MATHEMATICS OF COMPUTATION
Volume 73, Number 247, Pages 1345–1364
S 0025-5718(03)01585-0
Article electronically published on July 28, 2003

LOCAL BEHAVIOUR OF POLYNOMIALS

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Abstract. In this paper we study the local behaviour of a trigonometric polynomial $t(\theta) := \sum_{\nu=-n}^{n} a_\nu\, e^{\mathrm{i}\nu\theta}$ around any of its zeros in terms of its estimated values at an adequate number of freely chosen points in $[0\,,\,2\pi)$. The freedom in the choice of sample points makes our results particularly convenient for numerical calculations. Analogous results for polynomials of the form $\sum_{\nu=0}^{n} a_\nu\, x^\nu$ are also proved.

1. Introduction

Let p be a polynomial of degree at most n and suppose that $|p(z)|$ attains its maximum on the unit circle at the point $z = 1$. How near to this point can there be a zero α of p? This question was asked by P. Turán [11]. He himself pointed out that necessarily $|\alpha - 1| \geq 1/n$. In addition, he proved that if α is a point on the unit circle, the nearest positions of a zero are $\alpha = e^{\pm \mathrm{i}\pi/n}$, and that $p(z)$ must be of the form $c(1 + z^n)$, $c \neq 0$ if $p(z)$ vanishes at one of these points. Note that $t(\theta) := e^{-\mathrm{i}n\theta}\, p(e^{2\mathrm{i}\theta})$ is a trigonometric polynomial of degree at most n such that $|t(\theta)|$ attains its maximum on $[-\pi\,,\,\pi]$ at the point 0. Generalizing the question of Turán we may ask: how near to the point 0 can a zero of a trigonometric polynomial t of degree at most n be if $|t(\theta)|$ attains its maximum M on $[-\pi\,,\,\pi]$ at the point 0? It was shown by Hyltén-Cavalius [6] that

$$|\,t(\theta)\,| \;\geq\; M\,\cos n\theta \qquad \Big(-\frac{\pi}{2n} \leq \theta \leq \frac{\pi}{2n}\Big)\,, \tag{1.1}$$

with equality at some point $\theta \in [-\pi/2n\,,\,0) \cup (0\,,\,\pi/2n]$ only when $t(\theta) \equiv M e^{\mathrm{i}\gamma} \cos n\theta$ for some $\gamma \in \mathbb{R}$. In particular, $t(\theta) \neq 0$ in $(-\pi/2n\,,\,\pi/2n)$. An equivalent problem is to consider a trigonometric polynomial t of degree at most n vanishing at the origin and to determine how near to this point can $|\,t(\theta)\,|$ attain its maximum on $[-\pi\,,\,\pi]$. To see the equivalence, we may consider the nonnegative trigonometric polynomial $M^2 - t(\theta)\overline{t(\theta)}$ bounded above by M^2 on $[-\pi\,,\,\pi]$. By a well-known result of L. Fejér and F. Riesz [7, p. 77, problem 40], there exists a polynomial p of degree at most $2n$ such that

$$M^2 - t(\theta)\,\overline{t(\theta)} \equiv \big|\,p\,(e^{\mathrm{i}\theta})\big|^2 \qquad (-\pi \leq \theta \leq \pi)\,. \tag{1.2}$$

It is clear that $|\,p(z)\,| \leq M$ for $|z| = 1$ and that $|\,p(1)\,| = M$. Thus, $e^{-\mathrm{i}n\theta}\, p\,(e^{\mathrm{i}\theta})$ is a trigonometric polynomial of degree at most n, which attains its maximum

Received by the editor August 20, 2002 and, in revised form, December 22, 2002.
2000 *Mathematics Subject Classification.* Primary 42A05, 26D05, 26D10, 30C10, 30A10.
Key words and phrases. Trigonometric polynomials, algebraic polynomials, M. Riesz's interpolation formula, Schur's inequality, Bernstein's inequality.

modulus M on $[-\pi\,,\,\pi]$ at the point 0. Hence, by (1.1),

$$\left|\,p\left(e^{\mathrm{i}\theta}\right)\right| \;\geq\; M\,\cos n\theta \qquad \left(-\frac{\pi}{2n}\;\leq\;\theta\;\leq\;\frac{\pi}{2n}\right).$$

Now, (1.2) implies that

$$M^2\;-\;|\,t(\theta)|^2\;\geq\;M^2\,\cos^2\,n\theta \qquad \left(-\frac{\pi}{2n}\;\leq\;\theta\;\leq\;\frac{\pi}{2n}\right),$$

that is,

$$|\,t(\theta)\,|\;\leq\;M\,|\,\sin n\theta\,| \qquad \left(-\frac{\pi}{2n}\;\leq\;\theta\;\leq\;\frac{\pi}{2n}\right). \tag{1.3}$$

The estimate is best possible as the example $t(\theta)\;:=\;M\sin n\theta$ shows.

Conversely, we shall show that (1.3) implies (1.1). Let t be a real trigonometric polynomial of degree at most n such that $t(0)=M=\max_{\theta\in\mathbb{R}}|t(\theta)|$. Then, $t_1(\theta):=(M-t(2\theta))\,/2$ is a nonnegative trigonometric polynomial of degree at most $2n$ such that $t_1(\theta)\leq M$ for all real θ and $t_1(0)=0$. Hence, $t_1(\theta)\equiv|t_2(\theta)|^2$, where t_2 is a trigonometric polynomial of degree at most n such that $|t_2(\theta)|\leq\sqrt{M}$ for all real θ and $t_2(0)=0$. Applying (1.3) to t_2, we conclude that

$$M-t(2\theta)\leq 2M\,\sin^2 n\theta \qquad \left(-\frac{\pi}{n}\;\leq\;2\theta\;\leq\;\frac{\pi}{n}\right),$$

which proves (1.1) for *real* trigonometric polynomials. However, this additional restriction on t is superfluous. To see this, let $t(0)\,e^{-i\gamma}=M:=\max_{\theta\in\mathbb{R}}|t(\theta)|$, where $0\leq\gamma<2\pi$. Then, $\sigma(\theta):=\Re\left(e^{-i\gamma}\,t(\theta)\right)$ is a *real* trigonometric polynomial of degree at most n such that $\sigma(0)=M=\max_{\theta\in\mathbb{R}}|\sigma(\theta)|$. Hence,

$$|t(\theta)|\geq|\sigma(\theta)|\geq M\,\cos n\theta \qquad \left(-\frac{\pi}{2n}\leq\theta\leq\frac{\pi}{2n}\right).$$

Over the years, several people, including the late Professor P. Turán and the late Professor R. P. Boas, have independently asked one of us the following question. *What form would* (1.3) *take if t had a zero of multiplicity k at the origin, where $k\leq 2n$?* Although, we are not able to give an explicit upper bound for $|\,t(\theta)\,|$ in some interval around the origin, except in the case where $k\in\{1,2,2n-1,2n\}$, we can characterize the trigonometric polynomial that *maximizes* the bound. Before we do this, we recall the following result of Boas [4] according to which (1.3) holds if simply $|\,t(\theta)|\;\leq\;M$ for $\theta\;=\;\pi/2n,\,3\pi/2n,\,\ldots,\,(4n-1)\pi/2n$.

Theorem A. *Let t be a trigonometric polynomial of degree at most n such that*

$$t(0)\;=\;0 \quad\text{and}\quad |\,t\left((2\nu\;+\;1)\pi/2n\right)|\;\leq\;M \quad\text{for } \nu\;=\;0,1,\ldots,2n-1\,. \tag{1.4}$$

Then,

$$|\,t(\theta)|\;<\;M\;|\sin n\theta| \qquad (0\;<\;|\theta|\;<\;\pi/2n)\;,$$

unless $t(\theta)\;\equiv\;M\,e^{\mathrm{i}\gamma}\,\sin n\theta$ for some $\gamma\;\in\mathbb{R}$.

Here, it is important to note that the points $\pi/2n,\,3\pi/2n,\,\ldots,\,(4n-1)\pi/2n$ are exactly the points in $(0\,,\,2\pi)$, where the extremal polynomial $M\,\sin n\theta$ attains the values M and $-M$, alternately. In the case where t is assumed to have a zero of multiplicity k at the origin, a logical extremal would be a trigonometric polynomial of degree n which attains the upper bound M and the lower bound $-M$, alternately, at $2n-k+1$ points in $(0\,,\,2\pi)$. Do we know if such a trigonometric polynomial even exists? We claim that the answer is yes for any $k\;\in\;\mathbb{N}$. We do not know of any ready reference, if there is any. Instead of digressing to include an adequate

justification for our assertion at this place, we have chosen to present it as an "Appendix" in Section 6.

With the presumption of our claim having been substantiated, we state the following generalization of Theorem A.

Proposition 1.1. *Let $t_\star = t_{n,k,\star}$ be the unique trigonometric polynomial of degree n having a zero of multiplicity k at the origin such that $|t_\star(\theta)| \le M$ for $0 \le \theta \le 2\pi$ and taking the values M and $-M$, alternately, at $2n-k+1$ points in $(0\,,\,2\pi)$, say $\theta_0^* < \theta_1^* < \cdots < \theta_{2n-k}^*$. Furthermore, let t be any trigonometric polynomial of degree at most n having a zero of multiplicity k at the origin and satisfying the condition*

$$|\,t(\theta_\nu^*)\,| \;\le\; M \qquad (\nu \,=\, 0, 1, \ldots, 2n-k)\;.$$

Then, $|\,t(\theta)\,| \;<\; |\,t_\star(\theta)|$ in $\big(\theta_{2n-k}^* \,-\, 2\pi\,,\,0\big) \,\cup\, (0\,,\,\theta_0^*)$, unless $t(\theta) \;\equiv\; e^{\mathrm{i}\gamma}\, t_*(\theta)$ for some $\gamma \in \mathbb{R}$.

We omit the proof of Proposition 1.1 since we shall prove a considerably more general result in Theorem 2.2.

We know $t_{n,k,\star}$ explicitly for $k \in \{1,\,2,\,2n-1,\,2n\}$ but not for any other value of k. Thus, Proposition 1.1 is of little practical value if $3 \le k \le 2n-2$. As long as we do not know the points $\theta_0^* < \theta_1^* < \cdots < \theta_{2n-k}^*$, we shall take any convenient set of $2n-k+1$ points $\theta_0 < \theta_1 < \cdots < \theta_{2n-k}$ in $(0\,,\,2\pi)$ and see how large $|\,t(\theta)\,|$ can be at any given point of $(\theta_{2n-k} - 2\pi\,,\,0) \cup (0\,,\,\theta_0)$ if we know upper bounds for $|\,t(\theta_0)\,|\,,\,\ldots\,,\,|\,t(\theta_{2n-k})\,|$.

We shall also consider the analogous problem for polynomials of the form $\sum_{\nu=0}^{n} a_\nu x^\nu$ having a zero of multiplicity k at a given point $\xi \in \mathbb{R}$.

2. Trigonometric polynomials

Let t be a trigonometric polynomial of degree n having a zero of multiplicity k at the origin, which means that $t(0) = \cdots = t^{(k-1)}(0) = 0\,,\; t^{(k)}(0) \ne 0$. Suppose, in addition, that we know estimates for $|t(\theta)|$ at $2n-k+1$ distinct points $\theta_0 < \cdots < \theta_{2n-k}$ in $(0\,,\,2\pi)$. Then, how large can $|t(\theta)|$ be at any given point of $(\theta_{2n-k} - 2\pi\,,\,\theta_0)$? We shall find the answer to this question.

It is convenient to first introduce some notation.

Notation 2.1. Let $\mathfrak{T}_n$ denote *the set of all trigonometric polynomials* of degree at most n.

Notation 2.2. Let k and n be integers such that $2n \ge k \ge 1$. In addition, let $\theta_0, \ldots, \theta_{2n-k}$ be $2n-k+1$ distinct points in $(0\,,\,2\pi)$ such that

$$0 \;<\; \theta_0 \;<\; \theta_1 \;<\; \cdots \;<\; \theta_{2n-k} \;<\; 2\pi\,. \tag{2.1}$$

Furthermore, let $\{y_0, \ldots, y_{2n-k}\}$ be an arbitrary set of $2n-k+1$ nonnegative numbers *not all zeros*. We shall say that t *belongs to* $\mathfrak{T}_{n,k}(\theta_0, \ldots, \theta_{2n-k};\, y_0, \ldots, y_{2n-k})$ (or $\mathfrak{T}_{n,k}$ for short) if $t \in \mathfrak{T}_n$ and

$$t(0) = \cdots = t^{(k-1)}(0) = 0 \quad \text{and} \quad |t(\theta_\nu)| \;\le\; y_\nu \qquad (\nu = 0,\,1,\,\ldots,\,2n-k)\,. \tag{2.2}$$

Notation 2.3. Let $\theta_0, \dots, \theta_{2n-k}$ and $y_0, \dots, y_{2n-k}$ be as above. We shall use $\tau_{n,k}\,(\theta_0, \dots, \theta_{2n-k};\, y_0, \dots, y_{2n-k})$, or simply $\tau_{n,k}$, to denote *the unique trigonometric polynomial* of degree at most n such that

(2.3)
$$\tau_{n,k}(0) = \cdots = \tau_{n,k}^{(k-1)}(0) = 0 \quad \text{and} \quad \tau_{n,k}(\theta_\nu) \,=\, (-1)^\nu\, y_\nu \quad (\nu = 0,\, 1,\, \dots,\, 2n-k)\,.$$

Notation 2.4. In the case where $y_0 = \cdots = y_{2n-k} = 1$, we shall write $\tau_{n,k,*}$ for the trigonometric polynomial $\tau_{n,k}(\theta_0, \dots, \theta_{2n-k}\,;\, y_0, \dots, y_{2n-k})$ of Notation 2.3. It is easily checked that $\tau_{n,k,*}$ can be represented as

$$\tau_{n,k,*}(\theta) = \sum_{\nu=0}^{2n-k} (-1)^\nu \left\{ \frac{\sin\,\theta/2}{\sin\,\theta_\nu/2} \right\}^k \frac{s_{n,k}(\theta)}{2\, s'_{n,k}(\theta_\nu) \sin\,(\theta-\theta_\nu)/2}$$
$$\left(s_{n,k}(\theta) := \prod_{\nu=0}^{2n-k} \sin\frac{\theta-\theta_\nu}{2} \right).$$

Because of its relevance, we shall first determine the sharp upper bound for $|t^{(k)}(0)|$ as t varies in the class $\mathfrak{T}_{n,k}(\theta_0, \dots, \theta_{2n-k};\, y_0, \dots, y_{2n-k})$ of Notation 2.2. Our results and the proofs will reveal a new property of the Hermite type *interpolatory polynomial* $\tau_{n,k}$ introduced in Notation 2.3.

The following interpolation formula for the k-th derivative of a trigonometric polynomial, at the point 0 where it vanishes along with its derivatives of order $j \le k-1$, plays a crucial role in our investigation.

Theorem 2.1. *For any integer $k \in \{1, \dots, 2n\}$, let $\theta_0, \dots, \theta_{2n-k}$ be an increasing sequence of $2n-k+1$ numbers in $(0\,, 2\pi)$. Furthermore, let*

$$s_{n,k}(\theta) := \prod_{\nu=0}^{2n-k} \sin\frac{\theta-\theta_\nu}{2} \qquad \text{and} \qquad S_{n,k}(\theta) := \left(\sin\frac{\theta}{2}\right)^{k-1} s_{n,k}(\theta)\,.$$

Then, for any trigonometric polynomial t of degree at most n such that

$$t(0) = \cdots = t^{(k-1)}(0) = 0\,,$$

we have

$$t^{(k)}(0) = \frac{k!}{2^{k+1}}\, |s_{n,k}(0)| \sum_{\nu=0}^{2n-k} (-1)^\nu \frac{t(\theta_\nu)}{\left|S'_{n,k}(\theta_\nu)\right|} \frac{1}{\sin^2\theta_\nu/2}\,. \tag{2.4}$$

In the case where $k \ge 2$, the preceding formula also holds for any trigonometric polynomial t of degree at most n, which has a zero of multiplicity $k-1$ at 0, provided that $s'_{n,k}(0) = 0$. When $k = 1$, formula (2.4) (for $t'(0)$) holds for any trigonometric polynomial of degree at most n (whether $t(0)$ vanishes or not), provided that $S'_{n,1}(0) = s'_{n,1}(0) = 0$.

For any $k \in \mathbb{N}$, if β is any of the zeros of $S'_{n,k}$ in $(0, 2\pi)$, then

$$t'(\beta) \,=\, -\frac{1}{4}\, S_{n,k}(\beta) \sum_{\nu=0}^{2n-k} \frac{t(\theta_\nu)}{S'_{n,k}(\theta_\nu)} \frac{1}{\sin^2\,(\beta-\theta_\nu)/2}\,. \tag{2.5}$$

Example 2.1. If $\theta_\nu = (2\nu+1)\pi/2n$ for $\nu = 0, 1, \ldots, 2n-1$, then

$$S_{n,1}(\theta) \,=\, s_{n,1}(\theta) \,=\, 2^{-(2n-1)}\,\cos n\theta$$

and so, $S'_{n,1}(0) \,=\, s'_{n,1}(0) \,=\, 0$. Hence, Theorem 2.1 says in particular that for any trigonometric polynomial t of degree at most n, we have

$$t'(0) = \frac{1}{4n}\sum_{\nu=0}^{2n-1}(-1)^\nu \frac{1}{\sin^2\{(2\nu+1)\pi/4n\}}\, t\left(\frac{(2\nu+1)\pi}{2n}\right).$$

Applying this formula to the trigonometric polynomial $t(\alpha\,+\,\cdot)$, $\alpha\in\mathbb{R}$, we obtain the well-known interpolation formula

$$t'(\alpha) = \frac{1}{4n}\sum_{\nu=0}^{2n-1}\frac{(-1)^\nu}{\sin^2\{(2\nu+1)\pi/4n\}}\, t\left(\alpha\,+\,\frac{(2\nu+1)\pi}{2n}\right) \quad (\alpha\in\mathbb{R}) \tag{2.6}$$

of M. Riesz ([8], [9]; also see [3, p. 211]). Thus, (2.4) may be seen as an extension of this classical formula.

Remark 2.1. Applying (2.4) to the trigonometric polynomial $\tau = \tau_{n,k}$ of Notation 2.3 and taking note of (2.3), we see that

$$\tau^{(k)}(0) \,=\, \frac{k!}{2^{k+1}}\,|\,s_{n,k}(0)\,|\sum_{\nu=0}^{2n-k}\frac{1}{|S'_{n,k}(\theta_\nu)|}\,\frac{y_\nu}{\sin^2\,\theta_\nu/2} \,>\, 0\,.$$

The following consequence of Theorem 2.1 is an important first step towards the solution of the problem formulated in the third paragraph of the Introduction.

Corollary 2.1. *Let $\theta_0, \ldots, \theta_{2n-k}$ be as in* (2.1)*, and let $\{y_0, \ldots, y_{2n-k}\}$ be an arbitrary set of $2n-k+1$ nonnegative numbers* not all zeros*. Furthermore, let $\tau = \tau_{n,k}$ be the trigonometric polynomial of Notation* 2.3*. Then, for any trigonometric polynomial t belonging to the class $\mathfrak{T}_{n,k}$ of Notation* 2.2*, we have*

$$|\,t^{(k)}(0)\,| \,<\, |\,\tau^{(k)}(0)\,| = \tau^{(k)}(0)\,, \tag{2.7}$$

unless $t(\theta) \,\equiv\, e^{\mathrm{i}\gamma}\,\tau(\theta)$ for some $\gamma\in\mathbb{R}$.

In the case where $k\ge 2$ and $\theta_0, \ldots, \theta_{2n-k}$ are such that $s'_{n,k}(0) = 0$, inequality (2.7) *also holds if t is a trigonometric polynomial of degree at most n, having a zero of multiplicity $k-1$ at* 0*, such that $|t(\theta_\nu)| \le y_\nu$ for $\nu = 0, 1, \ldots, 2n-k$. When $k = 1$ and $\theta_0, \ldots, \theta_{2n-1}$ are such that $s'_{n,1}(0) = 0$, the estimate* (2.7) (for $|t'(0)|$) *holds for any trigonometric polynomial t of degree at most n such that $|t(\theta_\nu)| \le y_\nu$ for $\nu = 0, 1, \ldots, 2n-k$, unless $t(\theta) \equiv e^{\mathrm{i}\gamma}\,\tau(\theta)$ for some $\gamma\in\mathbb{R}$.*

In addition, if β is any of the critical points of the trigonometric polynomial $S_{n,k}$ (introduced in the statement of Theorem 2.1) lying in $(0\,,\,2\pi)$*, then*

$$|\,t'(\beta)\,| \,<\, |\,\tau'(\beta)\,|\,, \tag{2.8}$$

unless $t(\theta) \,\equiv\, e^{\mathrm{i}\gamma}\,\tau(\theta)$ for some $\gamma\in\mathbb{R}$.

Remark 2.2. We claim that the trigonometric polynomial $\tau = \tau_{n,k}$ of Notation 2.3 is positive in $(0\,,\,\theta_0)$. For this, let $\tau_{n,k,*}$ be as in Notation 2.4, and note that for any $\varepsilon > 0$ the trigonometric polynomial

$$A_\varepsilon(\theta) \,:=\, \tau(\theta) \,+\, \varepsilon\,\tau_{n,k,*}(\theta)$$

has a *simple* zero in each of the intervals $(\theta_0\,,\,\theta_1), \ldots, (\theta_{2n-k-1}\,,\,\theta_{2n-k})$. Letting ε tend to zero, we see that τ has *at least one zero* in each of the intervals

$[\theta_0\,,\,\theta_1],\dots,[\theta_{2n-k-1}\,,\,\theta_{2n-k}]$. It may be added that θ_0 and θ_{2n-k} *cannot be multiple zeros* of τ and that $\theta_1,\dots,\theta_{2n-k-1}$ cannot be zeros of multiplicity greater than 2. Counting each zero as often as its multiplicity, we see that τ has at least $2n-k$ zeros in $[\theta_0\,,\,\theta_{2n-k}]$. Since it has a zero of multiplicity k at the origin, it must have exactly $2n-k$ zeros in $[\theta_0\,,\,\theta_{2n-k}]$ and no zero in $(0\,,\,\theta_0)$. Thus, $\tau(\theta)$ is of constant sign in $(0\,,\,\theta_0)$ and the sign can only be positive because $\tau^{(k)}(0)>0$ and

$$\tau(\theta)\;=\;\frac{1}{k!}\,\tau^{(k)}(0)\,\theta^k\;+\;O\left(\theta^{k+1}\right)\quad\text{as }\theta\to 0\,.$$

The answer to the question asked in the third paragraph of the Introduction is contained in the next result.

Theorem 2.2. *Let $\theta_0,\dots,\theta_{2n-k}$ be as in* (2.1)*, and let $\{y_0,\dots,y_{2n-k}\}$ be an arbitrary set of $2n-k+1$ nonnegative numbers* not all zeros*. Furthermore, let $\tau=\tau_{n,k}$ be the trigonometric polynomial of Notation* 2.3*. Then, for any t belonging to the class $\mathfrak{T}_{n,k}$ of Notation* 2.2*, we have*

$$|\,t(\theta)\,|\;\le\;|\,\tau(\theta)\,|\qquad(\theta\in(\theta_{2n-k}-2\pi\,,\,0)\cup(0\,,\,\theta_0))\;. \tag{2.9}$$

There is equality in (2.9) *for some $\theta\in(\theta_{2n-k}-2\pi\,,\,0)\cup(0\,,\,\theta_0)$ if and only if $t(\theta)\equiv e^{\mathrm{i}\gamma}\,\tau(\theta)$ for some $\gamma\in\mathbb{R}$.*

Besides, if $k\ge 2$ and $\varphi_0<\dots<\varphi_{2n-k+1}$ are the critical points of the trigonometric polynomial $S_{n,k}$ (introduced in the statement of Theorem 2.1) in $(0\,,\,2\pi)$*, then*

$$|\,t'(\theta)\,|\;<\;|\,\tau'(\theta)\,|\qquad(\theta\in(\varphi_{2n-k+1}-2\pi\,,\,0)\cup(0\,,\,\varphi_0))\;,$$

unless $t(\theta)\equiv e^{\mathrm{i}\gamma}\tau(\theta)$ for some $\gamma\in\mathbb{R}$.

Remark 2.3. Theorem 2.2 extends Theorem A of R. P. Boas. According to that result, if t is a trigonometric polynomial of degree n with $t(0)=0$ and $|t((2\nu+1)\pi/2n)|\le 1$ for $\nu=0,\dots,2n-1$, then

$$|\,t(\theta)\,|\;\le\;|\sin n\theta|\qquad\left(|\theta|\le\frac{\pi}{2n}\right)\;.$$

Moreover, $|\,t(\theta)\,|=|\sin n\theta|$ for some θ in $(-\pi/2n\,,\,\pi/2n)\backslash\{0\}$ if and only if $t(\theta)\equiv e^{\mathrm{i}\gamma}\sin n\theta$ for some $\gamma\in\mathbb{R}$. Boas [4, p. 44] simply gives a brief outline of the proof and does not even bother to explain why $|\,t(\theta)\,|=|\sin n\theta|$ for some θ in $(-\pi/2n\,,\,\pi/2n)\backslash\{0\}$ only if $t(\theta)\equiv e^{\mathrm{i}\gamma}\sin n\theta$ for some $\gamma\in\mathbb{R}$. However, the reader will soon realize that this part of the result does not even come close to being trivial.

3. Algebraic polynomials

Here again we start with some notation.

Notation 3.1. Let $\mathfrak{P}_N$ denote the set of all polynomials of degree at most N.

Notation 3.2. For some $k\in\mathbb{N}$ and $N\ge k$ let $x_0<\dots<x_{N-k}$ be $N-k+1$ real numbers. In addition, let $\{y_0,\dots,y_{N-k}\}$ be a set of $N-k+1$ nonnegative numbers such that $\sum_{\nu=0}^{N-k}y_\nu>0$. Furthermore, let ξ be any real number other than the numbers $x_0,\dots,x_{N-k}$. We shall say that a polynomial $p\in\mathfrak{P}_N$ belongs to $\mathfrak{P}_{N,k}(x_0,\dots,x_{N-k};\,y_0,\dots,y_{N-k};\xi)$ (or $\mathfrak{P}_{N,k}$ for short) if $p(\xi)=\dots=p^{(k-1)}(\xi)=0$ and $|p(x_\nu)|\le y_\nu$ for $\nu=0,\,1,\,\dots,\,N-k$.

Notation 3.3. Let $x_0 < \cdots < x_{N-k}$ and $y_0, \ldots, y_{N-k}$ be as above. Setting $x_{-1} := -\infty$ and $x_{N-k+1} := +\infty$, let ξ belong to the interval $(x_{\mu-1}, x_\mu)$, where $\mu \in \{0, \ldots, N-k+1\}$. We shall use $\pi_{N,k}(x_0, \ldots, x_{N-k}\,;\, y_0, \ldots, y_{N-k})$ or simply $\pi_{N,k}$ to denote the unique polynomial belonging to $\mathfrak{P}_{N,k}\,(x_0, \ldots, x_{N-k}\,;\, y_0, \ldots, y_{N-k}\,;\, \xi)$ such that

$$\pi_{N,k}(x_\nu) \;:=\; \begin{cases} (-1)^{\nu-\mu} y_\nu & \text{for } \nu = \mu, \ldots, N-k, \\ (-1)^{\nu-\mu-k+1} y_\nu & \text{for } \nu = 0, \ldots, \mu-1 \end{cases}$$

if $\mu \in \{1, \ldots, N-k\}$, and

$$\pi_{N,k}(x_\nu) \;:=\; (-1)^{N-\nu} y_\nu \quad \text{for } \nu = 0, \ldots, N-k$$

if $\mu = N-k+1$, whereas

$$\pi_{N,k}(x_\nu) \;:=\; (-1)^{\nu} y_\nu \quad \text{for } \nu = 0, \ldots, N-k$$

if $\mu = 0$.

Notation 3.4. In the case where $y_0 = \cdots = y_{N-k} = 1$, we shall write $\pi_{N,k,*}$ for the polynomial $\pi_{N,k}(x_0, \ldots, x_{N-k}\,;\, y_0, \ldots, y_{N-k})$ of Notation 3.3.

It is easily checked that $\pi_{N,k,*}$ can be represented as

$$\begin{aligned} \pi_{N,k,*}(x) = \sum_{\nu=0}^{\mu-1} (-1)^{\nu-\mu-k+1} \left(\frac{x-\xi}{x_\nu-\xi}\right)^k \frac{p_{N,k}(x)}{p'_{N,k}(x_\nu)(x-x_\nu)} \\ + \sum_{\nu=\mu}^{N-k} (-1)^{\nu-\mu} \left(\frac{x-\xi}{x_\nu-\xi}\right)^k \frac{p_{N,k}(x)}{p'_{N,k}(x_\nu)(x-x_\nu)} \\ \left(p_{N,k}(x) := \prod_{\nu=0}^{N-k} (x-x_\nu)\right) \end{aligned}$$

if ξ belongs to $(x_{\mu-1}\,,\, x_\mu)$ for some $\mu \in \{1, \ldots, N-k\}$, as

$$\pi_{N,k,*}(x) = \sum_{\nu=0}^{N-k} (-1)^{N-\nu} \left(\frac{x-\xi}{x_\nu-\xi}\right)^k \frac{p_{N,k}(x)}{p'_{N,k}(x_\nu)(x-x_\nu)}$$

if ξ belongs to $(x_{N-k}\,,\, +\infty)$, and as

$$\pi_{N,k,*}(x) = \sum_{\nu=0}^{N-k} (-1)^{\nu} \left(\frac{x-\xi}{x_\nu-\xi}\right)^k \frac{p_{N,k}(x)}{p'_{N,k}(x_\nu)(x-x_\nu)}$$

if ξ belongs to $(-\infty\,,\, x_0)$.

Analogously to Theorem 2.1, we prove the following result.

Theorem 3.1. *With $x_0, \ldots, x_{N-k}$ as above, let*

$$p_{N,k}(x) \;:=\; \prod_{\nu=0}^{N-k} (x-x_\nu) \quad \text{and} \quad P_{N,k}(x) \;:=\; (x-\xi)^{k-1}\, p_{N,k}(x)\,.$$

Then, for any polynomial p of degree at most N such that $p(\xi) = \cdots = p^{(k-1)}(\xi) = 0$, we have

$$p^{(k)}(\xi) \;=\; -k!\; p_{N,k}(\xi) \sum_{\nu=0}^{N-k} \frac{p(x_\nu)}{P'_{N,k}(x_\nu)} \frac{1}{(\xi-x_\nu)^2}\,. \tag{3.1}$$

In the case where $k \geq 2$, the preceding formula also holds for any polynomial p of degree at most N, which has a zero of multiplicity $k-1$ at ξ, provided that $p'_{N,k}(\xi) = 0$. If $k = 1$, then formula (3.1) (for $p'(\xi)$) *holds for any polynomial p of degree at most N, whether $p(\xi)$ is zero or not, provided that $P'_{N,1}(\xi) = p'_{N,1}(\xi) = 0$.*

Besides, if η is any of the critical points of $P_{N,k}$, other than ξ, then

$$p'(\eta) = -P_{N,k}(\eta) \sum_{\nu=0}^{N-k} \frac{p(x_\nu)}{P'_{N,k}(x_\nu)} \frac{1}{(\eta - x_\nu)^2} . \tag{3.2}$$

Remark 3.1. It is easily seen that if ξ belongs to $(x_{\mu-1}, x_\mu)$, where $\mu \in \{1, \dots, N-k\}$, then

$$P'_{N,k}(x_\nu) := \begin{cases} (-1)^{N-k-\nu} |P'_{N,k}(x_\nu)| & \text{for } \nu = \mu, \dots, N-k, \\ (-1)^{N-\nu+1} |P'_{N,k}(x_\nu)| & \text{for } \nu = 0, \dots, \mu-1, \end{cases}$$

and $p_{N,k}(\xi) = (-1)^{N-k-\mu+1} |p_{N,k}(\xi)|$. Hence, applying formula (3.1) to the polynomial $\pi_{N,k}$ of Notation 3.3, it is easily checked that if $\xi \in (x_{\mu-1}, x_\mu)$, where $\mu \in \{1, \dots, N-k\}$, then

$$\pi^{(k)}_{N,k}(\xi) = k! |p_{N,k}(\xi)| \sum_{\nu=0}^{N-k} \frac{y_\nu}{|P'_{N,k}(x_\nu)|} \frac{1}{(\xi - x_\nu)^2} > 0 . \tag{3.3}$$

If ξ belongs to $(x_{N-k}, +\infty)$, then

$$P'_{N,k}(x_\nu) = (-1)^{N-\nu-1} |P'_{N,k}(x_\nu)| \qquad (\nu = 0, \dots, N-k) ,$$

and $p_{N,k}(\xi) = |p_{N,k}(\xi)|$. Hence, (3.3) remains true in this case.

Finally, if ξ belongs to $(-\infty, x_0)$, then

$$P'_{N,k}(x_\nu) = (-1)^{N-k+\nu} |P'_{N,k}(x_\nu)| \qquad (\nu = 0, \dots, N-k) ,$$

and $p_{N,k}(\xi) = (-1)^{N-k+1} |p_{N,k}(\xi)|$, which shows that (3.3) is true in this case also.

Corollary 3.1. *For $k \in \mathbb{N}$, let $x_0 < \dots < x_{N-k}$ be $N-k+1$ real numbers, and let $\{y_0, \dots, y_{N-k}\}$ be any set of $N-k+1$ nonnegative numbers such that $\sum_{\nu=0}^{N-k} y_\nu > 0$. Furthermore, let ξ be a real number different from the numbers $x_0, \dots, x_{N-k}$, and let $\pi_{N,k}$ be the corresponding polynomial of Notation* 3.3. *Then, for any polynomial p belonging to the class $\mathfrak{P}_{N,k}(x_0, \dots, x_{N-k}; y_0, \dots, y_{N-k}; \xi\}$ of Notation* 3.2,

$$|\, p^{(k)}(\xi) \,| < |\, \pi^{(k)}_{N,k}(\xi) \,| = \pi^{(k)}_{N,k}(\xi) \tag{3.4}$$

unless $p(x) \equiv e^{i\gamma} \pi_{N,k}(x)$ for some $\gamma \in \mathbb{R}$.

In the case where $k \geq 2$, inequality (3.4) *holds for any polynomial p of degree at most N such that $|p(x_\nu)| \leq y_\nu$, $\nu = 0, \dots, N-k$ and $p(\xi) = \dots = p^{(k-2)}(\xi) = 0$ (whether $p^{(k-1)}(\xi)$ is zero or not), provided that $p'_{N,k}(\xi) = 0$ and $p(x) \neq e^{i\gamma} \pi_{N,k}(x)$ for all $\gamma \in \mathbb{R}$. If $k = 1$, then inequality* (3.4) (for $|p'(\xi)|$) *holds for any polynomial p of degree at most N such that $|p(x_\nu)| \leq y_\nu$ for $\nu = 0, \dots, N-1$ (whether $p(\xi)$ is zero or not), provided that $p'_{N,1}(\xi) = 0$ and $p(x) \neq e^{i\gamma} \pi_{N,1}(x)$ for all $\gamma \in \mathbb{R}$.*

Besides, if ω is any of the critical points of $P_{N,k}(x) = (x-\xi)^{k-1} \prod_{\nu=0}^{N-k} (x - x_\nu)$, other than ξ, then

$$|\, p'(\omega) \,| < |\, \pi'_{N,k}(\omega) \,| , \tag{3.5}$$

whatever k may be.

Remark 3.2. We claim that the polynomial $\pi_{N,k}$ of Notation 3.3 is positive in $(\xi\,,\,x_\mu)$.

First let $\mu \in \{1,\dots,N-k\}$. With $\pi_{N,k,*}$ as in Notation 3.4, let

$$C_\varepsilon(x) \,:=\, \pi_{N,k}(x) \,+\, \varepsilon\,\pi_{N,k,*}(x) \qquad (\varepsilon > 0)\,.$$

Clearly, $C_\varepsilon(x_{\nu-1})C_\varepsilon(x_\nu) < 0$ for any $\nu \in \{1,\dots,N-k\}\backslash\{\mu\}$. Hence, C_ε vanishes at least once in $\big(x_{\nu-1}\,,\,x_\nu\big)$ for $\nu \in \{1,\dots,N-k\}\backslash\{\mu\}$. Taking its k-fold zero at ξ into account, we see that C_ε, being a real polynomial of degree N, cannot have any non-real zeros. Furthermore, it can have *at most one zero in the open interval* $(\xi\,,\,x_\mu)$. In order to show that it has none, it suffices to prove that $C_\varepsilon(x)$ must have an even number of zeros in this interval if any. However, by (3.3), there exists a positive number δ such that $C_\varepsilon(x) > 0$ in $(\xi\,,\,\xi+\delta)$, and then, since $C_\varepsilon(x_\mu) > 0$, the polynomial C_ε cannot have an odd number of zeros in $(\xi\,,\,x_\mu)$. Thus, $C_\varepsilon(x)$ does not have any zeros in $(\xi\,,\,x_\mu)$, and so it must remain positive throughout this interval. Similarly, $C_\varepsilon(x)$ does not have any zeros in $(x_{\mu-1}\,,\,\xi)$. So, apart from the k-fold zero at ξ, all the other $N-k$ zeros of C_ε lie in $(-\infty\,,\,x_{\mu-1}) \cup (x_\mu\,,\,\infty)$. Letting ε tend to zero, we see that the polynomial $\pi_{N,k}$ must have all its zeros in $(-\infty\,,\,x_{\mu-1}] \cup \{\xi\} \cup [x_\mu\,,\,\infty)$. From (3.3) it follows that $\pi_{N,k}(x)$ must be positive throughout the open interval $(\xi\,,\,x_\mu)$.

If $\xi \in (x_{N-k}\,,\,\infty)$, then C_ε vanishes at least once in $\big(x_{\nu-1}\,,\,x_\nu\big)$ for $\nu = 1,\dots,N-k$, and so, apart from the k-fold zero at ξ, the polynomial $\pi_{N,k}$ must have all its zeros in $(x_0\,,\,x_{N-k})$. Again, (3.3) implies that $\pi_{N,k}$ is positive throughout $(\xi\,,\,x_{N,k+1})$, where $x_{N,k+1} := \infty$. Similarly, $\pi_{N,k}$ is positive in $(\xi\,,\,x_0)$ if $\xi \in (-\infty\,,\,x_0)$.

Analogously to Theorem 2.2, we have the following result.

Theorem 3.2. *For $k \in \mathbb{N}$, let $x_0 < \cdots < x_{N-k}$ be $N-k+1$ real numbers. Setting $x_{-1} := -\infty$ and $x_{N-k+1} := +\infty$, let ξ belong to the interval $(x_{\mu-1}\,,\,x_\mu)$, where $\mu \in \{0,\dots,N-k+1\}$. In addition, let $\{y_0,\dots,y_{N-k}\}$ be a set of $N-k+1$ nonnegative numbers such that $\sum_{\nu=0}^{N-k} y_\nu > 0$, and let $\pi_{N,k}$ be the corresponding polynomial of Notation 3.3. Then, for any polynomial p belonging to the class $\mathfrak{P}_{N,k}(\,x_0,\dots,x_{N-k}\,;\,y_0,\dots,y_{N-k}\,;\,\xi\}$ of Notation 3.2,*

$$|\,p(x)\,| \,<\, |\,\pi_{N,k}(x)\,| \qquad (x \in (x_{\mu-1}\,,\,\xi) \cup (\xi\,,\,x_\mu)) \tag{3.6}$$

unless $p(x) \equiv e^{\mathrm{i}\gamma}\,\pi_{N,k}(x)$ for some $\gamma \in \mathbb{R}$.

The following result is due to Bernstein [2]. It remained unknown until it was rediscovered by Erdős [5].

Theorem B. *Let p be a polynomial of degree at most n with real coefficients such that $|p(x)| \le 1$ for $-1 \le x \le 1$. Furthermore, let T_n be the Chebyshev polynomial of the first kind of degree n. Then*

$$|p(z)| \le |T_n(z)| \qquad (|z| \ge 1)\,.$$

Clearly, equality holds for $p(z) \equiv \pm T_n(z)$.

The argument used to prove Theorem B can be modified to prove the following more general result.

Theorem C. *Let $x_0,\dots,x_n$ be an arbitrary set of $n+1$ real numbers and let $\mathfrak{y}_0,\dots,\mathfrak{y}_n$ be any set of $n+1$ nonnegative numbers, not all zero. Denote by Π_n*

the unique polynomial of degree at most n such that $\Pi_n(x_\nu) = (-1)^{n-\nu}\mathfrak{y}_\nu$ for $\nu = 0, \dots, n$. Furthermore, let P be a real polynomial of degree at most n such that $|P(x_\nu)| \le \mathfrak{y}_\nu$ for $\nu = 0, \dots, n$. Then,

$$|P(z)| < |\Pi_n(z)| \qquad \left(\left|z - \frac{x_0 + x_n}{2}\right| \ge \frac{x_n - x_0}{2}\,,\ z \notin \{x_0\,,\, x_n\}\right)\,, \tag{3.7}$$

unless $P(z) \equiv \pm\Pi_n(z)$. In particular, if P is a real polynomial of degree at most n such that $|P(\cos\nu\pi/n)| \le 1$ for $\nu = 0, \dots, n$, then

$$|P(z)| < |T_n(z)| \qquad (|z| \ge 1\,,\ z \ne \pm 1)$$

unless $P(z) \equiv \pm T_n(z)$.

If p satisfies the conditions of Theorem 3.2, then the polynomial $P(z) := p(z)/(z-\xi)^k$ satisfies the conditions of Theorem C with $n := N-k$ and $\mathfrak{y}_\nu = y_\nu/|x_\nu - \xi|^k$ for $\nu = 0, \dots, N-k$. Taking note of the fact that the role of Π_n, $n = N-k$, is to be played by the polynomial $\pi^{\sim}_{N,k}(x)/(x-\xi)^k$, where $\pi^{\sim}_{N,k}$ belongs to $\mathfrak{P}_{N,k}$ and satisfies the iterpolation conditions

$$\pi^{\sim}_{N,k}(x_\nu) := (-1)^{N-k-\nu}(x_\nu - \xi)^k \frac{y_\nu}{|x_\nu - \xi|^k} \qquad (\nu = 0,\, 1,\, \dots,\, N-k)\,,$$

we obtain the following addendum to Theorem 3.2.

Proposition 3.1. *Let p be a real polynomial satisfying the conditions of Theorem 3.2. Furthermore, let $\pi^{\sim}_{N,k}$ be as above. Then,*

$$|p(z)| < |\pi^{\sim}_{N,k}(z)| \qquad \left(\left|z - \frac{x_0 + x_{N-k}}{2}\right| \ge \frac{x_{N-k} - x_0}{2},\ z \notin \{x_0, x_{N-k}\}\right), \tag{3.8}$$

unless $p(z) \equiv \pm\pi^{\sim}_{N,k}(z)$.

As an application of Theorem C and the interpolation formula of M. Riesz mentioned in Example 2.1, we prove the following result for real cosine polynomials.

Theorem 3.3. *Let t be a real cosine polynomial of degree at most n such that*

$$\left|t\left(\frac{\nu\pi}{n}\right)\right| \le 1 \qquad (\nu = 0, \dots, n)\,. \tag{3.9}$$

In addition, let $t(z) \not\equiv \pm\cos nz$. Then, for $k = 0, 1, \dots,$

$$\left|t^{(2k)}(z)\right| < n^{2k}|\cos nz| \qquad \left(\left\{z = x + iy\,:\, \cosh y \ge \sqrt{1 + \sin^2 x}\right\} \setminus \{0,\, \pm\pi,\, \pm 2\pi,\, \dots\}\right)\,,$$

and

$$\left|t^{(2k+1)}(z)\right| < n^{2k+1}|\sin nz| \qquad \left(\left\{z = x + iy\,:\, \cosh y \ge \sqrt{1 + \sin^2 x}\right\} \setminus \{0,\, \pm\pi,\, \pm 2\pi,\, \dots\}\right)\,.$$

4. Proofs of the results stated in Section 2

Proof of Theorem 2.1. For any $\mu \in \{0, \dots, 2n-k\}$ the function

$$u_{k,\mu}(\theta) \;=\; u_{n,k,\mu}(\theta) \;:=\; \frac{S_{n,k}(\theta)}{S'_{n.k}(\theta_\mu)} \frac{\cos\,(\theta-\theta_\mu)/2}{2\sin\,(\theta-\theta_\mu)/2} \tag{4.1}$$

is a trigonometric polynomial of degree n. Furthermore, it is clear that

$$u_{k,\mu}(\theta_\nu) \;=\; \delta_{\mu,\nu} \;:=\; \begin{cases} 1 & \text{if } \nu=\mu, \\ 0 & \text{if } \nu \neq \mu \end{cases}$$

and that if $k \geq 2$, then $u_{k,\mu}(0) = \cdots = u_{k,\mu}^{(k-2)}(0) = 0$. Since

$$S'_{n,k}(\theta_\nu) \;=\; (-1)^{2n-k-\nu} |S'_{n,k}(\theta_\nu)| \qquad (\nu = 0, 1, \dots, 2n-k)\,,$$

we conclude that

$$U_{n,k}(\theta) \;:=\; \sum_{\nu=0}^{2n-k} t(\theta_\nu)\, u_{k,\nu}(\theta) \;=\; S_{n,k}(\theta) \sum_{\nu=0}^{2n-k} (-1)^{k+\nu} \frac{t(\theta_\nu)}{|S'_{n,k}(\theta_\nu)|} \frac{1}{2} \cot \frac{\theta-\theta_\nu}{2}$$

is a trigonometric polynomial of degree at most n interpolating t at the points $\theta_0, \dots, \theta_{2n-k}$, that is,

$$U_{n,k}(\theta_\nu) \;=\; t(\theta_\nu) \qquad (\nu = 0, 1, \dots, 2n-k)\,.$$

Besides, $U_{n,k}(0) \;=\; \cdots \;=\; U_{n,k}^{(k-2)}(0) \;=\; 0$ if $k \geq 2$.

Thus, if $k = 1$, then there exists a constant c such that the representation

$$t(\theta) \;\equiv\; c\, S_{n,k}(\theta) \;+\; U_{n,k}(\theta) \;=\; c \left(\sin \frac{\theta}{2}\right)^{k-1} s_{n,k}(\theta) \;+\; U_{n,k}(\theta) \tag{4.2}$$

holds for any trigonometric polynomial t of degree at most n; if $k \geq 2$, then the same formula holds for any trigonometric polynomial t of degree at most n, which has a zero of multiplicity at least $k-1$ at 0.

Now, let $\psi(\theta) := t(\theta)/s_{n,k}(\theta)$. Then, by (4.2),

$$\psi(\theta) = \left(\sin \frac{\theta}{2}\right)^{k-1} \left\{ c + \sum_{\nu=0}^{2n-k} (-1)^{k+\nu} \frac{t(\theta_\nu)}{|S'_{n,k}(\theta_\nu)|} \frac{1}{2} \cot \frac{\theta-\theta_\nu}{2} \right\} \qquad (\theta \notin \{\theta_0, \dots, \theta_{2n-k}\})\,.$$

Since

$$\phi(\theta) \;:=\; \left(\sin \frac{\theta}{2}\right)^{k-1} \;=\; \left(\frac{\theta}{2}\right)^{k-1} \left\{ 1 - \frac{k-1}{3!} \left(\frac{\theta}{2}\right)^2 \;+\; O\left(\theta^4\right) \right\}$$

as $\theta \to 0$, we see that

$$\phi^{(k)}(0) \;=\; 0\,, \qquad \phi^{(k-1)}(0) \;=\; \frac{(k-1)!}{2^{k-1}}\,.$$

Furthermore, if $k \geq 2$, then

$$\phi^{(j)}(0) \;=\; 0 \quad (j = 0, \dots, k-2)\,.$$

Hence, using Leibniz's rule to calculate the k-th derivative of ψ, as a product of ϕ and the function

$$\lambda(\theta) \;:=\; c \;+\; \sum_{\nu=0}^{2n-k} (-1)^{k+\nu} \frac{t(\theta_\nu)}{|S'_{n,k}(\theta_\nu)|} \;\frac{1}{2}\; \cot \frac{\theta-\theta_\nu}{2}\,,$$

we see that

$$\psi^{(k)}(0) = k\,\phi^{(k-1)}(0)\,\lambda'(0) = (-1)^{k-1}\frac{k!}{2^{k+1}}\sum_{\nu=0}^{2n-k}(-1)^{\nu}\,\frac{t(\theta_\nu)}{|S'_{n,k}(\theta_\nu)|}\,\frac{1}{\sin^2\theta_\nu/2}\,.$$

If we consider ψ as a product of t and $1/s_{n,k}$ and use Leibniz's rule once again to calculate the k-th derivative of ψ, then we see that

$$\psi^{(k)}(0) = \frac{t^{(k)}(0)}{s_{n,k}(0)}$$

if $t(0) = \cdots = t^{(k-1)}(0) = 0$ and also if t has a zero of multiplicity $k-1$ at 0, for some $k \geq 2$, provided that $s'_{n,k}(0) = 0$. Hence, in either of these two situations

$$\frac{t^{(k)}(0)}{s_{n,k}(0)} = (-1)^{k-1}\frac{k!}{2^{k+1}}\sum_{\nu=0}^{2n-k}(-1)^{\nu}\,\frac{t(\theta_\nu)}{|S'_{n,k}(\theta_\nu)|}\,\frac{1}{\sin^2\theta_\nu/2}\,.$$

However, $s_{n,k}(0) = (-1)^{2n-k+1}\,|\,s_{n,k}(0)\,|$, and so (2.4) holds for any polynomial t of degree at most n such that $t(0) = \cdots = t^{(k-1)}(0) = 0$ and also if t has a zero of multiplicity $k-1$ at 0, for some $k \geq 2$, provided that $s'_{n,k}(0) = 0$.

If $k = 1$ and the points $\theta_0, \ldots, \theta_{2n-1}$ are such that $S'_{n,1}(0) = s'_{n,1}(0) = 0$, then, differentiating the two sides of (4.2), we obtain

$$t'(0) = U'_{n,1}(0) = \frac{1}{4}\,|\,S_{n,1}(0)\,|\sum_{\nu=0}^{2n-1}(-1)^{\nu}\frac{t(\theta_\nu)}{|\,S'_{n,1}(\theta_\nu)\,|}\,\frac{1}{\sin^2\theta_\nu/2}\,.$$

In order to obtain (2.5), we may simply differentiate the two sides of (4.2) and use the fact that $S'_{n,k}(\beta) = 0$. □

Proof of Corollary 2.1. Since $|t(\theta_\nu)| \leq y_\nu$ for $\nu = 0, 1, \ldots, 2n-k$, it follows from (2.4) that

$$|\,t^{(k)}(0)\,| < \frac{k!}{2^{k+1}}\,|\,s_{n,k}(0)\,|\sum_{\nu=0}^{2n-k}\frac{1}{|\,S'_{n,k}(\theta_\nu)\,|}\,\frac{y_\nu}{\sin^2\theta_\nu/2} \tag{4.3}$$

unless there exists a real number γ such that

$$(-1)^{\nu}\,t(\theta_\nu) = y_\nu\,e^{\mathrm{i}\gamma} \qquad (\nu = 0, 1, \ldots, 2n-k)\,,$$

that is, unless $t(\theta) \equiv e^{\mathrm{i}\gamma}\,\tau(\theta)$. This does it because, in view of Remark 2.1, the right-hand side of (4.3) is simply $\tau^{(k)}(0)$.

The remaining assertions of Corollary 2.1 can be proved in exactly the same way; of course, for the proof of (2.8) we need to recall that

$$S'_{n,k}(\theta_\nu) = (-1)^{2n-k-\nu}|\,S'_{n,k}(\theta_\nu)\,|$$

for $\nu = 0, 1, \ldots, 2n-k$. □

Proof of Theorem 2.2. Let $|\,t(\theta^*)| > \tau(\theta^*)$ for some $\theta^* \in (0\,,\theta_0)$. If $t(\theta^*) = |\,t(\theta^*)\,|\,e^{\mathrm{i}\gamma}$, then

$$\eta(\theta) := \Re\left\{e^{-\mathrm{i}\gamma}\,t(\theta)\right\}$$

is a real trigonometric polynomial belonging to

$$\mathfrak{T}_{n,k}(\theta_0, \ldots, \theta_{2n-k}\,;\,y_0, \ldots, y_{2n-k})$$

such that

$$\eta(\theta^*) = |\,t(\theta^*)\,| > \tau(\theta^*)\,. \tag{4.4}$$

With $\tau_{n,k,*}$ as in Notation 2.4 let

$$B_\varepsilon(\theta) := \tau(\theta) + \varepsilon\,\tau_{n,k,*}(\theta) - \eta(\theta) \qquad (\varepsilon > 0)\,.$$

Then, clearly, there exists a positive number ε_0 such that $B_\varepsilon(\theta^*) < 0$ for $0 < \varepsilon < \varepsilon_0$. However, Corollary 2.1 implies that $B_\varepsilon(\theta)$ is positive immediately to the right of 0. Hence, B_ε has at least one zero in $(0\,,\,\theta^*)$ for $\varepsilon \in (0\,,\,\varepsilon_0)$. Besides, for such values of ε, the trigonometric polynomial B_ε has at least one zero in each of the intervals

$$(\theta^*\,,\,\theta_0),\,(\theta_0\,,\,\theta_1),\ldots,(\theta_{2n-k-1}\,,\,\theta_{2n-k})\,.$$

Thus, together with the k-fold zero at 0, it has *at least* $2n+2$ zeros in $[0\,,\,2\pi)$. Hence, $\tau(\theta)+\varepsilon\,\tau_{n,k,*}(\theta) \equiv \eta(\theta)$ for all sufficiently small $\varepsilon > 0$. However, η does not depend on ε, and so, $\eta(\theta) \equiv \tau(\theta)$, contradicting the assumption that $|\,t(\theta^*)\,| > \tau(\theta^*)$. Consequently,

$$|\,t(\theta)\,| \le \tau(\theta) \qquad (\theta \in (0\,,\,\theta_0))\,. \tag{4.5}$$

We shall now show that equality can hold in (2.9) at some point θ^* in $(0\,,\,\theta_0)$ only if

$$\eta(\theta) := \Re\left\{t(\theta)\,e^{-\mathrm{i}\gamma}\right\} \equiv \tau(\theta) \qquad (\gamma := \arg\,t(\theta^*))\,.$$

As indicated earlier, η belongs to $\mathfrak{T}_{n,k}(\theta_0,\ldots,\theta_{2n-k}\,;\,y_0,\ldots,y_{2n-k})$. Assume that $\eta \ne \tau$. Then, there must exist $j \in \{0,\ldots,2n-k\}$ such that $\eta(\theta_j) \ne (-1)^j\,y_j$, that is, $(-1)^j\eta(\theta_j)$ is different from y_j and $|\eta(\theta_j)| \le y_j$. This implies that

$$y_j > 0 \quad\text{and}\quad (-1)^j\,\{\tau(\theta_j) - \eta(\theta_j)\} = y_j - (-1)^j\,\eta(\theta_j) > 0\,. \tag{4.6}$$

Let δ_n be the trigonometric polynomial of degree at most n having a zero of multiplicity k at 0 such that $\delta_n(\theta^*) = -1$ and $\delta_n(\theta_\nu) = (-1)^\nu$ for $\nu \in \{0,1,\ldots,2n-k\}\backslash\{j\}$. Now define

$$C_\varepsilon(\theta) := \tau(\theta) - \eta(\theta) + \varepsilon\,\delta_n(\theta) \qquad (\varepsilon > 0)\,. \tag{4.7}$$

In view of (4.6) there exists a positive number ε_1 such that

$$(-1)^j\,C_\varepsilon(\theta_j) > 0 \qquad (\,0 < \varepsilon < \varepsilon_1\,)\,. \tag{4.8}$$

Since $\eta \ne \tau$, it follows from Corollary 1 that $\tau^{(k)}(0) - \eta^{(k)}(0) > 0$, and so also $C_\varepsilon^{(k)}(0) > 0$ for all sufficiently small $\varepsilon > 0$, making sure that $C_\varepsilon(\theta)$ is positive immediately to the right of 0. However, $\tau(\theta^*) = \eta(\theta^*)$ by our assumption and so

$$C_\varepsilon(\theta^*) = \varepsilon\,\delta_n(\theta^*) = -\varepsilon < 0\,. \tag{4.9}$$

Consequently, C_ε has at least one zero in $(0,\theta^*)$ provided that ε is positive and small.

From (4.7) it follows that

$$(-1)^\nu\,C_\varepsilon(\theta_\nu) = y_\nu - (-1)^\nu\,\eta(\theta_\nu) + \varepsilon > 0 \qquad (\nu \in \{0,\,1,\,\ldots,2n-k\}\backslash\{j\})\,.$$

Thus taking (4.8) into account, we see that

$$(-1)^\nu C_\varepsilon(\theta_\nu) > 0 \qquad (\nu = 0,\,1,\,\ldots,\,2n-k) \tag{4.10}$$

for all sufficiently small positive values of ε. From (4.9) and (4.10) it follows that C_ε has at least one zero in each of the $2n-k+1$ intervals

$$(\theta^*\,,\,\theta_0),\,(\theta_0\,,\,\theta_1),\,\ldots,(\theta_{2n-k-1}\,,\,\theta_{2n-k})$$

provided that ε is positive and sufficiently small. Thus, according to our tally, C_ε has at least $2n+2$ zeros in $[0\,,\,2\pi)$ for all small $\varepsilon > 0$, and so for such values of ε, we have

$$\eta(\theta) \equiv \tau(\theta) + \varepsilon\,\delta_n(\theta)\,.$$

Since η does not depend on ε, we must have $\eta(\theta) \equiv \tau(\theta)$, which is a contradiction. Hence, $\eta(\theta) < \tau(\theta)$ for all $\theta \in (0\,,\,\theta_0)$ unless $\eta(\theta) := \Re\left\{e^{-\mathrm{i}\,\gamma}t(\theta)\right\} \equiv \tau(\theta)$. We claim that $\eta(\theta) \equiv \tau(\theta)$ only if $t(\theta) \equiv e^{\mathrm{i}\,\gamma}\tau(\theta)$. Indeed, $H(\theta) := \Im\left\{e^{-\mathrm{i}\,\gamma}t(\theta)\right\}$ is a real trigonometric polynomial of degree at most n, having at the point 0 a zero of multiplicity at least k, and vanishing at the points $\theta_0, \dots, \theta_{2n-k}$, which is possible only if $H(\theta)$ is identically zero.

In order to prove that $|\,t(\theta)\,| < |\,\tau(\theta)\,|$ for all $\theta \in (\theta_{2n-k} - 2\pi\,,\,0)$ unless $t(\theta)$ is identically equal to $e^{\mathrm{i}\,\gamma}\tau(\theta)$ for some $\gamma \in \mathbb{R}$, we may simply observe that $t(-\,\cdot)$ belongs to $\mathfrak{T}_{n,k}(2\pi - \theta_{2n-k}, \dots, 2\pi - \theta_0\,;\, y_{2n-k}, \dots, y_0)$ and that, for any $\theta \in \mathbb{R}$, the trigonometric polynomial $\tau(2\pi - \theta_{2n-k}, \dots, 2\pi - \theta_0\,;\, y_{2n-k}, \dots, y_0)$ takes at the point θ the same value as $(-1)^k\,\tau(\theta_0, \dots, \theta_{2n-k}\,;\, y_0, \dots, y_{2n-k})$ does at $-\theta$.

The second part of Theorem 2.2 follows from the first part applied to t', which has a zero of multiplicity at least $k-1$. In addition, by Corollary 2.1, $|\,t'(\varphi_j)\,| < |\,\tau'(\varphi_j)\,|$ for $j = 0,\,1,\,\dots,\,2n-(k-1)$, unless $t(\theta) \equiv e^{\mathrm{i}\,\gamma}\tau(\theta)$ for some $\gamma \in \mathbb{R}$. $\square$

5. Proofs of the results stated in Section 3

Proof of Theorem 3.1. For any $\mu \in \{0, \dots, N-k\}$ the function

$$v_{k,\mu}(x) = v_{N,k,\mu}(x) := \frac{1}{P'_{N,k}(x_\mu)}\,\frac{P_{N,k}(x)}{x - x_\mu} \tag{5.1}$$

is a polynomial of degree $N-1$. It is clear that

$$v_{k,\mu}(x_\nu) = \delta_{\mu,\nu} := \begin{cases} 1 & \text{if} \quad \nu = \mu, \\ 0 & \text{if} \quad \nu \neq \mu \end{cases}$$

and that if $k \geq 2$, then $v_{k,\mu}(\xi) = \cdots = v_{k,\mu}^{(k-2)}(\xi) = 0$. Hence,

$$V_{N,k} := \sum_{\nu=0}^{N-k} p(x_\nu) v_{N,k,\mu}(x) = P_{N,k}(x) \sum_{\nu=0}^{N-k} \frac{p(x_\nu)}{P'_{N,k}(x_\nu)}\,\frac{1}{x - x_\nu}$$

is a polynomial of degree at most $N-1$ interpolating p at the points $x_0, \dots, x_{N-k}$, that is,

$$V_{N,k}(x_\nu) = p(x_\nu) \quad \text{for } 0 \leq \nu \leq N-k\,.$$

In addition, $V_{N,k}(\xi) = \cdots = V_{N,k}^{(k-2)}(\xi) = 0$ if $k \geq 2$.

Thus, if $k = 1$, then there exists a constant c such that the representation

$$p(x) \equiv c\,P_{N,k}(x) + V_{N,k}(x) \tag{5.2}$$

holds for any polynomial of degree at most N; the same formula with $k \geq 2$ holds for any polynomial p of degree at most N, which has a zero of multiplicity at least $k-1$ at ξ.

Now, let $\Psi(x) := p(x)/p_{N,k}(x)$. Then, by (5.2),

$$\Psi(x) = x^{k-1}\left\{c + \sum_{\nu=0}^{N-k} \frac{p(x_\nu)}{P'_{N,k}(x_\nu)}\,\frac{1}{x - x_\nu}\right\} \qquad (x \notin \{x_0, \dots, x_{N-k}\})\,.$$

Clearly,

$$\Psi^{(k)}(\xi) = \frac{p^{(k)}(\xi)}{p_{N,k}(\xi)} \tag{5.3}$$

if $p(\xi) = \cdots = p^{(k-1)}(\xi) = 0$. Hence, if $k \in \mathbb{N}$ and p is a polynomial of degree at most n such that $p(\xi) = \cdots = p^{(k-1)}(\xi) = 0$, then

$$p^{(k)}(\xi)\,\frac{1}{p_{N,k}(\xi)} = \Psi^{(k)}(\xi) = -k!\sum_{\nu=0}^{N-k}\frac{p(x_\nu)}{P'_{N,k}(x_\nu)}\,\frac{1}{(\xi - x_\nu)^2}\,,$$

and so (3.1) holds.

Evidently, (5.3) also holds if $p(\xi) = \cdots = p^{(k-2)}(\xi) = 0$ and $p'_{N,k}(\xi) = 0$ even if $p^{(k-1)}(\xi) \neq 0$. Hence, (3.1) remains true if p has a zero of multiplicity $k-1$ at ξ and $p'_{N,k}(\xi)$ vanishes.

In the special case where $k = 1$, formula (5.2) holds for *any* polynomial of degree at most N, whether $p(\xi)$ is zero or not. In other words

$$p(x) \equiv cP_{N,1}(x) + V_{N,1}(x) \qquad (p \in \mathfrak{P}_N)\ .$$

Differentiating the two sides of this formula and using the fact that $P'_{N,1}(\xi) = 0$, we see that the case $k = 1$ of formula (3.1) holds for any polynomial p of degree at most N if the points $x_0, \ldots, x_{N-k}$ are such that $P'_{N,1}(\xi) = p'_{N,1}(\xi) = 0$.

In order to obtain (3.2), we may simply differentiate the two sides of (5.2) and use the fact that $P'_{N,k}(\eta) = 0$. □

Proof of Theorem 3.3. Clearly, $t(\theta) \equiv P(\cos\theta)$, where P is a real polynomial of degree at most n such that

$$\left|P\left(\cos\frac{\nu\pi}{n}\right)\right| \le \left|T_n\left(\cos\frac{\nu\pi}{n}\right)\right| \qquad (\nu = 0, \ldots, n)\,.$$

Hence, by Theorem C,

$$|P(\cos z)| < |T_n(\cos z)| = |\cos nz| \qquad (\{z\,:\,|\cos z| \ge 1\}\backslash\{z\,:\,\cos z = \pm 1\})\ ,$$

that is,

$$|t(z)| < |\cos nz| \qquad (\{z\,:\,|\cos z| \ge 1\}\backslash\{0,\,\pm\pi,\,\pm 2\pi,\,\ldots\})\ . \tag{5.4}$$

Using (3.9) in (2.6), we conclude that $\left|t'\big((2\nu+1)\pi/2n\big)\right| \le n$ for $\nu = 0, \ldots, n-1$. Hence, $\left|P'\big(\cos(2\nu+1)\pi/2n\big)\right| \le \left|T'_n\big(\cos(2\nu+1)\pi/2n\big)\right|$ for $\nu = 0, \ldots, n-1$, and so, once again, Theorem C may be applied to conclude that $|P'(\cos z)| < |T'_n(\cos z)|$ if $|\cos z| > \cos\pi/2n$. In particular, this estimate for $|P'(\cos z)|$ holds if $|\cos z| \ge 1$, and consequently

$$|\sin z||P'(\cos z)| < |\sin z||T'_n(\cos z)| \qquad (\{z\,:\,|\cos z| \ge 1\}\backslash\{z\,:\,\sin z = 0\})\ ,$$

that is,

$$|t'(z)| < n|\sin nz| \qquad (\{z\,:\,|\cos z| \ge 1\}\backslash\{0,\,\pm\pi,\,\pm 2\pi,\,\ldots\})\ , \tag{5.5}$$

Since t' is a sine polynomial of degree at most n and $\left|t'\big((2\nu+1)\pi/2n\big)\right| \le n$ for $\nu = 0, \ldots, n-1$, it follows from (2.6) that $\left|t''\big(\nu\pi/n\big)\right| \le n^2$ for $\nu = 0, \ldots, n$. Thus, $n^{-2}t''$ is not only a real cosine polynomial of degree at most n but also $\left|n^{-2}t''(\cos\nu\pi/n)\right| \le 1$ for $\nu = 0, \ldots, n$. Hence, in view of (5.4) and (5.5),

$$|t''(z)| < n^2|\cos nz| \quad \text{and} \quad |t'''(z)| < n^3|\sin nz|$$

for any z such that $|\cos z| \geq 1$, except $z = 0, \pm\pi, \pm 2\pi, \ldots$. This argument can be repeatedly applied to conclude that for any $k \in \{0, 1, \ldots\}$,

$$\left|t^{(2k)}(z)\right| < n^{2k}|\cos nz| \qquad (\{z \,:\, |\cos z| \geq 1\} \setminus \{0, \pm\pi, \pm 2\pi, \ldots\})$$

and

$$\left|t^{(2k+1)}(z)\right| < n^{2k+1}|\sin nz| \qquad (\{z \,:\, |\cos z| \geq 1\} \setminus \{0, \pm\pi, \pm 2\pi, \ldots\}) \,.$$

Now, it may be added that $|\cos(x+iy)|^2 = \cosh^2 y - \sin^2 x$ if x and y are real. □

6. Appendix

In closing, we shall provide a proof of the existence theorem that was alluded to in the Introduction.

Proposition 6.1. *Let $M > 0$ and $n \in \mathbb{N}$. Then, for any positive integer $k \leq 2n$, there exists a unique trigonometric polynomial $t_\star = t_{n,k,\star}$ of degree n having a zero of multiplicity k at the origin such that $|t_\star(\theta)| \leq M$ for $0 \leq \theta \leq 2\pi$. In addition, $t_\star = t_{n,k,\star}$ takes the values M and $-M$, alternately, at $2n-k+1$ points in $(0\,,\,2\pi)$, say $\theta_0^* < \theta_1^* < \cdots < \theta_{2n-k}^*$.*

For the proof of this result, we use the solution of the following problem, which goes back to P. L. Chebyshev (see [1, pp. 51–57]): Given two real-valued functions f and s, both continuous on the finite interval $[a\,,\,b]$, how small can the deviation $\max_{a\leq x\leq b}|f(x) - s(x)\,p(x)|$ be as p varies in the class $\mathfrak{P}_N$ of all polynomials of degree at most N?

The next lemma, which is a special case of a result in [1, pp. 55 – 57], gives a characterization of the polynomial p^* that minimizes $\max_{a\leq x\leq b}|f(x) - s(x)p(x)|$ over all polynomials p of degree at most N. It may be noted that $s(x)$ *is required* (see [1, p. 51]) *to be different from* 0 *throughout* $[a\,,\,b]$.

Lemma 6.1. *Let f and s be two real valued continuous functions on $[a\,,\,b]$. Suppose, in addition, that $s(x) \neq 0$ for $a \leq x \leq b$. Then, there exists a unique polynomial $p^* \in \mathfrak{P}_N$ such that*

$$\max_{a\leq x\leq b}|f(x) - s(x)\,p^*(x)| = H := \inf_{p\in\mathfrak{P}_N}\max_{a\leq x\leq b}|f(x) - s(x)\,p(x)| \,.$$

Furthermore, there exist at least $N+2$ points on $[a\,,\,b]$ at which $f(x) - s(x)\,p^*(x)$ takes the values H and $-H$, alternately.

We shall also use the following well-known result [7, p. 84, problem 80].

Lemma 6.2. *Let p be a polynomial of degree at most $n-1$ such that $\sqrt{1-x^2}\,|p(x)| \leq 1$ on the interval $[-1\,,\,1]$. Then, $|p(x)| \leq n$ on the same interval.*

The following supplement [10, p. 280] to Lemma 6.2 will also come in handy.

Lemma 6.3. *Let p be a polynomial of degree at most n such that $p(1) = 0$ and $|p(x)| \leq 1$ on the interval $[-1\,,\,1]$. Then,*

$$\left|\frac{p(x)}{1-x}\right| \;\leq\; \frac{n}{2}\,\cot\frac{\pi}{4n} \;<\; n^2 \qquad (-1 \leq x \leq 1)\,.$$

Proof of Proposition 6.1. First, let k be an *odd* integer. Lemma 6.1, applied with

$$f(x) := \sqrt{1-x^2}\,(1-x)^{\frac{k-1}{2}}\,x^{n-\frac{k+1}{2}}, \qquad s(x) := \sqrt{1-x^2}\,(1-x)^{\frac{k-1}{2}}$$

and

$$N := n - \frac{k+3}{2}\,,$$

shows that for any $\delta \in (0\,,\,1)$,

$$H_{k,\delta} := \inf_{p\in\mathfrak{P}_{n-\frac{k+3}{2}}} \max_{-1+\delta\le x\le 1-\delta} \left\{\sqrt{1-x^2}\,(1-x)^{\frac{k-1}{2}}\left|x^{n-\frac{k+1}{2}} - p(x)\right|\right\} \tag{6.1}$$

is attained, say for p^*_δ. Furthermore, if

$$d_k(x) := \sqrt{1-x^2}\,(1-x)^{\frac{k-1}{2}}\left(x^{n-\frac{k+1}{2}} - p^*_\delta(x)\right) \qquad (k \text{ odd}),$$

then, by Lemma 6.1, there exist at least $n - \dfrac{k-1}{2}$ consecutive points on the interval $[-1+\delta\,,\,1-\delta]$ at which $d_k(x)$ takes the values $H_{k,\delta}$ and $-H_{k,\delta}$, alternately.

In (6.1), we may take $p(x) := x^{n-\frac{k+1}{2}} - (1+x)^{\frac{k-1}{2}}\,x^{n-k}$, which clearly belongs to $\mathfrak{P}_{n-\frac{k+3}{2}}$, to see that

$$\begin{aligned} H_{k,\delta} &\le \max_{-1+\delta\le x\le 1-\delta} \sqrt{1-x^2}(1-x)^{\frac{k-1}{2}}(1+x)^{\frac{k-1}{2}}\,x^{n-k} \\ &= \max_{-1+\delta\le x\le 1-\delta}(1-x^2)^{\frac{k}{2}}\,x^{n-k} \le 1\,. \end{aligned}$$

Thus, d_k^2 is a polynomial of degree $2n$ such that $\max_{-1+\delta\le x\le 1-\delta}|d_k^2(x)| \le 1$, and so, by Theorem B of Section 3,

$$\left|d_k^2\big((1-\delta)x\big)\right| \le T_{2n}(|x|) \qquad \big(x\in\mathbb{R}\backslash(-1\,,\,1)\big)\,.$$

Consequently, $\max\limits_{-1\le x\le 1}\left|d_k^2(x)\right| \le T_{2n}\left(\dfrac{1}{1-\delta}\right)$, which implies that

$$\sqrt{1-x^2}\,(1-x)^{\frac{k-1}{2}}\left|x^{n-\frac{k+1}{2}} - p^*_\delta(x)\right| \le \sqrt{T_{2n}\left(\frac{1}{1-\delta}\right)} \qquad (-1\le x\le 1)\,.$$

Using Lemmas 6.2 and 6.3, we conclude that

$$\left|x^{n-\frac{k+1}{2}} - p^*_\delta(x)\right| \le \sqrt{T_{2n}\left(\frac{1}{1-\delta}\right)}\;n^k \qquad (-1\le x\le 1)\,.$$

Hence, there exists a positive number δ' such that $|p^*_\delta(x)| < 2\,n^k + 1$ for all $x \in [-1\,,\,1]$ if $\delta\in(0\,,\delta')$. This implies that, if $\delta\in(0\,,\delta')$, then $d_k(x)\to 0$ as $x\to\pm 1$. It also shows that the search for the infimum in (6.1) may be restricted to those polynomials in $\mathfrak{P}_{n-\frac{k+3}{2}}$ whose modulus does not exceed $2\,n^k+1$ at any point in $[-1\,,\,1]$.

Now, note that for any polynomial p of degree at most $n - \dfrac{k+3}{2}$,

$$\begin{aligned} &\max_{-1\le x\le 1}\sqrt{1-x^2}\,(1-x)^{\frac{k-1}{2}}\left|x^{n-\frac{k+1}{2}} - p(x)\right| \\ &\qquad = \sqrt{\max_{-1\le x\le 1}\left|(x^2-1)\,(x-1)^{k-1}\left(x^{n-\frac{k+1}{2}} - p(x)\right)^2\right|} \ge \frac{1}{2^{n-\frac{1}{2}}} \end{aligned}$$

since the maximum modulus of a monic polynomial of degree $2n$ on $[-1\,,\,1]$ cannot be smaller than 2^{-2n+1}. Consequently, there exists $\delta_0 \in (0\,,\,\delta')$, depending on n and k, such that, for any polynomial p of degree at most $n - \dfrac{k+3}{2}$ with $\max_{-1\le x\le 1}|p(x)| \le 2n^k + 1$, the maximum of $\sqrt{1-x^2}\,(1-x)^{\frac{k-1}{2}}\left|x^{n-\frac{k+1}{2}} - p(x)\right|$ over $[-1+\delta\,,\,1-\delta]$ is the same as its maximum over $[-1\,,\,1]$ if $\delta < \delta_0$. Thus, for any such δ, the quantity $H_{k,\delta}$ is equal to

$$H_{k,0} := \inf_{p\in\mathfrak{P}_{n-\frac{k+3}{2}}} \max_{-1\le x\le 1}\left\{\sqrt{1-x^2}\,(1-x)^{\frac{k-1}{2}}\left|x^{n-\frac{k+1}{2}} - p(x)\right|\right\}.$$

In addition, there exist $n - \dfrac{k-1}{2}$ consecutive points in $(-1\,,\,1)$ at which $d_k(x)$ takes the values $H_{k,0}$ and $-H_{k,0}$, alternately. Then, clearly,

$$d_k(\cos\theta) = 2^{-n+k+1}(\sin\theta)\left(\sin\frac{\theta}{2}\right)^{k-1}\left(\cos\left(n-\frac{k+1}{2}\right)\theta - p^*(\cos\theta)\right)$$

is an odd trigonometric polynomial of degree n having a zero of multiplicity k at the origin such that $\max_{0\le\theta<2\pi}|d_k(\cos\theta)| = H_{k,0}$. Furthermore, there exist $2n-k+1$ points in $(0\,,\,2\pi)$ at which $d_k(\cos\theta)$ takes the values $H_{k,0}$ and $-H_{k,0}$, alternately.

In the case where k is *even*, Lemma 6.1, applied with

$$f(x) := (1-x)^{\frac{k}{2}}\,x^{n-\frac{k}{2}}, \qquad s(x) := (1-x)^{\frac{k}{2}} \quad \text{and} \quad N := n - \frac{k+2}{2}\,,$$

shows that for any $\delta \in (0\,,\,1)$,

$$h_{k,\delta} := \inf_{p\in\mathfrak{P}_{n-\frac{k+2}{2}}} \max_{-1\le x\le 1-\delta}(1-x)^{\frac{k}{2}}\left|x^{n-\frac{k}{2}} - p(x)\right| \tag{6.2}$$

is attained, say for p_δ^{**}. In (6.2), we may take $p(x) := x^{n-\frac{k}{2}} - (1+x)^{\frac{k}{2}}\,x^{n-k}$, which clearly belongs to $\mathfrak{P}_{n-\frac{k+2}{2}}$, to see that

$$\max_{-1\le x\le 1-\delta}(1-x)^{\frac{k}{2}}\left|x^{n-\frac{k}{2}} - p_\delta^{**}(x)\right| \le 1\,.$$

As in the earlier case, we can show that $\max_{-1\le x\le 1}|p_\delta^{**}(x)| \le 2n^k + 1$ for all sufficiently small δ.

This time, there exist at least $n - \dfrac{k-2}{2}$ consecutive points on $[-1\,,\,1-\delta]$ at which

$$d_k(x) := (1-x)^{\frac{k}{2}}\left(x^{n-\frac{k}{2}} - p_\delta^{**}(x)\right) \qquad (k \text{ even})$$

takes the values $h_{k,\delta}$ and $-h_{k,\delta}$, alternately. The point -1 must necessarily be amongst the points where $d_k(x) = \pm h_{k,\delta}$ since, otherwise, d_k' (a polynomial of degree $n-1$), would have at least n zeros. Analogously to the case where k is odd, we can now show that

$$h_{k,0} := \inf_{p\in\mathfrak{P}_{n-\frac{k+2}{2}}} \max_{-1\le x\le 1}(1-x)^{\frac{k}{2}}\left|x^{n-\frac{k}{2}} - p(x)\right|$$

is attained. In addition, there exist $n - \dfrac{k-2}{2}$ consecutive points in $[-1\,,\,1)$, at which $d_k(x)$ takes the values $h_{k,0}$ and $-h_{k,0}$, alternately; the point -1 is necessarily

amongst these points where $d_k(x) = \pm h_{k,0}$. It is clear that

$$d_k(\cos\theta) = 2^{-n+k+1}\left(\sin\frac{\theta}{2}\right)^k\left(\cos\left(n-\frac{k}{2}\right)\theta - p^*(\cos\theta)\right)$$

is an even trigonometric polynomial of degree n having a zero of multiplicity k at the origin such that $\max_{0\le\theta<2\pi}|d_k(\cos\theta)| = h_{k,0}$. Furthermore, there exist $2n-k+1$ points in $(0\,,\,2\pi)$ at which $d_k(\cos\theta)$ takes the values $h_{k,0}$ and $-h_{k,0}$, alternately. The point π is always amongst these $2n-k+1$ points.

We may use Corollary 2.1 to see the uniqueness of t_*. □

Remark 6.1. We are able to identify the trigonometric polynomial $t_{n,k,*}$ of Proposition 6.1 for $k \in \{1, 2, 2n-1, 2n\}$. In fact, $t_{n,1,*}(\theta) \equiv M\,\sin n\theta$ has the desired property for $k = 1$.

If, as usual, T_n denotes the Chebyshev polynomial of the first kind of degree n, then

$$t(\theta) \;:=\; -M\,T_n\left(\cos\frac{\pi}{2n}\,\cos\theta\right)$$

is a trigonometric polynomial of degree n such that $t(0) = t'(0) = 0$, $|t(\theta)| \le M$ for $0 \le \theta \le 2\pi$, and if $\theta_0 < \theta_1 < \cdots < \theta_{2n-2}$ are the points in $(0\,,\,2\pi)$ such that

$$\left(\cos\frac{\pi}{2n}\right)\cos\theta_\nu \;=\; \cos\frac{(\nu+1)\pi}{n} \qquad (\nu \;=\; 0, 1, \ldots, 2n-2)\ ,$$

then $t(\theta_\nu) \;=\; (-1)^\nu\,M$ for $\nu = 0, 1, \ldots, 2n-2$. Thus,

$$t_{n,2,*}(\theta) \equiv -M\,T_n\left(\cos\frac{\pi}{2n}\,\cos\theta\right).$$

We leave it to the reader to check that

$$t_{n,2n-1,*}(\theta) \equiv \frac{M\sqrt{2n}}{\left(1-\frac{1}{2n}\right)^{n-\frac{1}{2}}}\left(\sin\frac{\theta}{2}\right)^{2n-1}\cos\frac{\theta}{2} \quad\text{and}\quad t_{n,2n,*} \equiv M\left(\sin\frac{\theta}{2}\right)^{2n}.$$

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