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SOLUTIONS OF THE CONGRUENCE $a^{p-1} \equiv 1 \pmod{p^r}$

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Abstract. To supplement existing data, solutions of $a^{p-1} \equiv 1 \pmod{p^2}$ are tabulated for primes a, p with $100 < a < 1000$ and $10^4 < p < 10^{11}$. For $a < 100$, five new solutions $p > 2^{32}$ are presented. One of these, $p = 188748146801$ for $a = 5$, also satisfies the "reverse" congruence $p^{a-1} \equiv 1 \pmod{a^2}$. An effective procedure for searching for such "double solutions" is described and applied to the range $a < 10^6$, $p < \max(10^{11}, a^2)$. Previous to this, congruences $a^{p-1} \equiv 1 \pmod{p^r}$ are generally considered for any $r \geq 2$ and fixed prime p to see where the smallest prime solution a occurs.

1. Introduction

In this paper we will be concerned with solutions of the congruence $a^{p-1} \equiv 1 \pmod{p^r}$, where the base a is not a power of another integer, p is an odd prime, and $r \geq 2$. The congruence is of interest in relation to several number-theoretical questions, as is summarized in [4]; see also the bibliography therein.

Historically, much computational effort has been devoted to finding solutions p in the particular case of $r = 2$ and small fixed bases a. Especially for $a = 2$, where only the two celebrated solutions $p = 1093$ and $p = 3511$ are known, the search has been pushed to considerably high limits, more recently in [6] up to $4 \cdot 10^{12}$, and then extended to $8 \cdot 10^{12}$ by R. McIntosh [14] and to $4.8 \cdot 10^{13}$ by R. Brown [5]. Finally, an Internet based search conducted by J. Knauer and the second author [13] attained the limit of $1.25 \cdot 10^{15}$. For bases in the range $2 < a < 100$, the first substantial tabulation is found in [4], which covers, at least, all solutions $p < 10^6$, and was subsequently extended to $2 \cdot 10^8$ in [11] and to 2^{32} in [18].

For the range $100 < a < 1000$, solutions $p < 10^4$ are given in Table 1 of [1] for prime bases a only. In this table, the omission of three consecutive lines should be noted, corresponding (in our notation) to $a = 709$, with solutions $p = 17$, 199, 1663, to $a = 719$ with $p = 3$, 41, and to $a = 727$ with $p = 11$. M. Aaltonen has kindly informed us that these additions conform with the original data produced for the paper [1], so the missing lines were obviously lost during the typesetting process.

Extending those data, in our Table 1 we present all 133 solutions p existing for $10^4 < p < 10^{11}$. For the bases $a = 103$, 167, 211, 281, 283, 383, 409, 563, 661, 769, 853, 877, and 929 no solution was previously known, so in each case the smallest one is shown in the table.

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Table 1. Solutions of $a^{p-1} \equiv 1 \pmod{p^2}$ for primes a, p with $100 < a < 1000$ and $10^4 < p < 10^{11}$

a	p	a	p	a	p
101	1050139	283	46301	641	24481
103	24490789	313	1259389	643	460609
107	613181	317	2227301		7354807
109	20252173	331	6134718817	647	15266862761
127	13778951	337	30137417	653	22171
131	754480919	353	465989		637699
137	18951271		17283818861	659	65983
	4483681903	359	24350087	661	441583073
149	29573	383	28067251	691	84131
	121456243	389	29569		10843045487
	2283131621		211850543	719	4414200313
151	14107	397	279421	739	5681059
	5288341		13315373041	757	242789
	15697215641	401	115849	769	1305827821
157	122327	409	34583	773	787711
	4242923		1894600969		26259199
	5857727461	419	22891217		142719149
163	3898031	421	350677	787	427541
167	64661497	431	12755833	797	14607661
173	56087	433	129497	809	448110371
179	35059		244403	821	37871
	126443	439	170899693		209140301
191	379133	443	3406223	839	11840951
197	6237773	457	1589513	853	1125407
199	77263	479	500239	857	32478247
	1843757	491	661763933	863	12049
211	279311	499	81307	877	78926821
227	40277		24117560837	881	22385723
233	86735239	509	7215975149		94626144313
239	74047	521	8938997	887	60623
	212855197	523	19289	907	3497891
	361552687	547	1691778551	911	318917
	12502228667	557	39829	929	62199604679
241	35407	563	18920521	937	22343
251	395696461	569	25359067		500861
257	49559	571	308383		1031299
	648258371	577	1381277		258469889
263	267541	587	22091	953	513405611
	159838801		6343317671	967	44830663
269	65684482177	599	35771	971	401839
271	168629	607	40303229		7672759
	16774141	613	81371669	977	37589
	235558417		18419352383	991	26437
	12145092821	619	11682481		
281	3443059		52649183399		

For prime bases $a < 100$ we also searched the interval $2^{32} < p < 10^{11}$, obtaining four new solutions characterized by the following pairs (a, p):

$$(5, 6692367337),\ (23, 15546404183),\ (37, 76407520781),\ (97, 76704103313).$$

While in [18] the only prime bases $a < 100$ that remained without a solution $p < 2^{32}$ were $a = 29$, 47, 61, we can now assert that for $a = 29$, 47, 61, 113, 139, 311, 347, 983 the smallest solution, if one exists, must be greater than 10^{11}.

The particular cases of $a = 3$ and $a = 5$ were further examined up to $p < 10^{13}$, which revealed one more solution for $a = 5$. Thus, the congruence $5^{p-1} \equiv 1 \pmod{p^2}$ is now known to hold for

$$p = 20771,\ 40487,\ 53471161,\ 1645333507,\ 6692367337,\ 188748146801,$$

and for no other $p < 10^{13}$. The first two of these solutions were found by Riesel in 1961 (as reported in [20]), the third was first published in [4], and the fourth was found by Montgomery [18].

In [8] it was noted that $p = 1645333507$ produced 14 solutions (a, p) with $a < p$, the highest number of solutions known for a prime p. This is a consequence of the fact that for a small basis a satisfying the congruence, the power a^n, which also gives a solution, remains below p for several successive exponents n, and additional solutions with $a < p$ might also occur. Accordingly, we observed that $p = 6692367337$ has $5^n < p$ for $n = 1, 2, \ldots, 14$, and $a^{p-1} \equiv 1 \pmod{p^2}$ is also satisfied for $a = 4961139411$ and for $a = 6462265338$, giving a total of 16 solutions $a < p$. For $p = 188748146801$, instead, we have $5^n < p$ for $n = 1, 2, \ldots, 16$, but no further solution $a < p$ exists.

The solution $(a, p) = (5, 188748146801)$ turned out to be one of those exceptional instances where the "reverse" congruence $p^{a-1} \equiv 1 \pmod{a^2}$ is also satisfied. Aside from $(a, p) = (3, 1006003)$ and $(a, p) = (5, 1645333507)$, presented in [1] and [18], respectively, only three such pairs of odd primes with $a, p > 5$ were previously known; see [8] and §4. All of them had $a, p < 10^6$. Through a systematic search restricted to such occurrences, we were able to show that no other pair of this kind exists in the range $a < 10^6$, $p < \max(10^{11}, a^2)$. The details will also be given in §4.

As is usual, for all the solutions found, we checked if they satisfied $a^{p-1} \equiv 1 \pmod{p^3}$, but this was never the case. One might suspect that, except for the smallest odd primes $p = 3$ and $p = 5$, this would generally be a rare event. In fact, in the range of Table 1 in [18] only the pairs $(a, p) = (18, 7)$, $(19, 7)$, $(42, 23)$ and $(68, 113)$ lead to such a solution, apparently supporting that impression. None of these pairs satisfies the congruence for the modulus p^4.

Nevertheless, it has been known for more than a century [7] that for any power p^r of an arbitrarily chosen prime p, infinitely many bases a exist for which $a^{p-1} \equiv 1 \pmod{p^r}$ is satisfied and that a complete incongruent set of them may be determined quite easily. Moreover, the totality of those bases a happens to include an infinitude of primes. On the other hand, for increasing exponents r the first appropriate a, prime or not, may be quite a large number. We will be exhibiting the smallest prime solution for $p = 3$ and $r = 165896$, which is a number of 79153 decimal digits.

Finally, we note that for bases $a \leq 1000$ including composite values of a, a complete table of solutions has been produced for $p < 10^{10}$ and is available from the authors. For easier reference, an excerpt covering prime bases only can be seen at [12]. The complete table contains 2735 solutions (a, p). The number of solutions observed for each a gives the frequencies 60, 145, 273, 229, 171, 70, 37, 11, and 3 for the occurrence of 0, 1, 2, 3, 4, 5, 6, 7, or 8 solutions, respectively. The highest number of eight known solutions corresponds to the composite bases $a = 260$, 476, and to the prime base $a = 937$. For this prime base the solutions are $p = 3$, 41, 113, 853, 22343, 500861, 1031299, 258469889.

2. Solutions a for fixed modulus p^r

Probably the first concise statement about the solutions of $a^{p-1} \equiv 1 \pmod{p^r}$ for a fixed modulus p^r is the following theorem proved by Meyer [16] in 1902.

Theorem 1. *Let p be a prime, $r \geq 2$, and consider the set of $p^{r-1}(p-1)$ integers $a < p^r$ such that $(a,p) = 1$. Then $a^{p-1} - 1$ is divisible by p^s, $s = 1, 2, \ldots, r-1$, but not by p^{s+1}, for exactly $p^{r-1-s}(p-1)^2$ of these integers a, and is divisible by p^r for the remaining $p-1$ such integers.*

As a corollary, we see that $a^{p-1} \equiv 1 \pmod{p^r}$ has exactly $p-1$ solutions that are incongruent modulo p^r, independent of the exponent r. These solutions occur in pairs a, $p^r - a$ and include the trivial solutions 1, $p^r - 1$. Therefore a listing of the $(p-3)/2$ solutions in the interval $1 < a < (p^r - 1)/2$ would suffice to describe the complete set.

Remarkably, the algorithmic determination of these solutions was also mastered more than a century ago. Thus, Cunningham [7] tabulated the solutions a for $r = 2$ and all $p \leq 101$ in 1900. But he also gave (with perfect accuracy) the solutions for higher powers p^r, which include 5^r up to $r = 8$ and 7^r up to $r = 9$. From his tables we learn, in particular, that the four nontrivial solutions of $a^6 \equiv 1 \pmod{7^9}$ are

$$a = 14906455,\ 14906456,\ 25447151,\ 25447152.$$

The tabulation for $r = 2$ was extended to $p < 200$ by Beeger [2] and to $p < 500$ by Meissner [15], both in 1914. Meissner's table, however, was not included in his paper.

Apparently the first comprehensive table of solutions produced in the computer age is the recent one of Ernvall and Metsänkylä [8], who listed all pairs $a, p < 10^6$ satisfying $a^{p-1} \equiv 1 \pmod{p^2}$, with inclusion of bases a that are congruent modulo p^2. Also in [8] it was shown that the closest possible proximity of two solutions as observed in the Cunningham example above is a quite general phenomenon. In fact, for every prime $p \equiv 1 \pmod{6}$ and each $r \geq 2$ there exists a solution a such that $a + 1$ also satisfies the congruence. For $r = 2$ this had already been proven by Beeger [2].

As to the computational aspect, we note that the clue for an effective determination of the solutions for $r = 2$ was given, and exemplified for $p = 11$, by Worms de Romilly [21] in a charming little note of 1901. The procedure was restated and extensively used by Beeger in [2]. It can be expressed more generally as follows.

Theorem 2. *Let a_1 be a primitive root of the prime p and define $a_r = a_1^{p^{r-1}} \bmod p^r$ for any $r \geq 2$. Then $\left\{a_r^m \bmod p^r : m = 0, 1, \ldots, p-2\right\}$ represents a complete set of incongruent solutions of $a^{p-1} \equiv 1 \pmod{p^r}$, each of which generates an infinite sequence of solutions in arithmetic progression with difference p^r.*

The listed solutions $a_r^m \bmod p^r$ may be given in a computationally more convenient form, which is derived from the fact that for a primitive root a_1 of p we always have $a_r^{(p-1)/2} \equiv -1 \pmod{p^r}$. As a consequence, $a_r^{(p-1)/2+m} \equiv -a_r^m \pmod{p^r}$ for $m = 0,\ 1,\ \ldots (p-3)/2$. Hence the set of incongruent solutions is equivalently described by $\{\pm\, a_r^m \bmod p^r : m = 0, 1, \ldots, (p-3)/2\}$.

As an example, consider $p = 7$, $r = 2$, $a_1 = 3$, $a_2 = 31$. The expression $a_r^m \bmod p^r$ gives $a_2^m \equiv 1,\ 31,\ 30 \pmod{49}$ for $m = 0, 1, 2$, the companion solutions being $-1 \equiv 48$, $-31 \equiv 18$, $-30 \equiv 19$. In increasing order we then have the solutions $a = 1,\ 18,\ 19,\ 30,\ 31,\ 48$, and all those obtained by successively adding $7^2 = 49$, starting with $a = 50,\ 67,\ 68,\ 79,\ 80,\ 97,\ 99$. Thus, in the range of Table 1 in [4], which is $1 < a < 100$, the prime $p = 7$ must occur as a solution for 12 different bases a, which is in accordance with the table.

By using the same procedure, we can now determine for which of those bases the congruence $a^{p-1} \equiv 1 \pmod{p^3}$ is also satisfied. With $a_3 = 325$ we get the solutions $a = 1,\ 18,\ 19,\ 324,\ 325,\ 342, \ldots$, two of which are within the range of the considered table.

For the larger table covering $1 < a \le 1000$, mentioned in §1, we could have predicted that it contains $999 \cdot 2/3^2 = 222$ pairs $(a, 3)$, as well as $\lfloor 999 \cdot 4/5^2 \rfloor = 159$ pairs $(a, 5)$ and $\lfloor 999 \cdot 6/7^2 \rfloor = 122$ pairs $(a, 7)$. Moreover, the expression $999 \cdot \sum (p-1)/p^2$, extended to all odd primes $p < 10^5$, should give a good estimate for the number of solutions (a, p) with p in that range. In fact, while about 2001 solutions are predicted, 2020 are actually counted in the table.

3. The smallest prime solution a

Turning our attention now to prime solutions a, we know by Dirichlet's theorem that infinitely many do exist for any modulus p^r. In our examples for $p = 7$, $r = 2$, 3, we readily found a prime base, $a = 19$, which, incidentally, was the same for both exponents. But this cannot always be expected. For instance, in the case $p = 11$, $r = 3$, by the above procedure we would have had to list 19 composite solutions a before encountering the first prime solution $q = 2663$. It is therefore desirable to have an easily computable upper bound for the first occurrence of a prime solution q, which may be obtained by exhibiting a well-defined example.

Theorem 3. *Let q_{+1} be the smallest prime of the form $2hp^r + 1$, let q_{-1} be the smallest prime of the form $2hp^r - 1$, and define $q_0 = \min(q_{+1}, q_{-1})$. Then $a = q_0$ is a prime solution of $a^{p-1} \equiv 1 \pmod{p^r}$. For $p = 3$, this is always the smallest prime solution.*

For the proof we assume $p - 1 = d \cdot 2^s$ with d odd. Then we have the factorization

$$a^{d\cdot 2^s} - 1 = (a^d - 1)(a^d + 1) \prod_{i=1}^{s-1} (a^{d\cdot 2^i} + 1),$$

where $a^d - 1$ is always divisible by $a - 1$ and $a^d + 1$ is divisible by $a + 1$ whenever d is odd, which we are assuming. Therefore, if either $a - 1$ or $a + 1$ is an even multiple of p^r, say $2hp^r$, then $a = 2hp^r + 1$ or $a = 2hp^r - 1$, respectively, is an odd solution of the congruence $a^{p-1} \equiv 1 \pmod{p^r}$. Since these solutions are in arithmetic progression, we have only to search, in each case, for the first h making a a prime to settle the main statement of the theorem.

For $p = 3$ the above factorization simplifies to $a^2 - 1 = (a-1)(a+1)$, implying that every prime solution $a > 2$ must be of one of the forms $2h \cdot 3^r + 1$ or $2h \cdot 3^r - 1$.

TABLE 2. Smallest solutions $a = q_{\min} \le q_0$ of $a^{p-1} \equiv 1 \pmod{p^r}$

	p	$q_{\min}$	q_0	p	$q_{\min}$	q_0
$r = 2$	3	17	17	43	19	3697
	5	7	101	47	53	8837
	7	19	97	53	521	56179
	11	3	241	59	53	6961
	13	19	337	61	601	44651
	17	131	577	67	1301	17957
	19	127	2887	71	11	50411
	23	263	4231	73	619	10657
	29	41	10091	79	31	37447
	31	229	7687	83	269	41333
	37	691	5477	89	3187	47527
	41	313	3361	97	53	56453
$r = 3$	3	53	53	43	3623	159013
	5	193	251	47	6397	1245877
	7	19	1373	53	9283	893261
	11	2663	2663	59	63463	410759
	13	239	13183	61	38447	453961
	17	653	78607	67	36809	1804577
	19	2819	27437	71	21499	715823
	23	8401	194671	73	75227	1556069
	29	10133	48779	79	1523	2958233
	31	6287	59581	83	55933	9148591
	37	691	202613	89	42937	8459629
	41	10399	413527	97	341293	5476039

When $p > 3$, the prime q_0 defined by the theorem can sometimes also be the smallest prime solution of all, as in the above example of $p = 11$, $r = 3$, where $q_0 = q_{+1} = 2 \cdot 11^3 + 1 = 2663$ while $2 \cdot 11^3 - 1$ is divisible by 3.

In Table 2 and Table 3 some specific results illustrating the above can be seen for primes $p < 100$ and $2 \le r \le 5$, as well as for a few small primes and moduli up to $r = 10$. Note the case of $p = 7$, $r = 9$ in relation to Cunningham's example.

Table 4 reflects the fact that with current computational means the smallest solution with $p = 3$ can easily be determined for quite arbitrary large exponents r. Considerably larger examples are found for some isolated values of r. In fact, if $a = 2 \cdot 3^r - 1$ is a prime, then the congruence $a^2 \equiv 1 \pmod{3^r}$ is satisfied for this but for no smaller prime base a. Similarly, if $a = 2 \cdot 3^r + 1$ is a prime and the companion number $2 \cdot 3^r - 1$ is not, then again we have at once the smallest prime solution a for the corresponding power p^r.

We have shown that $2 \cdot 3^r - 1$ is prime for $r = 1$, 2, 3, 7, 8, 12, 20, 23, 27, 35, 56, 62, 68, 131, 222, 384, 387, 579, 644, 1772, 3751, 5270, 6335, 8544, 9204, 12312, 18806, 21114, 49340, 75551, 90012, and for no other $r \le 100000$.

Also, it is known that $2 \cdot 3^r + 1$ is prime for $r = 1$, 2, 4, 5, 6, 9, 16, 17, 30, 54, 57, 60, 65, 132, 180, 320, 696, 782, 822, 897, 1252, 1454, 4217, 5480, 6225, 7842, 12096, 13782, 17720, 43956, 64822, 82780, 105106, 152529, 165896, and for no other $r \le 170000$.

These findings yield, in particular, the explicit expression for the smallest prime solution modulo the power 3^r for each of the exponents $r = 1252$, 1454, 1772, 3751,

Table 3. Smallest solutions $a = q_{\min}$ of $a^{p-1} \equiv 1 \pmod{p^r}$

	p	$q_{\min}$	p	$q_{\min}$	p	$q_{\min}$
$r=4$	3	163	29	78017	61	8065789
	5	443	31	690143	67	3246107
	7	3449	37	398023	71	1353383
	11	45989	41	1977343	73	5934307
	13	239	43	574081	79	15631613
	17	15541	47	1513367	83	2864371
	19	2819	53	4388179	89	14754769
	23	60793	59	3198427	97	15012733
$r=5$	3	487	29	24639193	61	130702609
	5	14557	31	40373093	67	304154189
	7	32261	37	70697317	71	143584109
	11	275393	41	31851901	73	183298237
	13	220861	43	47289133	79	79451167
	17	15541	47	456330179	83	1058782027
	19	2342959	53	10000453	89	352845203
	23	1051847	59	154075723	97	567620413
$r=6$	3	1459	11	2120879	19	2342959
	5	14557	13	7654109	23	90603883
	7	152617	17	24527681		
$r=7$	3	4373	7	3294173	11	28723679
	5	735443				
$r=8$	3	13121	7	3376853	11	174625993
	5	3124999				
$r=9$	3	39367	5	7812499	7	135967277
$r=10$	3	472391	5	78124999	7	135967277

Table 4. Smallest solutions $a = q_0 = 2h \cdot 3^r + \varepsilon$ of $a^2 \equiv 1 \pmod{3^r}$

r	h	ε	r	h	ε	r	h	ε
2	1	−1	20	1	−1	200	6	+1
3	1	−1	30	1	+1	300	79	+1
4	1	+1	40	20	−1	400	56	+1
5	1	+1	50	4	−1	500	39	+1
6	1	+1	60	1	+1	600	602	−1
7	1	−1	70	5	−1	700	11	+1
8	1	−1	80	7	−1	800	35	+1
9	1	+1	90	22	+1	900	61	+1
10	4	−1	100	45	−1	1000	51	+1

4217, 5270, 5480, 6225, 6335, 7842, 8544, 9204, 12096, 12312, 13782, 17720, 18806, 21114, 43956, 49340, 64822, 75551, 82780, 90012, 105106, 152529, 165896. The prime $a = 2 \cdot 3^{165896} + 1$ has 79153 digits.

It should be remarked that most of the primality proofs were accomplished with Gallot's excellent program `Proth.exe` [9]. The seven primes having $r > 50000$ were discovered by I. Buechel and the first author in the period 1999–2003, using that program.

Table 5. Values of r where the smallest prime solution of $a^4 \equiv 1 \pmod{5^r}$ is $a = q_0 = 2h \cdot 5^r + \varepsilon$

r	h	ε	r	h	ε	r	h	ε
8	4	-1	40	11	$+1$	63	3	$+1$
9	2	-1	41	28	$+1$	65	3	-1
10	4	-1	42	8	$+1$	66	9	-1
11	3	-1	44	5	$+1$	69	2	-1
13	1	$+1$	45	1	$+1$	71	15	-1
15	2	-1	46	6	$+1$	72	3	-1
23	3	$+1$	47	12	$+1$	81	6	-1
24	1	-1	48	6	-1	83	18	$+1$
26	8	$+1$	49	9	-1	84	28	-1
27	3	$+1$	50	2	$+1$	85	27	-1
28	3	-1	51	6	-1	88	55	-1
30	1	-1	52	25	-1	89	11	-1
33	3	$+1$	53	5	-1	95	5	-1
36	13	-1	54	1	-1	96	1	-1
38	10	-1	60	12	-1	98	24	$+1$
39	2	-1	62	15	$+1$	99	13	$+1$

Beyond $p = 3$, the case of $p = 5$ might also be of special interest. Since for this prime we have

$$a^{p-1} - 1 = a^4 - 1 = (a-1)(a+1)(a^2+1),$$

the smallest prime solution of $a^4 \equiv 1 \pmod{5^r}$ must be of the form $a = 2h \cdot 5^r + 1$ or of the form $a = 2h \cdot 5^r - 1$, or it must be a solution of the congruence $a^2 \equiv -1 \pmod{5^r}$. The latter has two different roots for each r that can be determined recursively (cf. [3], p. 198). However, to see whether the smallest prime base a satisfying $a^4 \equiv 1 \pmod{5^r}$ is one of these roots, it is not necessary to know them in advance. Instead, we can use our general procedure for finding the smallest appropriate prime a of all and then compare with the smallest primes of the forms $a = 2h \cdot 5^r \pm 1$. This was actually carried out for all $r \le 300$. The result for $r \le 100$ is shown in Table 5. Exponents that are not listed have their least prime solution satisfying $a^2 \equiv -1 \pmod{5^r}$. This is observed for a total of 191 exponents $r \le 300$.

Returning to the more modest dimensions of Table 2, we note in its first segment devoted to $r = 2$ that the smallest prime solution $q_{\min}$ may sometimes be smaller than p, as is the case for $p = 11$, 43, 59, 71, 79, and 97. It has generally been asked [19, p. 345] how many prime solutions $a < p$ or even $a < \sqrt{p}$ of $a^{p-1} \equiv 1 \pmod{p^2}$ may exist for a fixed prime p.

We have examined all 664577 primes p with $5 \le p < 10^7$ in this regard. It turned out that $618178 = 93.02\%$ of these primes do not have one single prime solution $a < p$, or, in other terms, for all these primes we have $q_{\min} > p$. For the remaining 46399 primes with $q_{\min} < p$, the exact number of prime solutions $a < p$ is 1, 2, 3, or 4 for 44784, 1575, 37 and 3 primes p, respectively. Four such solutions were found for $p = 24329$, with $a = 1777$, 3301, 4919, 13691, for $p = 2105669$, with $a = 248891$, 654923, 1296877, 1865299, and for $p = 9656869$, with $a = 788393$, 1639607, 1786913, 7860337. Only 76 primes p show just one prime solution $a < \sqrt{p}$, the smallest being $p = 11$, as seen in Table 2.

4. Search for prime bases a satisfying the "reverse" congruence

As was mentioned in §1, only three pairs (a, p) of primes with $a, p > 5$ are currently known which simultaneously satisfy both congruences

$$a^{p-1} \equiv 1 \pmod{p^2} \qquad \text{and} \qquad p^{a-1} \equiv 1 \pmod{a^2}.$$

These pairs are $(a, p) = (4871, 83)$, $(18787, 2903)$, and $(318917, 911)$. The first one was discovered by M. Aaltonen (see [10]) and the other two by Mignotte and Roy [17]. Note that the last pair also appeared in our Table 1 (in reverse order). The paper [10] and the report [17] point to the most important application of those pairs, which is the study of Catalan's equation.

Consistently using the procedure outlined in Theorem 2, the search for pairs (a, p) satisfying both of the above congruences can be carried out with great efficiency, and some insights about the chances of finding a new one might possibly be obtained.

We know how to generate a complete sequence of bases a satisfying the first congruence, up to some arbitrary limit. Since we are interested in prime bases only, the composite ones can be eliminated by some convenient sieving process. Then only the remaining bases a have to be checked out as possible solutions of the second congruence. As an example, let us consider the case of $p = 83$. The first prime solutions a are

$$a = 269, 293, 401, 821, 1451, 1453, 2161, 2633, 3181, 3851, \mathbf{4871}, 5839, \ldots\ ,$$

the eleventh of which yields the known solution. Up to this point, the other 641 existing odd primes could implicitly be ignored. More generally, to the limit of $a < 10^7$, of the existing 664578 odd primes only 8001 had to be tested. Similarly, for $p = 2903$ the prime bases a to be taken into account are

$$a = 5347, 11593, \mathbf{18787}, 35437, 38651, 45821, 205991, 213611, 252667, \ldots\ ,$$

altogether 231 of the 664578 odd primes existing below 10^7.

The quantitative version of Dirichlet's theorem about primes in an arithmetic progression (see [19, pp. 274–275]) leads to the general statement that for a fixed prime p and a high limit N for a, about $\pi(N)/(p-1)$ primes would have to be tested to reach that limit. Thus, for $p = 83$ and $p = 2903$ we obtain an estimate of 8104 and 229 primes, respectively, in good agreement with the actual frequencies.

Based on this approach, it would be desirable to derive some heuristic result about the expectation of encountering some new "double solution" (a, p).

If a uniform upper limit N is envisaged for the practical search, as p becomes larger and larger, the complete set of incongruent basic solutions $a < p^2$ extends over an interval that eventually exceeds that limit. Since we cannot discern in advance which of the basic solutions to be generated would surpass the prescribed limit N, their totality has to be calculated anyway. Under these circumstances we found it reasonable always to sieve through the complete set of available "candidates" and to test the remainder for the desired property.

By actual computation we have determined that no further pair of those in question exists for $p < 10^6$ and $a < \max(10^{11}, p^2)$ or vice versa. The crossing point is $p \approx 316228$, and the highest of the varying limits attained was about 10^{12}. In practice, multiples of 3, 5, 7, 11, or 13 among the generated odd values of a were first eliminated, the remaining ones being subjected to a simple Fermat test. Although this procedure fails to detect a few composite values of a, testing them unnecessarily does not really affect the efficiency of the program.

We have also tested (or re-tested) the $44784 \cdot 1 + 1575 \cdot 2 + 37 \cdot 3 + 3 \cdot 4 = 48057$ pairs (a, p) with $a < p < 10^7$ referred to at the end of §3, which slightly extends the covered range.

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