MAXIMUM-NORM ESTIMATES FOR RESOLVENTS OF ELLIPTIC FINITE ELEMENT OPERATORS

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ABSTRACT. Let Ω be a convex domain with smooth boundary in R^d. It has been shown recently that the semigroup generated by the discrete Laplacian for quasi-uniform families of piecewise linear finite element spaces on Ω is analytic with respect to the maximum-norm, uniformly in the mesh-width. This implies a resolvent estimate of standard form in the maximum-norm outside *some* sector in the right halfplane, and conversely. Here we show directly that such a resolvent estimate holds outside *any* sector around the positive real axis, with *arbitrarily small angle*. This is useful in the study of fully discrete approximations based on $A(\theta)$-stable rational functions, with θ small.

1. INTRODUCTION

Resolvent estimates for elliptic partial differential operators may be utilized in the study, via semigroup theory, of the corresponding parabolic equations, and this holds also for spatially semidiscrete approximations of such equations, e.g., by finite element methods. Estimates of the type needed may often be obtained easily in the L_2 norm, whereas estimates in L_p for $p \neq 2$ are more difficult to derive; this paper is devoted to resolvent estimates in the maximum-norm for the discrete Laplacian associated with piecewise linear finite element spaces.

Let Ω be a bounded domain with smooth boundary in d-dimensional Euclidean space R^d, and consider the positive definite operator $A = -\Delta = -\sum_{j=1}^{d} \partial^2 u/\partial x_j^2$ with homogeneous Dirichlet boundary conditions. Let $\mathcal{C} = \mathcal{C}(\bar{\Omega})$ be the Banach space of continuous complex-valued functions on $\bar{\Omega}$, with norm $|v| = \sup_{x \in \Omega} |v(s)|$, and let $\mathcal{C}_0 = \{v \in \mathcal{C};\ v = 0 \text{ on } \partial\Omega\}$. A special case of a result by Stewart [26] (see Theorem 2.1 below) shows that for any $\varphi \in (0, \frac{1}{2}\pi)$ there is a constant C such that, if $D(A) = \mathcal{C}^2(\bar{\Omega}) \cap \mathcal{C}_0$, then

(1.1) $|(\lambda I - A)^{-1} v| \leq C(1 + |\lambda|)^{-1}|v|, \quad \text{for } \lambda \notin \Sigma_\varphi = \{\lambda;\ |\arg \lambda| \leq \varphi\},\ v \in \mathcal{C}.$

If we were to consider A as an operator in $\mathcal{C}$, it would not be densely defined, and the standard results of semigroup theory therefore would not automatically apply

(cf., in this respect, Da Prato and Sinestrari [9]). However, it was shown in [26] that if we restrict the considerations to $\mathcal{C}_0$, and thus also require $Av \in \mathcal{C}_0$ for v to be in $D(A)$, then A generates an analytic semigroup. This semigroup $E(t) = e^{-tA}$ may then be represented as

$$E(t) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma e^{-\lambda t} (\lambda I - A)^{-1} \, d\lambda,$$

where the contour Γ may be chosen, e.g., as $\Gamma = \{\lambda; \, |\arg \lambda| = \alpha\}$, where $\alpha \in (\varphi, \frac{1}{2}\pi)$, with $\operatorname{Im} \lambda$ increasing from $-\infty$ to ∞. The semigroup $E(t)$ is the solution operator for the initial-value problem

(1.2) $u_t = \Delta u$ in Ω, $u = 0$ on $\partial\Omega$,   with $u(0) = v$,   for $v \in \mathcal{C}_0$.

The maximum principle for the heat equation shows that $E(t)$ is a contraction semigroup in $\mathcal{C}_0$ with respect to $|\cdot|$, and the resolvent estimate (1.1) implies the smoothing property

$$|E'(t)v| = |\Delta E(t)v| \leq Ct^{-1}|v|, \quad \text{for } t > 0,$$

by a classical theorem of Hille, cf. Pazy [17], Theorem 2.5.2. Conversely, by the same theorem, the smoothing property implies (1.1) for some $\varphi \in (0, \frac{1}{2}\pi)$.

We are interested in finite element analogues of these properties. From now on we assume that the basic domain Ω is convex. For $h \in (0, h_0]$, $h_0 > 0$, we denote by $\mathcal{T}_h = \{\tau_j\}_{j=1}^{J_h}$ a triangulation of $\Omega_h = \operatorname{Int}\left(\bigcup_j \overline{\tau_j}\right) \subset \Omega$ into mutually disjoint open face-to-face simplices τ_j. We assume that any vertex of a simplex τ_j which is on $\partial\Omega_h$ is also on $\partial\Omega$. Letting $h = \max_j \operatorname{diam} \tau_j$, we also assume that the family $\{\mathcal{T}_h\}$ of triangulations is globally quasi-uniform, so that $\min_j \operatorname{vol}(\tau_j) \geq ch^d$, with $c > 0$. We note that then $\operatorname{dist}(x, \partial\Omega) \leq Ch^2$ for $x \in \partial\Omega_h$.

Let S_h be the finite dimensional space of continuous complex-valued piecewise linear functions associated with $\mathcal{T}_h$ that vanish outside Ω_h. With the operator A we associate its finite element version $A_h : S_h \to S_h$ by

$$(A_h\psi, \chi) = (\nabla\psi, \nabla\chi), \quad \forall \, \psi, \chi \in S_h, \quad \text{where } (v, w) = \int_\Omega v\,\bar{w}\,dx.$$

We remark that, since $A_h\psi \in S_h$, this function vanishes on $\partial\Omega$, which is analogous to the case when $v \in D(A)$ requires $Av \in \mathcal{C}_0$.

Consider now the spatially semidiscrete version of the initial-value problem (1.2),

$$(u_{h,t}, \chi) + (\nabla u_h, \nabla\chi) = 0, \quad \forall\chi \in S_h, t > 0, \quad \text{with } u_h(0) = v_h,$$

or

(1.3) $u_{h,t} + A_h u_h = 0$,   for $t > 0$,   with $u_h(0) = v_h$.

The solution operator $E_h(t)$ of this problem, defined by $u_h(t) = E_h(t)v_h$, is then the semigroup $E_h(t) = e^{-A_h t}$. A maximum-principle is not valid for the semidiscrete problem, and this raises the question about the boundedness of $E_h(t)$ in maximum-norm. In this regard it was shown in Thomée and Wahlbin [29] (following Schatz, Thomée, and Wahlbin [22]) that

(1.4) $|E_h(t)v_h| + t|E_h'(t)v_h| \leq C|v_h|$,   for $t > 0$.

By the theorem of Hille referred to above, this implies that the analogue of (1.1) is satisfied for A_h and some $\varphi \in (0, \pi/2)$. We note that the existence of a resolvent estimate for A_h would provide an alternative proof of the stability and smoothing property (1.4).

Our main result in this paper is now the following, with Ω a bounded, convex domain with smooth boundary.

Theorem 1.1. *For any $\varphi \in (0, \frac{1}{2}\pi)$ there exists a constant C such that*

$$(1.5) \qquad |(\lambda I - A_h)^{-1}\chi| \leq C(1 + |\lambda|)^{-1}|\chi|, \quad \text{for } \lambda \notin \Sigma_\varphi, \quad \chi \in S_h.$$

For earlier work concerning maximum-norm stability, smoothing and resolvent estimates of finite element operators, we also refer to Fujii [11], Schatz, Thomée, and Wahlbin [21], Nitsche and Wheeler [15], Wahlbin [31], Thomée and Wahlbin [28], Rannacher [18], Chen [5], Crouzeix, Larsson, and Thomée [6], Palencia [16], Bakaev, Larsson, and Thomée [3], Bakaev [2], Crouzeix and Thomée [7]; brief discussions of the above work are given in [22] and [2]. We note, in particular, that the result in the case $d = 1$ follows from [6], and we may therefore restrict our considerations to $d \geq 2$ below.

We emphasize that our Theorem 1.1 is an improvement of the resolvent estimate that follows from (1.4) in that the angle φ can be chosen arbitrarily small. This is of interest, e.g., in the study of such single step time stepping methods for (1.3) for which the fully discrete solution at time $t_n = nk$ can be thought of as $U^n = r(kA_h)^n v_h$, where k denotes a constant time step. This approximation may then be represented in terms of the resolvent as

$$r(kA_h)^n = r(\infty)^n I + \frac{1}{2\pi\mathrm{i}} \int_\Gamma (r(k\lambda)^n - r(\infty)^n)(\lambda I - A_h)^{-1} \, d\lambda,$$

where $\Gamma = \{\lambda; \ |\arg \lambda| = \alpha\}$ with $\alpha \in (0, \theta)$, if $r(\lambda)$ is $A(\theta)-$stable. Our result thus makes the analysis of time stepping schemes possible for schemes based on $A(\theta)-$stable rational functions with small θ, cf., e.g., Thomée [27], Chapter 8.

The analogue of Theorem 1.1 was shown in [2] in the case of Lagrange elements of polynomial degree at least two for $d = 2$ and 3. Using the techniques for large $|\lambda|$ in Section 3 below, the restriction to $d = 2, 3$ in [2] is easily removed. Moreover, the bound in the piecewise linear case in [2] contains a factor $(\log\log(1/h))^{1/2+d/4}$, and an essential contribution of the present paper is to remove this factor.

Our approach is inspired by Palencia [16], and uses resolvent estimates for the continuous problem together with certain properties of the L_2 and Ritz projections onto S_h. Such preliminary material is collected in Section 2 below, and the proof of Theorem 1.1 is then given in Section 3. An important ingredient in our proof is a maximum-norm stability property of the gradient of the Ritz projection. A proof of this is given in an Appendix. The restriction to $A = -\Delta$ is made for simplicity of presentation only, and the modifications needed for more general positive definite second order elliptic operators with smooth real-valued coefficients would not present any difficulties.

2. Preliminaries

In this section we collect some properties related to the finite element spaces S_h and the above operator $A = -\Delta$ which will be used below in the proofs of our main results.

In what follows, we shall repeatedly apply the well-known inverse property

$$(2.1) \qquad |\chi|_{1,\tau} \leq Ch^{-1}|\chi|_\tau \quad \text{for } h \in (0, h_0], \ \tau \in \mathcal{T}_h, \text{ and } \chi \in S_h,$$

which follows by the quasi-uniformity of the triangulations $\mathcal{T}_h$. As a consequence, the same inequality holds for the full domain Ω_h as well. Here and below, when s

is an integer, $|\cdot|_{s,V}$ denotes the standard norm in $\mathcal{C}^s(V)$ or $W^s_\infty(V)$, the functions in V with derivatives of order s in $\mathcal{C}(\bar{V})$ or $L_\infty(V)$, respectively; V will normally be omitted when $V = \Omega$. We will use the norm $|\cdot|_{s+\alpha}$ in $\mathcal{C}^{s+\alpha}$ when s is an integer and $0 < \alpha < 1$. We shall also use the notation $\dot{\mathcal{C}}^s = \mathcal{C}^s \cap \mathcal{C}_0$ and $\dot{W}^s_\infty = W^s_\infty \cap \mathcal{C}_0$ for $s > 0$.

Let P_h denote the orthogonal L_2-projection onto S_h, so that

$$(P_h v, \chi) = (v, \chi), \quad \forall\, \chi \in S_h.$$

We shall need the stability of P_h with respect to the maximum-norm (cf. Descloux [8], Douglas, Dupont and Wahlbin [10]),

$$(2.2) \qquad |P_h v| \le C|v|_{\Omega_h}, \quad \text{for } v \in L_\infty.$$

Using (2.1) and a well-known estimate for the standard Lagrange interpolant I_h, we easily find that

$$(2.3) \qquad |P_h v|_1 \le C|v|_{1,\Omega_h}, \quad \text{for } v \in \dot{W}^1_\infty.$$

Next let R_h be the standard Ritz projection onto S_h, given by

$$(\nabla R_h v, \nabla \chi) = (\nabla v, \nabla \chi), \quad \forall\, \chi \in S_h.$$

It is known from Schatz and Wahlbin [23] that, for small h,

$$(2.4) \qquad |R_h v| \le C \log(1/h)|v|_{\Omega_h}, \quad \text{for } v \in \mathcal{C},$$

where the logarithm cannot be removed in our case of piecewise linear finite elements, see Haverkamp [12]. An essential component in our approach is the following stability estimate for R_h in W^1_∞, without a logarithmic factor:

$$(2.5) \qquad |R_h v|_1 \le C|v|_{1,\Omega_h} + Ch|v|_{1,\Omega\setminus\Omega_h}, \quad \text{for } v \in \dot{W}^1_\infty.$$

This was shown in Rannacher and Scott [19] in the case of a convex polygonal domain in the plane, and also stated there in the case of a curvilinear domain (cf. also Chen and Huang [4] for the case $d = 3$). A proof in our case of arbitrary space dimension is given here in an Appendix.

Applying (2.5) to $I_h v - v$, we find, with $0 < \alpha < 1$, that

$$|R_h v - v|_{1,\Omega_h} \le C|I_h v - v|_{1,\Omega_h} + Ch|v|_{1,\Omega\setminus\Omega_h} \le Ch^\alpha |v|_{1+\alpha}, \quad \text{for } v \in \dot{\mathcal{C}}^{1+\alpha},$$

and, using the fact that $P_h v - R_h v = P_h(v - R_h v)$, it follows from (2.3) that

$$(2.6) \quad |(P_h - R_h)v|_1 \le C|R_h v - v|_{1,\Omega_h} \le Ch^\alpha |v|_{1+\alpha}, \quad \text{for } v \in \dot{\mathcal{C}}^{1+\alpha}, \quad 0 < \alpha < 1.$$

We now turn to some resolvent estimates for the operator A which will be needed below. We begin with the following result, which includes (1.1).

Theorem 2.1. *For any* $\varphi \in (0, \frac{1}{2}\pi)$, *there exists a constant C such that*

$$(2.7) \quad |(\lambda I - A)^{-1} v|_j \le C(1 + |\lambda|)^{-1+j/2}|v|, \quad \text{for } \lambda \notin \Sigma_\varphi, \quad j = 0, 1, \quad v \in \mathcal{C}.$$

Proof. It follows from Theorem 1 of [26] that the estimate of (2.7) holds for $\lambda \notin \Sigma_\varphi$, $|\lambda| \ge R$, for some $R \ge 0$ and some $\varphi \in (0, \frac{1}{2}\pi)$. To see that in the case $A = -\Delta$ the angle φ may be chosen arbitrarily small, we observe that by the proof in [26] it suffices to note that for any homogeneous positive definite polynomial $p(\xi)$ in R^d with real coefficients we have $|p(\xi) - e^{i\theta}\tau^2| \ge c > 0$ for $|\xi|^2 + \tau^2 = 1$, $0 < \varphi \le |\theta| \le \pi$. Here $p(\xi)$ is the characteristic polynomial of $A = -\Delta$ (i.e., $p(\xi) = |\xi|^2$) or that of the principal part of the operator after a transformation needed to make the boundary locally plane near a point $x \in \partial\Omega$.

To complete the proof we need to show that $|(\lambda I - A)^{-1}v|_1 \leq C|v|$ for $\lambda \notin \Sigma_\varphi$, $|\lambda| \leq R$. We first note that if

$$\|(\lambda I - A)^{-1}v\|_{L_p} \leq \|v\|_{L_p}, \quad \text{for } \lambda \notin \Sigma_\varphi, \ |\lambda| \leq R, \quad \text{with } p < \infty,$$

then, using elliptic regularity (cf., e.g., Agmon, Douglis, and Nirenberg [1]) and the identity $A(\lambda I - A)^{-1} = -I + \lambda(\lambda I - A)^{-1}$,

$$\|(\lambda I - A)^{-1}v\|_{W_p^2} \leq C\|A(\lambda I - A)^{-1}v\|_{L_p} \leq C\|v\|_{L_p}.$$

Hence, by Sobolev's inequality, for $p^{-1} + q^{-1} < 2d^{-1}$, $q > p$,

$$\|(\lambda I - A)^{-1}v\|_{L_q} \leq C\|(\lambda I - A)^{-1}v\|_{W_p^2} \leq C\|v\|_{L_p} \leq C\|v\|_{L_q}.$$

For $p = 2$ we have directly, with λ_0 the minimal eigenvalue of A, and $\lambda \notin \Sigma_\varphi$,

$$\|(\lambda I - A)^{-1}v\|_{L_2} \leq C \sup_{x \geq \lambda_0} \frac{1}{|\lambda - x|}\|v\|_{L_2} \leq C\|v\|_{L_2}.$$

Hence the above estimates follow for any $p < \infty$ in a finite number of steps. Finally, using the Sobolev inequality $|v|_1 \leq C\|v\|_{W_q^2}$ with q sufficiently large completes the proof of the desired result. $\square$

Theorem 2.2. *For any $\varphi \in (0, \pi/2)$, there exists a constant C such that*

$$(2.8) \qquad |A(\lambda I - A)^{-1}v| \leq C(1 + |\lambda|)^{-1/2}|v|_1, \quad \text{for } \lambda \notin \Sigma_\varphi, \ v \in \dot{W}_\infty^1.$$

Proof. Application of (2.7) with $j = 0$ shows that

$$|A(\lambda I - A)^{-1}v| \leq C|v|, \quad \text{for } v \in \mathcal{C},$$

and, replacing v by Av in (2.7),

$$|A(\lambda I - A)^{-1}v| \leq C(1 + |\lambda|)^{-1}|Av| \leq C(1 + |\lambda|)^{-1}|v|_2, \quad \text{for } v \in \dot{\mathcal{C}}^2.$$

We now claim that, for $v \in \dot{W}_\infty^1$ and $t > 0$ a parameter, we may write $v = w_0(t) + w_1(t)$, with $w_0(t) \in \mathcal{C}_0$, $w_1(t) \in \dot{\mathcal{C}}^2$, such that

$$(2.9) \qquad |w_0(t)| \leq Ct^{1/2}|v|_1, \quad |w_1(t)|_2 \leq Ct^{-1/2}|v|_1, \quad \text{for } t > 0.$$

In fact, using a partition of unity and a change of variables, it suffices to show this in a halfplane, $x_1 > 0$, say, and for v with compact support and with $v = 0$ for $x_1 = 0$. To treat the function near the boundary, one may then extend v by $v(-x_1, x_2, \ldots, x_d) = -v(x_1, x_2, \ldots, x_d)$ and then take for $w_1(t)$ a smoothing convolution of v over a disc with radius $t^{1/2}$ with even kernel. This gives $w_1(t) = 0$ for $x_1 = 0$, so that $w_1(t) \in \dot{\mathcal{C}}^2$ and (2.9) holds. With $t = (1 + |\lambda|)^{-1}$ this shows that

$$|A(\lambda I - A)^{-1}v| \leq C|w_0(t)| + C(1 + |\lambda|)^{-1}|w_1(t)|_2 \leq C(1 + |\lambda|)^{-1/2}|v|_1. \qquad \square$$

We shall also need a resolvent estimate in a Hölder norm. In the proof of this we shall use some notions from real interpolation theory, cf., e.g., Triebel [30]. When $A_1 \subseteq A_0$ are two Banach spaces with $\|v\|_{A_0} \leq C\|v\|_{A_1}$ for $v \in A_1$, the space $(A_0, A_1)_{1/2,\infty}$ is defined by the norm

$$\|v\|_{(A_0,A_1)_{1/2,\infty}} = \|v\|_{A_0} + \sup_{t>0} t^{-1/2} \inf_{\substack{v=w_0(t)+w_1(t) \\ w_0(t) \in A_0, \ w_1(t) \in A_1}} (\|w_0(t)\|_{A_0} + t\|w_1(t)\|_{A_1}).$$

If L is a linear operator $A_0 \to B_0$ and $A_1 \to B_1$ with norms N_0 and N_1, respectively, then also $L : (A_0, A_1)_{1/2,\infty} \to (B_0, B_1)_{1/2,\infty}$, with norm bounded by $CN_0^{1/2}N_1^{1/2}$. From [30], Section 4.5, we have that $(\mathcal{C}^\alpha, \mathcal{C}^{2+\alpha})_{1/2,\infty} = \mathcal{C}^{1+\alpha}$ for $0 < \alpha < 1$.

Also, from (2.9) it follows that $\dot{W}^1_\infty \subseteq (\mathcal{C}, \dot{\mathcal{C}}^2)_{1/2,\infty}$ and $\|v\|_{(\mathcal{C},\dot{\mathcal{C}}^2)_{1/2,\infty}} \leq C|v|_1$, for $v \in \dot{W}^1_\infty$.

Theorem 2.3. *For any $\varphi \in (0, \pi/2)$ and $\alpha \in (0,1)$ there exists a constant C such that*

$$(2.10) \qquad |(\lambda I - A)^{-1}v|_{1+\alpha} \leq C(1+|\lambda|)^{-1+\alpha/2}|v|_1, \quad \text{for } \lambda \notin \Sigma_\varphi, \ v \in \dot{W}^1_\infty.$$

Proof. We start with the estimate

$$(2.11) \qquad |(\lambda I - A)^{-1}v|_\alpha \leq C(1+|\lambda|)^{-1+\alpha/2}|v|, \quad \text{for } \lambda \notin \Sigma_\varphi, v \in \mathcal{C}.$$

In fact, this holds for $\alpha = 0$ and $\alpha = 1$ by (2.7), and hence follows for $0 < \alpha < 1$ since $|w|_\alpha \leq C|w|_1^\alpha |w|^{1-\alpha}$. Applying (2.11) with Av substituted for v, we find that

$$(2.12) \qquad |(\lambda I - A)^{-1}Av|_\alpha \leq C(1+|\lambda|)^{-1+\alpha/2}|Av|, \quad \text{for } \lambda \notin \Sigma_\varphi, \ v \in \dot{\mathcal{C}}^2.$$

By a Schauder estimate for the elliptic operator A, cf. [1], we have, for $0 < \alpha < 1$,

$$|(\lambda I - A)^{-1}v|_{2+\alpha} \leq C|A(\lambda I - A)^{-1}v|_\alpha = C|(\lambda I - A)^{-1}Av|_\alpha, \quad \text{for } v \in \dot{\mathcal{C}}^2,$$

so that by (2.12),

$$(2.13) \qquad |(\lambda I - A)^{-1}v|_{2+\alpha} \leq C(1+|\lambda|)^{-1+\alpha/2}|v|_2, \quad \text{for } \lambda \notin \Sigma_\varphi, \ v \in \dot{\mathcal{C}}^2.$$

Thus by (2.11) and (2.13) $(\lambda I - A)^{-1}$ is a bounded operator both $\mathcal{C} \to \mathcal{C}^\alpha$ and $\dot{\mathcal{C}}^2 \to \mathcal{C}^{2+\alpha}$, with norms bounded by the same number in the way shown. From our remarks above about real interpolation theory, (2.10) hence follows by taking $A_0 = \mathcal{C}$, $A_1 = \dot{\mathcal{C}}^2$, $B_0 = \mathcal{C}^\alpha$, and $B_1 = \mathcal{C}^{2+\alpha}$. $\qquad\square$

3. Proof of the main result

In this section we give the proof of Theorem 1.1 as stated in the introduction. The proof will be separated into two parts, involving different techniques. The first part concerns $|\lambda| \leq \omega_0 h^{-2}$, with ω_0 sufficiently small, and the second part treats $|\lambda| > \omega_0 h^{-2}$.

For the first part we shall show that there exists $\omega_0 > 0$ such that, for any $\varphi \in (0, \frac{1}{2}\pi)$ and $\chi \in S_h$,

$$(3.1) \qquad |A_h(\lambda I - A_h)^{-1}\chi| \leq C(1+|\lambda|)^{-1/2}|\chi|_1, \quad \text{for } \lambda \notin \Sigma_\varphi, \ |\lambda| \leq \omega_0 h^{-2}.$$

We now show that (3.1) implies the result of Theorem 1.1 for these λ. Since $P_h A = A_h R_h$, we have, for $\chi \in S_h$, the identity

$$(\lambda I - A_h)^{-1}\chi = P_h(\lambda I - A)^{-1}\chi + A_h(\lambda I - A_h)^{-1}(P_h - R_h)(\lambda I - A)^{-1}\chi,$$

where both sides are well defined for $\lambda \notin \Sigma_\varphi$. Using the estimates (2.2), (2.7) with $j = 0$, and (3.1), we find that

$$(3.2) \qquad |(\lambda I - A_h)^{-1}\chi| \leq C(1+|\lambda|)^{-1}|\chi|$$
$$+ C(1+|\lambda|)^{-1/2}|(P_h - R_h)(\lambda I - A)^{-1}\chi|_1, \quad \text{for } \lambda \notin \Sigma_\varphi, \ \chi \in S_h.$$

In view of (2.3), (2.5), and (2.7) with $j = 1$, this shows the desired result.

A similar approach was used in [16]; it is the fact that we use $|\cdot|_1$ in the last term in (3.2) that makes it possible to avoid the logarithm from (2.4).

To show (3.1) we now use the identity

$$(3.3) \qquad A_h(\lambda I - A_h)^{-1}\chi = P_h A(\lambda I - A)^{-1}\chi$$
$$+ \lambda A_h(\lambda I - A_h)^{-1}(P_h - R_h)(\lambda I - A)^{-1}\chi, \quad \text{for } \lambda \notin \Sigma_\varphi, \ \chi \in S_h.$$

Here, by (2.2) and (2.8),

$$|P_h A(\lambda I - A)^{-1}\chi| \le C(1 + |\lambda|)^{-1/2}|\chi|_1, \quad \text{for } \lambda \notin \Sigma_\varphi.$$

Using the operator norm $|B_h|_{S_h,1} = \sup_{\chi \in S_h}(|B_h\chi|/|\chi|_1)$, we see that the last term in (3.3) is bounded by

$$C|\lambda|\,|A_h(\lambda I - A_h)^{-1}|_{S_h,1}|(P_h - R_h)(\lambda I - A)^{-1}\chi|_1.$$

To bound the last factor we have, by (2.6) and (2.10),

$$|(P_h - R_h)(\lambda I - A)^{-1}\chi|_1 \le Ch^{1/2}|(\lambda I - A)^{-1}\chi|_{3/2} \le Ch^{1/2}(1 + |\lambda|)^{-3/4}|\chi|_1,$$

so that (3.3) implies, for $\lambda \notin \Sigma_\varphi$,

$$|A_h(\lambda I - A_h)^{-1}|_{S_h,1} \le C(1 + |\lambda|)^{-1/2} + C_1 h^{1/2}(1 + |\lambda|)^{1/4}|A_h(\lambda I - A_h)^{-1}|_{S_h,1}.$$

Hence, if $C_1 h^{1/2}(1 + |\lambda|)^{1/4} \le 1/2$, we find that

$$|A_h(\lambda I - A_h)^{-1}|_{S_h,1} \le C(1 + |\lambda|)^{-1/2}, \quad \lambda \notin \Sigma_\varphi,$$

which thus shows (3.1) for $|\lambda| \le \omega_0 h^{-2}$, for h small and some $\omega_0 > 0$.

We now turn to the case $|\lambda| \ge \omega_0 h^{-2}$, where ω_0 is determined by the above. Let us note in passing that by the well-known inverse estimate $|A_h\chi| \le Ch^{-2}|\chi|$ for $\chi \in S_h$ we have $|\lambda^{-1}A_h|_{S_h} \le 1/2$ for $|\lambda| \ge \omega_1 h^{-2}$ with ω_1 large enough, where $|A_h|_{S_h} = \sup_{\chi \in S_h}|A_h\chi|/|\chi|$. Hence $|(I - \lambda^{-1}A_h)^{-1}|_{S_h} \le 2$ for $|\lambda| \ge \omega_1 h^{-2}$, which implies (1.5) for $|\lambda| \ge \omega_1 h^{-2}$, but leaves the possible gap $\omega_0 h^{-2} \le |\lambda| \le \omega_1 h^{-2}$.

For $x \in \Omega_h$ fixed and $\varphi \in (0, \frac{1}{2}\pi)$, we will use the adjoint discrete Green's function

$$G_h^x(y, \bar{\lambda}) = ((\bar{\lambda}I - A_h)^{-1}\delta_h^x)(y), \quad \text{for } \lambda \notin \Sigma_\varphi,$$

where δ_h^x is the discrete delta-function defined by

$$(\chi, \delta_h^x) = \chi(x), \quad \forall \chi \in S_h.$$

We then have

$$((\lambda I - A_h)^{-1}\chi)(x) = (\chi, G_h^x(\cdot, \bar{\lambda})), \quad \forall \chi \in S_h.$$

Since the sector Σ_φ is symmetric around the real axis, it therefore suffices to show that for any $x \in \Omega_h$ and $\varphi \in (0, \pi/2)$, we have

(3.4) $$\|G_h^x(\cdot, \lambda)\|_{L_1} \le C|\lambda|^{-1}, \quad \text{for } |\lambda| \ge \omega_0 h^{-2},\ \lambda \notin \Sigma_\varphi.$$

Writing for brevity $\|\cdot\| = \|\cdot\|_{L_2(\Omega)}$, we introduce the weighted norm

(3.5) $$\|v\|_m = \|(\rho_h^x)^m v\|, \quad m \ge 0, \quad \text{where } \rho_h^x(y) = \left(|x - y|^2 + h^2\right)^{1/2}.$$

The estimate (3.4) will then be a consequence of the following lemma.

Lemma 3.1. *For each $m \ge 0$, $\varphi \in (0, \pi/2)$, and $\omega_0 > 0$ there is a C such that for any $x \in \Omega_h$ we have*

$$\|G_h^x(\cdot, \lambda)\|_m \le Ch^{m-d/2}|\lambda|^{-1}, \quad \text{for } \lambda \notin \Sigma_\varphi, \quad |\lambda| \ge \omega_0 h^{-2}.$$

In fact, choosing $m > d/2$, we have $\|(\rho_h^x)^{-m}\| \le Ch^{-m+d/2}$, and hence

$$\|G_h^x(\cdot, \lambda)\|_{L_1} \le C\|(\rho_h^x)^{-m}\|\,\|G_h^x(\cdot, \lambda)\|_m \le (Ch^{-m+d/2})\,(Ch^{m-d/2}|\lambda|^{-1}) = C|\lambda|^{-1}.$$

For the proof of Lemma 3.1 we shall need the following two lemmas.

Lemma 3.2. *For each $m \ge 0$ there is a C such that for any $x \in \Omega_h$ we have*

$$\|\delta_h^x\|_m \le Ch^{m-d/2}.$$

Proof. Using the exponential decay property of δ_h^x, see [8], [10], this follows easily as in [21], Lemma 1.5. $\qquad\square$

Lemma 3.3. *For each $m \geq 1$ there is a C such that for any $x \in \Omega_h$ and $\chi \in S_h$ we have*

$$\|\rho^{-m+1}\nabla(\rho^{2m}\chi - P_h(\rho^{2m}\chi))\| \leq Ch(\|\chi\|_{m-1} + \|\nabla\chi\|_m), \quad \text{where } \rho = \rho_h^x.$$

This is a special case of [2], Lemma 3.4 (with $j = 1$).

Proof of Lemma 3.1. For brevity we write G for G_h^x. We consider the expression
(3.6)
$$-\lambda\|G\|_m^2 + \|\nabla G\|_m^2 = -\lambda(G, \rho^{2m}G) + (\nabla G, \nabla(\rho^{2m}G)) - 2m(\nabla G, \rho^{2m-1}\nabla\rho\, G).$$

We note that G satisfies

$$-\lambda(G, \chi) + (\nabla G, \nabla\chi) = -(\delta_h^x, \chi), \quad \forall\chi \in S_h.$$

Choosing $\chi = P_h(\rho^{2m}G)$ and subtracting from (3.6), we have

(3.7)
$$-\lambda\|G\|_m^2 + \|\nabla G\|_m^2 = F := (\nabla G, \nabla(\rho^{2m}G - P_h(\rho^{2m}G)))$$
$$+ (\delta_h^x, \rho^{2m}G) - 2m(\nabla G, \rho^{2m-1}\nabla\rho\, G).$$

This equation is of the form

$$e^{i\alpha}a + b = f, \quad \text{with } a, b > 0, \ 0 \leq |\alpha| \leq \pi - \varphi,$$

and on multiplying by $e^{-i\alpha/2}$ and taking real parts we have

$$a + b \leq (\cos(\alpha/2))^{-1}|f| \leq (\sin(\varphi/2))^{-1}|f| = C_\varphi|f|.$$

From (3.7) we therefore conclude that

(3.8)
$$|\lambda|\,\|G\|_m^2 \leq |\lambda|\,\|G\|_m^2 + \|\nabla G\|_m^2 \leq C_\varphi|F|, \quad \text{for } \lambda \notin \Sigma_\varphi.$$

For $m = 0$ we note that $F = (\delta_h^x, G)$ and hence, by (3.8) and Lemma 3.2,

(3.9)
$$|\lambda|\,\|G\|_0 \leq C\|\delta_h^x\| \leq Ch^{-d/2}.$$

Now let $m \geq 1$. By Lemma 3.3 and using the easily shown inverse inequality $h\|\nabla\chi\|_m \leq C\|\chi\|_m$, and the fact that $h \leq \rho$, we then have

$$\|\rho^{-m+1}\nabla(\rho^{2m}G - P_h(\rho^{2m}G))\| \leq Ch(\|G\|_{m-1} + \|\nabla G\|_m) \leq C\|G\|_m,$$

and hence, for the first term in F,

$$|(\nabla G, \nabla(\rho^{2m}G - P_h(\rho^{2m}G)))| \leq C\|\nabla G\|_{m-1}\,\|G\|_m.$$

Further, by Lemma 3.2,

$$|(\delta_h^x, \rho^{2m}G)| \leq \|\delta_h^x\|_m\,\|G\|_m \leq Ch^{m-d/2}\,\|G\|_m,$$

and, since $\nabla\rho$ is bounded,

$$|(\nabla G, \rho^{2m-1}\nabla\rho\, G)| \leq C\|\nabla G\|_{m-1}\,\|G\|_m.$$

Thus, from (3.8),

(3.10)
$$|\lambda|\,\|G\|_m \leq C(\|\nabla G\|_{m-1} + h^{m-d/2}).$$

Using Hölder's and Young's inequalities, we next find that

$$C\|\nabla G\|_{m-1} \leq Ch^{-1}\|G\|_{m-1} \leq Ch^{-1}\|G\|_m^{1-1/m}\|G\|_0^{1/m}$$
$$\leq \tfrac{1}{2}|\lambda|\,\|G\|_m + C|\lambda|^{-m+1}h^{-m}\|G\|_0 \leq \tfrac{1}{2}|\lambda|\,\|G\|_m + Ch^{m-2}\|G\|_0,$$

where we have used $|\lambda| \geq \omega_0 h^{-2}$ in the last step. Hence, from (3.9) and (3.10),

$$|\lambda| \, \|G\|_m \leq C(h^{m-2}\|G\|_0 + h^{m-d/2}) \leq C(h^{m-2-d/2}|\lambda|^{-1} + h^{m-d/2}) \leq Ch^{m-d/2}.$$

This completes the proof for $m \geq 1$, and for $0 < m < 1$ the result is then obvious.
□

We remark that (3.1) is a discrete analogue of Theorem 2.2, which could possibly be of independent interest; the inequality holds also for $|\lambda| \geq \omega_0 h^{-2}$, which easily follows from Theorem 1.1 since, by the inverse estimate $|A_h\chi| \leq Ch^{-1}|\chi|_1$ on S_h, we may conclude

$$|A_h(\lambda I - A_h)^{-1}\chi| = |(\lambda I - A_h)^{-1}A_h\chi| \leq 2|\lambda|^{-1}|A_h\chi|$$
$$\leq Ch^{-1}|\lambda|^{-1}|\chi|_1 \leq C|\lambda|^{-1/2}|\chi|_1, \quad \text{for } |\lambda| \geq \omega_0 h^{-2}.$$

APPENDIX. MAXIMUM-NORM STABILITY
OF THE GRADIENT OF THE RITZ PROJECTION

We now show a stability estimate for ∇R_h which implies the estimate (2.5) needed in our proof of the resolvent estimate above. For the purpose of possible future reference we include, following Schatz [20], a weight function which shows a certain local character of the result, which is not needed here. As before, we use $\rho_h^x(y) = (|y - x|^2 + h^2)^{1/2}$.

Theorem A.1. *Let* $0 \leq s < 1$. *There exists a constant C such that for all $x \in \Omega_h$,*

(A.1) $\qquad |\nabla R_h v(x)| \leq C(|(h/\rho_h^x)^s \nabla v|_{\Omega_h} + \|(\rho_h^x)^{-d}v\|_{L_1(\partial\Omega_h)}), \quad \text{for } v \in W_\infty^1.$

To show (2.5) we note that, using polar coordinates,

(A.2) $\qquad \|(\rho_h^x)^{-d}\|_{L_1(\partial\Omega_h)} \leq C \int_0^\infty r^{d-2}(r^2 + h^2)^{-d/2}dr + C \leq Ch^{-1}.$

Since also $|v|_{\partial\Omega_h} \leq Ch^2|\nabla v|_{\Omega\backslash\Omega_h}$ for $v \in \dot{W}_\infty^1$, the last term in (A.1) is bounded as desired. Using (A.1) with $s = 0$ therefore shows (2.5).

Proof of Theorem A.1. Let x be an arbitrary point in Ω_h; this point will be fixed below. Denote by ∂ any derivative of first order, and let $\tau \in \mathcal{T}_h$ be such that $x \in \tau$. Let $\tilde{\delta}$ be a smooth nonnegative function with support in $\sigma \subset \tau$, with $\int_\sigma \tilde{\delta} \, dy = 1$ and such that

(A.3) $\qquad\qquad\qquad \text{dist}\,(\sigma, \partial\Omega_h) \geq ch, \quad \text{with } c > 0.$

Furthermore, $\tilde{\delta}$ can be taken so that σ contains a ball of radius $\geq ch$, and

(A.4) $\qquad\qquad\qquad\qquad |\tilde{\delta}(y)| \leq Ch^{-d}.$

Writing $R_h v = v_h$, we have, since ∂v_h is constant on τ,

(A.5) $\qquad\qquad\qquad \partial v_h(x) = (\partial v_h, \tilde{\delta}) = -(v_h, \partial\tilde{\delta}).$

Let $g = g^x(y)$ be the solution of

(A.6) $\qquad\qquad -\Delta g = \partial\tilde{\delta} \quad \text{in } \Omega, \quad \text{with } g = 0 \quad \text{on } \partial\Omega,$

and let $g_h = R_h g$ be its Ritz projection which satisfies $-\Delta_h g_h = P_h(\partial\tilde{\delta})$. Then, continuing from (A.5), with $(v, w)_V = \int_V v \, \bar{w} \, dx$,

(A.7) $\qquad \partial v_h(x) = (\nabla v_h, \nabla g_h) = (\nabla v, \nabla g_h) = (\nabla v, \nabla(g_h - g))_{\Omega_h} + (\nabla v, \nabla g)_{\Omega_h}.$

Here, by Green's formula and (A.6),

$$(A.8) \qquad (\nabla v, \nabla g)_{\Omega_h} = -(v, \partial \tilde{\delta})_{\Omega_h} + (v, \frac{\partial g}{\partial n})_{\partial \Omega_h} = (\partial v, \tilde{\delta})_\sigma + (v, \frac{\partial g}{\partial n})_{\partial \Omega_h}.$$

With $G = G^y(\cdot)$ now denoting the standard Green's function for Poisson's equation with homogeneous Dirichlet boundary conditions, we have

$$(A.9) \qquad g(y) = (G^y, \partial \tilde{\delta})_\sigma = -(\partial G^y, \tilde{\delta})_\sigma.$$

We shall use the following result from Krasovskii [13].

Lemma A.1. *For $|\alpha| + |\beta| > 0$ there exists C such that*

$$|D_y^\alpha D_z^\beta G^y(z)| \leq C|z - y|^{-d+2-|\alpha|-|\beta|}.$$

Combined with (A.9), we then find that, for $\text{dist}\,(y, \sigma) \geq ch$,

$$(A.10) \qquad |D^\alpha g(y)| \leq C|y - x|^{-d+1-|\alpha|}.$$

By (A.3) it follows that (for brevity we write $\rho = \rho_h^x$ below) $|\nabla g(y)| \leq C\rho(y)^{-d}$ for $y \in \partial \Omega_h$. Using this in (A.8), together with a trivial estimate for the first term on the right there, and then inserting the result into (A.7), we now have

$$|\partial v_h(x)| \leq |(\nabla v, \nabla(g_h - g))_{\Omega_h}| + |\partial v|_\sigma + C\|\rho^{-d} v\|_{L_1(\partial \Omega_h)}.$$

Since clearly $|\partial v|_\sigma \leq C|(h/\rho)^s \nabla v|_{\Omega_h}$, in order to conclude the proof of the theorem it remains to show that $|(\nabla v, \nabla(g_h - g))_{\Omega_h}| \leq C|(h/\rho)^s \nabla v|_{\Omega_h}$, or

$$(A.11) \qquad \|\omega^s \nabla(g_h - g)\|_{L_1(\Omega_h)} \leq C, \quad \text{where } \omega(y) = \omega_h(y) = \rho(y)/h.$$

For this we shall use the techniques developed in Scott [24], and in [23] and [20]. With x the central point, we introduce the dyadic annuli

$$\Omega_j = \Omega_h \cap A_j, \quad \text{where } A_j = \{y \in R^d : d_j \leq |y - x| \leq d_{j-1}\}, \quad \text{with } d_j = 2^{-j}.$$

This dyadic decomposition is broken off at a suitable distance from x; we let the innermost domain be $\Omega^I = \{y \in \Omega_h : |y - x| \leq Mh\}$, where M in the end is going to be chosen large enough. We thus have $\Omega_h = \Omega^I \cup (\bigcup_{j=j_0}^{\bar{j}_h} \Omega_j)$ with $j_0 = \lceil -\log \text{diam}\,(\Omega)/\log 2 \rceil$ fixed and $\bar{j}_h = \lceil -\log(Mh)/\log 2 \rceil$, where $\lceil a \rceil$ denotes the smallest integer $\geq a$. We also use the notation $A_j' = A_{j-1} \cup A_j \cup A_{j+1}$, $A_j'' = (A_j')'$, $A_j''' = (A_j'')'$, and analogously for Ω_j', Ω_j'', and Ω_j'''.

In order to show (A.11) we now consider the contributions of the various subregions Ω^I and Ω_j. For $y \in \Omega^I$ we have $\omega(y) = \rho(y)/h \leq 1 + M$. Thus, using also the Cauchy-Schwarz inequality (writing again $\|\cdot\| = \|\cdot\|_{L_2}$),

$$\|\omega^s \nabla(g_h - g)\|_{L_1(\Omega^I)} \leq C(1 + M)^s (Mh)^{d/2}(\|\nabla g_h\| + \|\nabla g\|).$$

An easy energy argument using (A.4) establishes that

$$(A.12) \qquad \|\nabla g_h\|^2 \leq \|\nabla g\|^2 = (g, \partial \tilde{\delta}) = -(\partial g, \tilde{\delta}) \leq \|\nabla g\|\,\|\tilde{\delta}\| \leq Ch^{-d/2}\|\nabla g\|,$$

and hence

$$(A.13) \qquad \|\omega^s \nabla(g_h - g)\|_{L_1(\Omega^I)} \leq K := C(1 + M)^s M^{d/2}.$$

We now turn to the contributions of the Ω_j. With $\omega_j = d_j/h$ we have, by the Cauchy-Schwarz inequality (all summations in j below are understood to be between j_0 and $\bar{j}_h$),

$$\sum_j \|\omega^s \nabla(g_h - g)\|_{L_1(\Omega_j)} \leq C \sum_j \omega_j^s d_j^{d/2} \|\nabla(g_h - g)\|_{\Omega_j}, \quad \text{with } \|\cdot\|_V = \|\cdot\|_{L_2(V)}.$$

We shall use the following result, which shows that, locally, the energy error in the Ritz projection is bounded by local best approximation and the error itself measured in a weaker norm. The result is essentially due to Nitsche and Schatz [14], and is proved in [23] in our case when domains may impinge on the mesh boundary $\partial\Omega_h$ (cf. in particular [23], Theorem 4.1). For a domain $V \subset \Omega_h$, we let $V_\gamma = \{y \in \Omega_h : \operatorname{dist}(y, V) \le \gamma\}$.

Lemma A.2. *There exist C and $c > 0$ such that for $\gamma \ge ch$ and any $\chi \in S_h$,*

$$\|\nabla(R_h v - v)\|_V \le C\big(\|\nabla(v - \chi)\|_{V_\gamma} + \gamma^{-1}\|v - \chi\|_{V_\gamma} + \gamma^{-1}\|R_h v - v\|_{V_\gamma}\big).$$

We shall see that Lemma A.2 and standard approximation theory imply

$$(A.14) \qquad \|\nabla(g_h - g)\|_{\Omega_j} \le Ch d_j^{-d/2-1} + C d_j^{-1}\|g_h - g\|_{\Omega_j'}.$$

After this we shall use a duality argument to show that

$$(A.15) \qquad \|g_h - g\|_{\Omega_j'} \le Ch d_j^{-d/2}(\|\nabla(g_h - g)\|_{L_1(\Omega_h)} + 1) + Ch\|\nabla(g_h - g)\|_{\Omega_j'''}.$$

Together, these inequalities show that

$$\sum_j \omega_j^s d_j^{d/2}\|\nabla(g_h - g)\|_{\Omega_j} \le Ch\sum_j \omega_j^s d_j^{-1}(\|\nabla(g_h - g)\|_{L_1(\Omega_h)} + 1)$$
$$+ Ch\sum_j \omega_j^s d_j^{d/2-1}\|\nabla(g_h - g)\|_{\Omega_j'''}.$$

Since $d_{\bar{j}_h} \ge Mh$ and $s < 1$, we have

$$h\sum_j \omega_j^s d_j^{-1} = \sum_j (d_j/h)^{s-1} \le CM^{s-1},$$

and we conclude that

$$\sum_j \omega_j^s d_j^{d/2}\|\nabla(g_h - g)\|_{\Omega_j} \le CM^{s-1}(\|\nabla(g_h - g)\|_{L_1(\Omega_h)} + 1)$$
$$+ CM^{-1}\sum_j \omega_j^s d_j^{d/2}\|\nabla(g_h - g)\|_{\Omega_j'''}.$$

In the last term we may replace Ω_j''' by Ω_j by increasing C and observing that for the Ω_j that intersect Ω^I, $\|\nabla(g_h - g)\|_{\Omega_j} \le Ch^{-d/2}$ by (A.12), so that their contribution may be added to the first term on the right. Thus

$$\sum_j \omega_j^s d_j^{d/2}\|\nabla(g_h - g)\|_{\Omega_j}$$
$$\le CM^{s-1}(\|\nabla(g_h - g)\|_{L_1(\Omega_h)} + M^{d/2}) + CM^{-1}\sum_j \omega^s d_j^{d/2}\|\nabla(g_h - g)\|_{\Omega_j}),$$

and choosing M large enough we can kick back the last term to obtain

$$\sum_j \omega_j^s d_j^{d/2}\|\nabla(g_h - g)\|_{\Omega_j} \le CM^{s-1}(\|\nabla(g_h - g)\|_{L_1(\Omega_h)} + M^{d/2}).$$

It follows that

$$\|\omega^s \nabla(g_h - g)\|_{L_1(\Omega_h)} \le K + C\sum_j \omega_j^s d_j^{d/2}\|\nabla(g_h - g)\|_{\Omega_j}$$
$$\le K + CM^{s-1}(\|\nabla(g_h - g)\|_{L_1(\Omega_h)} + M^{d/2}).$$

Since $\|\nabla(g_h - g)\|_{L_1(\Omega_h)} \leq \|\omega^s \nabla(g_h - g)\|_{L_1(\Omega_h)}$, we may now kick back also the weighted $L_1(\Omega_h)$-norm, which would complete the proof of (A.11) and hence of Theorem A.1.

It remains to prove (A.14) and (A.15), and we begin with (A.14). In Lemma A.2 we take $V = \Omega_j$, $\gamma = d_j$, $V_\gamma = \Omega_j'$, and $\chi = I_h g$, the standard interpolant into S_h. We have, using (A.10),

$$\|\nabla(g - I_h g)\|_{\Omega_j'} \leq C d_j^{d/2} |\nabla(g - I_h g)|_{\Omega_j'} \leq C d_j^{d/2} h \|g\|_{W_\infty^2(\Omega_j'')} \leq C d_j^{-d/2-1} h.$$

The same estimate is also valid for $d_j^{-1}\|g - I_h g\|_{L_2(\Omega_j')}$, which shows (A.14).

For (A.15) we consider $(g_h - g, \varphi)$ with $\|\varphi\| = 1$, supp $\varphi \subset \Omega_j'$. For each such φ, let w be the solution of

$$-\Delta w = \varphi \quad \text{in } \Omega, \quad \text{with } w = 0 \quad \text{on } \partial\Omega.$$

Then, for any $\chi \in S_h$, with $\Gamma_h = \Omega \setminus \Omega_h$ the "skin" layer,

$$
\begin{aligned}
\text{(A.16)} \qquad (g_h - g, \varphi) &= (\nabla(g_h - g), \nabla w) = (\nabla(g_h - g), \nabla(w - \chi)) \\
&= (\nabla(g_h - g), \nabla(w - \chi))_{\Omega_h} - (\nabla g, \nabla w)_{\Gamma_h}.
\end{aligned}
$$

The proof of (A.15) and hence our theorem will now follow from the following two estimates:

$$\text{(A.17)} \quad |(\nabla(g_h - g), \nabla(w - \chi))_{\Omega_h}| \leq C h d_j^{-d/2} \|\nabla(g_h - g)\|_{L_1(\Omega_h)} + C h \|\nabla(g_h - g)\|_{\Omega_j'''}$$

for χ suitable, and

$$\text{(A.18)} \qquad\qquad |(\nabla g, \nabla w)_{\Gamma_h}| \leq C h d_j^{-d/2}.$$

We begin with (A.17). Since functions in W_2^2 do not necessarily have point values when $d \geq 4$, we will not be able to use the standard interpolant $I_h w$ for χ. Instead we take χ to be a Scott-Zhang type [25] interpolant. The definition and results for this interpolant needed below for curved convex domains are found in [29], Lemma 2.1. We now have

$$|(\nabla(g_h - g), \nabla(w - \chi))_{\Omega_j'''}| \leq C \|\nabla(g_h - g)\|_{\Omega_j'''} h \|w\|_{W_2^2(\Omega)} \leq C h \|\nabla(g_h - g)\|_{\Omega_j'''},$$

where we have used the elliptic regularity result $\|w\|_{W_2^2(\Omega)} \leq C \|\varphi\| = C$.

On the complement of Ω_j''' we have

$$|(\nabla(g_h - g), \nabla(w - \chi))_{\Omega_h \setminus \Omega_j'''}| \leq C \|\nabla(g_h - g)\|_{L_1(\Omega_h \setminus \Omega_j''')} h \|w\|_{W_\infty^2(\Omega \setminus A_j'')}.$$

Since $w(y) = (G^y, \varphi)_{\Omega_j'}$ for $y \notin \Omega_j'$, we have, by Lemma A.1 and the Cauchy-Schwarz inequality, for $|\alpha| > 0$,

$$\text{(A.19)} \quad |D^\alpha w(y)| \leq C \operatorname{dist}(y, \Omega_j')^{-d+2-|\alpha|} \|\varphi\|_{L_1(\Omega_j')} \leq C \operatorname{dist}(y, \Omega_j')^{-d+2-|\alpha|} d_j^{d/2}.$$

In particular, $\|w\|_{W_\infty^2(\Omega \setminus A_j'')} \leq C d_j^{-d/2}$, and so

$$|(\nabla(g_h - g), \nabla(w - \chi))_{\Omega_h \setminus \Omega_j'''}| \leq C h d_j^{-d/2} \|\nabla(g_h - g)\|_{L_1(\Omega_h)},$$

which completes the proof of (A.17).

Similarly, for the proof of (A.18), we split our considerations over $\Gamma_h \cap A_j''$ and $\Gamma_h \setminus A_j''$, respectively. First, since the volume of $\Gamma_h \cap A_j''$ is bounded by $C h^2 d_j^{d-1}$,

we have by (A.10) and Sobolev's inequality, with $0 < \frac{1}{2} - \frac{1}{p} < \frac{1}{d}$ and $\frac{1}{p} + \frac{1}{q} = 1$,

$$(\nabla g, \nabla w)_{\Gamma_h \cap A_j''} \leq C d_j^{-d} \|\nabla w\|_{L_1(\Gamma_h \cap A_j'')} \leq C d_j^{-d} (h^2 d_j^{d-1})^{1/q} \|w\|_{W_p^1(\Omega)}$$
$$\leq C d_j^{-d} (h^2 d_j^{d-1})^{1/q} \|w\|_{W_2^2(\Omega)} \leq C d_j^{-d} (h^2 d_j^{d-1})^{1/q}.$$

Since $h \leq C d_j$, and choosing p such that $d(\frac{1}{2} - \frac{1}{p}) \geq \frac{1}{p}$, the right-hand side may be estimated as

$$C d_j^{-d} (h^2 d_j^{d-1})^{\frac{1}{q}} \leq C h d_j^{-d/2} d_j^{\frac{2}{q}-1-\frac{d}{2}+\frac{d-1}{q}} = C h d_j^{-d/2} d_j^{d(\frac{1}{2}-\frac{1}{p})-\frac{1}{p}} \leq C h d_j^{-d/2}.$$

We finally consider the contribution from $\Gamma_h \setminus A_j''$. Here, by (A.10), $|\nabla g(y)| \leq C\rho(y)^{-d}$. Also, since $\mathrm{dist}\,(\Gamma_h \setminus A_j'', \Omega_j') \geq d_j$, we have from (A.19) that, in the region under consideration, $|\nabla w(y)| \leq C d_j^{-d/2+1}$, and hence, cf. (A.2),

$$|(\nabla g, \nabla w)_{\Gamma_h \setminus A_j''}| \leq C d_j^{-d/2+1} \int_{\Gamma_h} \rho(y)^{-d} \, dy \leq C h d_j^{-d/2+1} \leq C h d_j^{-d/2}.$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

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