

LOCALIZATION OF THE FIRST ZERO OF THE DEDEKIND ZETA FUNCTION

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ABSTRACT. Using Weil's explicit formula, we propose a method to compute low zeros of the Dedekind zeta function. As an application of this method, we compute the first zero of the Dedekind zeta function associated to totally complex fields of degree less than or equal to 30 having the smallest known discriminant.

1. INTRODUCTION

Let K be a number field of degree n , of signature (r_1, r_2) and of discriminant d_K . The Dedekind zeta function of K is denoted by $\zeta_K(s)$. Currently there are no efficient methods to compute high zeros of Dedekind zeta functions of general non-abelian fields. For low zeros there is a method of Friedman [F] implemented by Tollis [T], but computations are restricted to number fields of small degree ($n \leq 6$). In this paper we show how to compute low zeros of the Dedekind zeta function by using Weil's explicit formula and its derivative. Actually, we compute the first zero by reversing the known Serre-Odlyzko construction to bounding discriminants when the field's discriminant is known. This method requires one to compute norms of prime ideals. For that purpose we use efficient algorithms to decompose the minimal polynomial modulo rational primes. As an application of this method, we estimate the first zero of the Dedekind zeta function of totally complex fields of degree less than 30 having the smallest known discriminant [CDO].

The functions $F(x)$ used in our computations are Serre's, which in fact give discriminant bounds less good than Odlyzko's but have the interesting property that their Fourier transforms $\hat{F}(x)$ decrease very rapidly and monotonically as x tends to $+\infty$. Therefore the sum running over the non-trivial zeros in Weil's explicit formula is concentrated on the very low zeros of $\zeta_K(s)$. We look then to find out how to separate these zeros and estimate the first one with a given precision.

2. WEIL'S IDENTITY

Let F be a real function of a real variable which we can suppose even and verifying the following conditions (A) and (B) [L]:

(A) F is continuous and continuously differentiable everywhere except at a finite number of points a_i , where $F(x)$ and $F'(x)$ have only a discontinuity of first kind, such that $F(a_i) = \frac{1}{2}(F(a_i + 0) + F(a_i - 0))$.

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(B) There is a number $b > 0$ such that $F(x)$ and $F'(x)$ are $O(e^{-(\frac{1}{2}+b)|x|})$ as $|x| \rightarrow \infty$.

Then the Mellin transform of F ,

$$\Phi(s) = \int_{-\infty}^{+\infty} F(x)e^{(s-\frac{1}{2})x} dx,$$

is holomorphic in every strip $-a \leq \sigma \leq 1+a$, where $0 < a < b$, $a < 1$, and we have the result established by Weil [Pt1], [Pt2]:

Theorem 1 (Weil). *Let F satisfy the conditions (A) and (B) above. Then the sum $\sum \Phi(\rho)$ running over the non-trivial zeros $\rho = \beta + i\gamma$ of $\zeta_K(s)$ with $|\gamma| < T$ tends to a limit as T tends to infinity, and this limit is given by the formula*

$$(1) \quad \sum_{\rho} \Phi(\rho) = \Phi(0) + \Phi(1) - 2 \sum_{\mathfrak{p}, m} \frac{\ln(N(\mathfrak{p}))}{N(\mathfrak{p})^{\frac{m}{2}}} F(m \ln(N(\mathfrak{p}))) + F(0) [\ln(|d_K|) - n \ln(2\pi)] - r_1 J(F) - nI(F),$$

where

$$J(F) = \int_0^{+\infty} \frac{F(x)}{2 \operatorname{ch}(\frac{x}{2})} dx \quad \text{and} \quad I(F) = \int_0^{+\infty} \left(\frac{F(x)}{2 \operatorname{sh}(\frac{x}{2})} - \frac{e^{-x}}{x} \right) dx.$$

Consequences. If we consider positive functions depending on a positive parameter y with $F(0) = 1$ and for which the Fourier transform is also positive, we get the inequality

$$(2) \quad \ln(|d_K|) \geq -\Phi(0) - \Phi(1) + n(\ln(2\pi) + I(F)) + r_1 J(F).$$

In the case where F is Serre's or Odlyzko's function, we determine the optimal value of y which gives the best lower bound for $\ln(|d_K|)$, obtaining the inequalities of Odlyzko and Serre.

3. COMPUTATION OF THE FIRST ZERO OF THE ZETA FUNCTION

Let us consider the identity (1) for the functions $F_y(x) = e^{-yx^2}$, where $y > 0$. The Mellin transform $\Phi_y(s)$ of F_y is

$$\Phi_y(s) = \sqrt{\frac{\pi}{y}} e^{\frac{1}{4y}(s-\frac{1}{2})^2},$$

and the Fourier transform φ_y of F_y is

$$\varphi_y(t) = \sqrt{\frac{\pi}{y}} e^{-\frac{1}{4y}t^2}.$$

If we assume the Generalized Riemann Hypothesis (GRH) for ζ_K , we have $\Phi_y(\rho) = \varphi_y(t)$, where $\rho = \frac{1}{2} + it$. For every $k \geq 1$, we denote by t_k the positive imaginary part of the k th zero of the Dedekind zeta function. Therefore we have the identity

$$\sum_{\rho} \Phi_y(\rho) = \sqrt{\frac{\pi}{y}} : \sum_{t=\Im m(\rho)} e^{-\frac{t^2}{4y}} = 2\sqrt{\frac{\pi}{y}} e^{\frac{1}{16y}} - 2 \sum_{\mathfrak{p}, m} \frac{\ln(N(\mathfrak{p}))}{N(\mathfrak{p})^{\frac{m}{2}}} e^{-y(m \ln(N(\mathfrak{p})))^2} + [\ln(|d_K|) - n \ln(2\pi)] - r_1 J(y) - nI(y),$$

where

$$J(y) = \int_0^{+\infty} \frac{e^{-yx^2}}{2 \operatorname{ch}(\frac{x}{2})} dx \quad \text{and} \quad I(y) = \int_0^{+\infty} \left(\frac{e^{-yx^2}}{2 \operatorname{sh}(\frac{x}{2})} - \frac{e^{-x}}{x} \right) dx.$$

In section 4, we will show how to compute the integrals $J(y)$, $I(y)$, and the sum

$$v(y) = \sum_{\mathfrak{p}, m} \frac{\ln(N(\mathfrak{p}))}{N(\mathfrak{p})^{\frac{m}{2}}} e^{-y(m \ln(N(\mathfrak{p})))^2}$$

with a given precision. This allows us to compute

$$S(y) = \sum_{t \in \Im m(\rho)} e^{-\frac{t^2}{4y}}$$

for every $y > 0$, as well as its derivative

$$S'(y) = \sum_{t \in \Im m(\rho)} \frac{t2}{4y^2} e^{-\frac{t^2}{4y}}.$$

One can easily see that $t_1 > 0$ if and only if $S(y) < 1$ for some $y > 0$. Therefore $S(y)$ can be written $S(y) = 2 \sum_{k=1}^{+\infty} e^{-t_k^2/4y}$, which tends to 0 as y tends to 0.

Let us start by supposing that there exists $\beta > 0$ such that $S(\beta) < 1$ (i.e. $t_1 > 0$). To compute the first zero of $\zeta_K(s)$ on the critical line, we proceed in the following manner:

1. **Lower bound for the first zero.** We use the following trivial bound:

Lemma 1. *For every $y > 0$ we have*

$$t_1 > \sqrt{\max\left(-4y \ln\left(\frac{S(y)}{2}\right), 0\right)}.$$

In practice we search for the value of y giving the best lower bound t_{1min} of t_1 .

2. **Upper bound for the first zero.** Let us define

$$H(y) = 4S'(y) - S(y) = 2 \sum_{k=1}^{+\infty} \left(\frac{t_k^2}{y^2} - 1\right) e^{-\frac{t_k^2}{4y}}.$$

To get an upper bound for t_1 , we use the lemma:

Lemma 2. *There exists $y_0 > 0$ such that for every $y \geq y_0$ we have $H(y) \leq 0$. It follows from the definition of $H(y)$ that $t_1 < y_0$.*

Proof. This is a consequence of the asymptotic behaviour of $S(y)$ and $S'(y)$:

$$\begin{aligned} \lim_{y \rightarrow 0^+} S(y) &= 0^+, & \lim_{y \rightarrow +\infty} S(y) &= +\infty, \\ \lim_{y \rightarrow 0^+} S'(y) &= 0^+, & \lim_{y \rightarrow +\infty} S'(y) &= 0^+. \end{aligned}$$

Therefore $\lim_{y \rightarrow 0^+} H(y) = 0^+$ and $\lim_{y \rightarrow +\infty} H(y) = -\infty$, proving lemma 2. We take for t_{1max} the value of y that is a solution of $H(y) = 0$.

3. **Improvement of the upper bound and separation of the zeros.** Let us consider, for every $\alpha > 0$,

$$M_\alpha(z) = 2 \sum_{k=1}^{+\infty} \left(\frac{t_k^2}{z^{16}} - 1\right) e^{-\alpha \frac{t_k^2}{z}}.$$

We have the following lemmas:

Lemma 3. *i) For every $z > 0$, $M_\alpha(z)$ can be written in terms of S and S' in the form*

$$M_\alpha(z) = \frac{1}{4\alpha^2 z^{14}} S'\left(\frac{z}{4\alpha}\right) - S\left(\frac{z}{4\alpha}\right).$$

ii) For every value of α there exists a constant $C_\alpha > 0$ such that $t_1 < C_\alpha$ (C_α is the solution of $M_\alpha(C_\alpha) = 0$).

The lemma below gives a lower bound for the second zero of ζ_K as a function of C_α .

Lemma 4. *We have the inequality*

$$t_2 > \sqrt{\max\left(-4y \ln\left(\frac{S(y)}{2} - e^{-\frac{C_\alpha^2}{4y}}\right), 0\right)}.$$

This inequality arises by ignoring all but the first two terms in $S(y)$.

In practice, we begin by computing the best constant C_α given by lemma 3; then we search for the optimal value of y which gives a lower bound of t_2 greater than C_α .

4. Improvement of the range for the first zero. An upper bound for t_2 can be obtained by the determination of the root of the function

$$H(y) - 2\left(\frac{t_{1min}^2}{y^2} - 1\right) e^{-\frac{t_{1min}^2}{4y}} = 0$$

where $y > t_{1max}$.

This upper bound of t_2 can be used to improve the value of t_{1min} given by lemma 1, but in practice, if we are only interested in slight precision for t_1 (only one or two decimal digits), the improvement of t_{1min} seems to be unnecessary because the desired precision can be obtained by simple improvement of the value of t_{1max} . This can be done in the following manner.

We define the sequences (b_n) and (c_n) by the recurrent formula

$$\begin{cases} b_0 = t_{1max} = C_\alpha, \\ c_0 = t_{2min} = \sup_{y>0} \sqrt{\max\left(-4y \ln\left(\frac{S(y)}{2} - e^{-\frac{C_\alpha^2}{4y}}\right), 0\right)} \end{cases}$$

and, for every integer n ,

$$\begin{cases} b_{n+1} = \inf_{y_0>y>0} \sqrt{-4y \ln\left(\frac{S(y)}{2} - e^{-\frac{c_n^2}{4y_0}\left(\frac{y_0}{y} - 1\right)\frac{S(y_0)}{2}}\right)} \\ c_{n+1} = \sup_{y>0} \sqrt{\max\left(-4y \ln\left(\frac{S(y)}{2} - e^{-\frac{b_{n+1}^2}{4y}}\right), 0\right)} \end{cases}.$$

We should remark that the formula for b_{n+1} arises by comparing $S(y)$ and $S(y_0)$ term by term and using the fact that, for $0 < y < y_0$ and $k \geq 2$,

$$e^{-\frac{c_n^2}{4y} + \frac{c_n^2}{4y_0}} e^{-\frac{t_k^2}{4y_0}} \geq e^{-\frac{t_k^2}{4y}}.$$

One can prove the lemma below:

Lemma 5. *i) For every integer $n \geq 1$, we have $t_1 \leq b_n$.
 ii) For every integer $n \geq 1$, we have $c_n \leq t_2$.*

Now, if we have $t_1 = 0$, that means in practice that $S(y) > 1$ for small values of y . In this case we don't apply lemma 1, we just find an upper bound near 0 for t_1 by combining $S(y)$ and $S'(y)$ as in lemma 2 and lemma 3.

The method we have just explained, allowing us to approximate the first zero, can be generalized for the computation of the other zeros (t_2, t_3, \dots) .

4. COMPUTATION OF $S(y)$

In this section we give a method to compute the different terms of $S(y)$, i.e., $J(y)$, $I(y)$ and $v(y)$.

4.1. **Computation of $J(y)$.** Let us write $G_y(x) = \frac{e^{-yx^2}}{2\text{ch}(\frac{x}{2})}$; then we have:

Lemma 6.

$$\lim_{h \rightarrow 0^+} \lim_{N \rightarrow +\infty} h \left[\frac{G_y(0)}{2} + \sum_{q=1}^N G_y(qh) \right] = \int_0^{+\infty} G_y(x) dx.$$

Proof. We apply Poisson's equality to the function defined by $x \mapsto G_y(hx)$, where $h > 0$.

In practice we consider two arbitrary sequences (N_i) and (h_i) such that $h_i N_i \rightarrow +\infty$ and $h_i \rightarrow 0$; we will take for example $N_i h_i^2 = \text{const}$. When this is the case, the following lemma [DaR] shows that the series decreases as $e^{-\text{const}\sqrt{N}}$.

Lemma 7 (Martensen). *Let*

$$E_h(G_y) = \int_0^{+\infty} G_y(x) dx - h \left[\frac{G_y(0)}{2} + \sum_{q=1}^N G_y(qh) \right].$$

Then there exists $s_0 > 0$ such that:

$$|E_h(G_y)| \leq \frac{1}{e^{\frac{2\pi s_0}{h}} - 1} \int_{-\infty + is_0}^{+\infty + is_0} |G_y(z)| dz.$$

4.2. **Computation of $I(y)$.** Let us consider the function \mathcal{G}_y defined by

$$\mathcal{G}_y(x) = \frac{1 - e^{-yx^2}}{2 \text{sh}(\frac{x}{2})}.$$

$I(y)$ can be decomposed as $I(y) = I_2 - I_1(y)$, where

$$I_1(y) = \int_0^{+\infty} \mathcal{G}_y(x) dx \quad \text{and} \quad I_2 = \int_0^{+\infty} \left(\frac{1}{2 \text{sh}(\frac{x}{2})} - \frac{e^{-x}}{x} \right) dx.$$

The value of I_2 can be computed by the Gauss formula [GR], which gives $I_2 = \gamma + 2 \ln(2)$. For $I_1(y)$ we write

$$I_1(y) = \underbrace{\int_0^1 \mathcal{G}_y(x) dx}_{K_1(y)} + \underbrace{\int_1^{+\infty} \frac{1}{2 \text{sh}(\frac{x}{2})} dx}_{K_2} - \underbrace{\int_1^{+\infty} \frac{e^{-yx^2}}{2 \text{sh}(\frac{x}{2})} dx}_{K_3(y)}.$$

We have $K_2 = \ln(\coth \frac{1}{4})$. To compute $K_3(y)$, we apply again lemma 6, which remains true for the function

$$G_y(x) = \frac{e^{-y(x+1)^2}}{2 \operatorname{sh}(\frac{x+1}{2})}.$$

For the computation of $K_1(y)$, we use the Clenshaw and Curtis method [DaR] based on the Fourier-Tschebyscheff expansion of G_y on the interval $[-1, 1]$.

4.3. Computation of $v(y)$. In practice, we compute

$$v_{p_0}(y) = \sum_{p \leq p_0} \sum_{\mathfrak{p} | (p)} \ln(N(\mathfrak{p})) \sum_{\substack{m \\ m \ln(N(\mathfrak{p})) \leq \text{cons}}} \frac{e^{-y(m \ln(N(\mathfrak{p})))^2}}{N(\mathfrak{p})^{\frac{m}{2}}},$$

where $\text{cons} = \sqrt{\frac{c \ln(10)}{y}}$. The number field being defined by a polynomial $P(x)$, for every prime number p prime to the index of the field, the decomposition of the ideal (p) into a product of prime ideals of the field is given by the decomposition of $P(x)$ modulo p [Co]. In the case where p divides the index, we use a stronger algorithm (see algorithm 6.2.5 in [Co]).

The condition $m \ln(N(\mathfrak{p})) < \text{cons}$ means that we don't take into account the terms of the series less than 10^{-c} . Generally, we take $c = 30$, and p_0 is the greatest prime number less than 10^7 (see section 5).

The error term

$$r_{p_0}(y) = v(y) - v_{p_0}(y)$$

is estimated using the following lemmas:

Lemma 8. *Let n be $[K : \mathbb{Q}]$. Then*

$$r_{p_0}(y) \leq r_0(n) + r_\infty(n),$$

where

$$r_0(n) = n^2 10^{-c} \sum_{p \leq p_0} \sum_{\substack{m \\ m \ln(p) > \frac{\text{cons}}{n}}} \frac{\ln(p)}{p^{\frac{m}{2}}},$$

and

$$r_\infty(n) = n^2 \sum_{p > p_0} \sum_{m \geq 1} \ln(p) \frac{e^{-y(m \ln(p))^2}}{p^{\frac{m}{2}}}.$$

Lemma 9 (Estimation of $r_0(n)$). *For every integer $p_0 \geq 2$ we have*

$$r_0(n) \leq 8 \ln(2) n^2 10^{-c} e^{-\frac{\text{cons}}{2n}} p_0.$$

Proof. We have

$$r_0(n) \leq n^2 10^{-c} \sum_{p \leq p_0} \frac{\ln(p)}{p^{\frac{g(p)}{2}} (p^{\frac{1}{2}} - 1)},$$

where $g(p) = E(\frac{\text{cons}}{n \ln(p)})$ (here $E(x)$ denotes the floor of x).

Thus $r_0(n) \leq 4n^2 10^{-c} e^{-\frac{cgn_s}{2n}} \theta(p_0)$, where θ is the Tschebyscheff function defined by

$$\theta(p_0) = \sum_{p \leq p_0} \ln(p).$$

According to Tschebyscheff's lemma [EM] we have $\theta(p_0) \leq 2 \ln(2) p_0$, and the desired result is obtained.

Proposition 1. *Let f be a decreasing, continuously differentiable and integrable positive function in $[a, +\infty[$ ($a \geq 0$) satisfying $f(x) = o(\frac{1}{x})$ and $xf(x)e^{-c\sqrt{\ln(x)}} = o(\int_x^{+\infty} f(t)dt)$ as $|x| \rightarrow \infty$, where c is a positive constant. Then*

$$\sum_{p \geq x} \ln(p)f(p) \sim \int_x^{+\infty} f(t)dt.$$

Proof. Let us consider the function $\theta(x) = \sum_{p \leq x} \ln(p)$. There exists a constant $c > 0$ such that $\theta(x) = x + R(x)$, where $R(x) = O(xe^{-c\sqrt{\ln(x)}})$ [La]. We can write the series $\sum_{p \geq x} \ln(p)f(p)$ as an integral:

$$\sum_{p \geq x} \ln(p)f(p) = \int_x^{+\infty} f(t)d\theta(t);$$

now by integration by parts, we get

$$\sum_{p \geq x} \ln(p)f(p) = -f(x)R(x) + \int_x^{+\infty} f(t)dt - \int_x^{+\infty} f'(t)R(t)dt.$$

It is not difficult to see that $f(x)R(x) = o(\int_x^{+\infty} f(t)dt)$ and

$$\int_x^{+\infty} |f'(t)R(t)| dt = O(e^{-c\sqrt{\ln(x)}} \int_x^{+\infty} |tf'(t)| dt).$$

After another integration by parts of this last integral, we find that

$$\int_x^{+\infty} |tf'(t)| dt = xf(x) + \int_x^{+\infty} f(t)dt,$$

and we deduce the result of the proposition.

Corollary 1 (Estimation of $r_\infty(n)$). *We have*

$$r_\infty(n) \leq s_1(n) + s_2(n) + s_3(n),$$

where

$$\begin{aligned} s_1(n) &= n^2 e^{-y(3\ln(p_0))^2} \sum_{p > p_0} \frac{\ln(p)}{p(p^{\frac{1}{2}} - 1)} \\ &\underset{p_0 \rightarrow \infty}{\sim} 2n^2 \frac{e^{-y(3\ln(p_0))^2}}{\sqrt{p_0}}, \\ s_2(n) &= n^2 \sum_{p > p_0} \frac{\ln(p)}{\sqrt{p}} e^{-y \ln(p)^2} \\ &\underset{p_0 \rightarrow \infty}{\sim} \frac{n^2 \sqrt{p_0} e^{-y \ln(p_0)^2}}{2y \ln(p_0)}, \end{aligned}$$

and

$$s_3(n) = n^2 \sum_{p > p_0} \frac{\ln(p)}{p} e^{-y(2\ln(p))^2}$$

$$\underset{p_0 \rightarrow \infty}{\sim} \frac{n^2 e^{-4y \ln(p_0)^2}}{8y \ln(p_0)}.$$

Example. In the case of the totally imaginary field of degree 20 having the smallest known discriminant, the value of y which maximizes $S(y)$ is 0.069. If we take $p_0 = 4 \cdot 10^7$ and $c = 30$, the error in the computation of $v(y)$ is bounded by 10^{-4} .

5. APPLICATION TO THE COMPUTATION
OF THE FIRST ZERO OF THE DEDEKIND ZETA FUNCTION
OF CERTAIN FIELDS OF SMALL DISCRIMINANT

In the following tables we indicate the value of the first zero $\frac{1}{2} + it_1$ of ζ_K , the limit p_0 of primes taken into consideration in the sum running over the prime ideals of the field, the value y giving the best lower bound for the first zero (cf. lemma 1), and a bound for the error term $r_{p_0}(y)$ in the sum $v(y)$ (cf section 4.3).

5.1. Totally real field of small discriminant of degree 7. This field having the smallest discriminant ($d_K = 20134393$) is defined by the polynomial [Ph]:

$$P(x) = x^7 + x^6 - 6x^5 - 5x^4 + 8x^3 + 5x^2 - 2x - 1;$$

and its root discriminant is 14.909% above the GRH bounds. See Table 1.

TABLE 1.

t_1	p_0	y	$r_{p_0}(y)$
1.81803	$8 \cdot 10^7$	0.082	10^{-7}

5.2. Totally imaginary fields of small discriminant of degree less than 30. We consider here the imaginary complex number fields given in [CDO] of degree less than or equal to 30 having a root discriminant near from Odlyzko’s lower bound.

Table 2 shows that the precision of the computations decreases with the degree; for example in degree 24, it was necessary to compute the decomposition for all the primes less than $15 \cdot 10^7$ to obtain an error on t_1 less than 10^{-2} . In degrees 28 and 30, the first zero is computed only to 10^{-1} but the indicated value is less than the exact value of the first zero. If we don’t assume (GRH), we compute $S(y) = \sum \Phi_y(\rho)$ for every value y and we obtain the following result:

Lemma 10. *The following assertions are equivalent:*

- i) $S(y)$ is bounded as $y \rightarrow 0$.*
- ii) $S(y) \rightarrow 0$ as $y \rightarrow 0$.*
- iii) There are no zeros of the Dedekind zeta function in the triangle $A(\frac{1}{2}, 0)$, $B(1, 0)$, $C(1, \frac{1}{2})$.*

Remark. Because $S(y)$ is bounded as $y \rightarrow 0$ for all the considered fields, we can deduce the non-existence of a Siegel’s zero for these fields.

TABLE 2.

n	t₁	p₀	y	r_{p₀}(y)
2	8.03	10 ⁷	0.4	10 ⁻⁴³
4	4.39	10 ⁷	0.3	10 ⁻³¹
6	3.30	10 ⁷	0.2	10 ⁻¹⁹
8	2.75	10 ⁷	0.11	10 ⁻⁸
10	2.40	10 ⁷	0.097	10 ⁻⁶
12	2.17	10 ⁷	0.090	10 ⁻⁵
14	2.10	10 ⁷	0.085	10 ⁻⁵
16	1.87	2 · 10 ⁷	0.075	10 ⁻⁴
18	1.76	2 · 10 ⁷	0.073	10 ⁻⁴
20	1.66	4 · 10 ⁷	0.069	10 ⁻⁴
22	1.63	9 · 10 ⁷	0.062	10 ⁻³
24	1.54	15 · 10 ⁷	0.058	10 ⁻³
26	1.23	6 · 10 ⁷	0.057	10 ⁻³
28	1.48*	5 · 10 ⁷	0.056	10 ⁻²
30	1.36*	5 · 10 ⁷	0.056	10 ⁻²

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