	0.920	1.0002	0.917	1.0008
$1^2 1$	0.836	0.9996	0.835	0.9997	0.832	1.0002	0.829	0.9985
1^3	0.909	1.0003	0.903	0.9998	0.895	1.0006	0.891	1.0025

Table 3 gives the comparison. This table illustrates how the slowness of the convergence $h_\alpha(x)/H_\alpha(x)$ to 1 seems to depend systematically on α_p, the totally split case $\alpha_p = 111$ being the slowest. On the other hand, $h_\alpha(x)/H_\alpha^*(x)$ is much closer to 1, with no apparent tendency to be above or below.

7. THREE CONCLUDING REMARKS

By using the five element set A, we are falling slightly short of the most locally refined conjecture, because there are different ways a prime can be ramified. Thus, for example, if a field K is 1^3 at $p = 3$, then there are actually nine possibilities for its 3-adic completion K_3; both $C_{3,1^3}$ and $K_{3,1^3}$ should be split into nine parts, in the proportions indicated in the $(p, \alpha_p) = (3, 1^3)$ slot in the following chart.

(7.1)

α_p	$p=2$	$p=3$	$p \geq 5$
$1^2 1$	$\frac{1}{4}\frac{1}{4}\frac{1}{8}\frac{1}{8}\frac{1}{8}\frac{1}{8}$	$\frac{1}{2}\frac{1}{2}$	$\frac{1}{2}\frac{1}{2}$
1^3	1	$\frac{1}{3}\frac{1}{3}\frac{1}{9}\frac{1}{81}\frac{1}{81}\frac{1}{81}\frac{1}{81}\frac{1}{81}\frac{1}{81}$	1 or $\frac{1}{3}\frac{1}{3}\frac{1}{3}$

Similarly, if a field K is 1^3 at $p \geq 5$, there are either 1 or 3 possibilities for K_3, according to whether $p \equiv 5$, or 1 modulo 6. In general, a p-adic algebra K_p of type α_p contributes to C_{p,α_p} and K_{p,α_p} in proportion to $1/(|\operatorname{Aut}(K_p)||D_{K_p}|)$.

Our basic theoretical references [5] and [6] work over arbitrary number fields as well as finite extensions of $\mathbf{F}_p(t)$ for $p \geq 5$. We expect that the analog of our fully refined Conjecture 3.1 holds there too, but at present there is no corresponding published computational data.

The most important remaining problem is to prove our conjecture. The generally pessimistic discussion in [6, page 124] and [2, page 620] suggests to us that the way may be difficult. However, one ingredient of a proof might be the functional equation of $\xi_{\alpha,S}$ with respect to $s \mapsto 1 - s$, studied in [15], [5], and [11]. Another ingredient might be [12, Thm. 3], which concerns growth of arithmetic functions whose associated Dirichlet series satisfy such a functional equation.

REFERENCES

[1] I. O. Angell, *A table of totally real cubic fields*, Math. Comput. **30** (1976), 184-187. MR **53:**5528

[2] K. Belabas, *Crible et 3-rang des corps quadratiques*, Ann. Inst. Fourier (Grenoble), **46** (1996), 909-949. MR **98b:**11112

[3] _____, *A fast algorithm to compute cubic fields*, Math. Comp. **66** (1997), 1213-1237. MR **97m:**11159

[4] H. Cohn, *The density of abelian cubic fields*, Proc. Amer. Math. Soc. **5** (1954), 476-477. MR **16:**222a

[5] B. Datskovsky and D. J. Wright, *The adelic zeta function associated with the space of binary cubic forms, II: local theory*, J. Reine Angew. Math. **367** (1986), 27-75. MR **87m:**11034

[6] ———, *Density of discriminants of cubic extensions*, J. Reine Angew. Math. **386** (1988) 116-138. MR **90b:**11112

[7] H. Davenport, *On the class number of binary cubic forms, I & II.*, J. London Math. Soc. **26** (1951), 183-198 (Corrigendum, ibid **27** (1952) 512). MR **13:**323e, MR **13:**323f

[8] H. Davenport and H. Heilbronn, *On the density of discriminants of cubic fields (II)*, Proc. Roy. Soc. Lond. A **322** (1971) 405-420 MR **58:**10816

[9] G. W. Fung and H. G. Williams, *On the computation of complex cubic fields, with discriminant $D \geq -10^6$*, Math. Comp. **55** (1990), 313-325. Errata Math. Comp. **63** (1994), 433. MR **94i:**11106

[10] P. Llorente and J. Quer, *On totally real cubic fields with discriminant $d < 10^7$*, Math. Comp. **50** (1988), 581-594. MR **89g:**11099

[11] Y. Ohno, *A conjecture on coincidence among the zeta functions associated with the space of binary cubic forms*, Amer. J. of Math. **119** (1997), 1083-1094. MR **98k:**11037

[12] M. Sato and T. Shintani, *On zeta functions associated with prehomogeneous vector spaces*, Ann. of Math. **100** (1974), 131-170. MR **49:**8969

[13] D. Shanks, *Review of* [1], Math. Comp. **30**, (1976), 670-673.

[14] ———, *A survey of quadratic, cubic and quartic algebraic number fields* (*from a computational point of view*), Cong. Numer. **17** (1976), 15–40. MR **56:**11951

[15] T. Shintani, *On Dirichlet series whose coefficients are class numbers of integral binary cubic forms*, J. Math. Soc. Japan, **24** (1972), 132-188. MR **44:**6619

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