INTEGER TRANSFINITE DIAMETER AND POLYNOMIALS WITH SMALL MAHLER MEASURE

VALÉRIE FLAMMANG, GEORGES RHIN, AND JEAN-MARC SAC-ÉPÉE

ABSTRACT. In this work, we show how suitable generalizations of the integer transfinite diameter of some compact sets in $\mathbb{C}$ give very good bounds for coefficients of polynomials with small Mahler measure. By this way, we give the list of all monic irreducible primitive polynomials of $\mathbb{Z}[X]$ of degree at most 36 with Mahler measure less than $1.324...$ and of degree 38 and 40 with Mahler measure less than 1.31.

1. INTRODUCTION

The Mahler measure of a polynomial

$$P = c_0 X^n + \cdots + c_n = c_0 \prod_{k=1}^{n}(X - \alpha_k) \in \mathbb{C}[X], \ c_0 \neq 0,$$

is

$$M(P) = |c_0| \prod_{k=1}^{n} \max(1, |\alpha_k|).$$

For an algebraic number α, we denote by $M(\alpha)$ the Mahler measure of its minimal polynomial in $\mathbb{Z}[X]$. If $P \in \mathbb{Z}[X]$, we have $M(P) \geq 1$ and the theorem of Kronecker implies that $M(P) = 1$ if and only if P is a product of cyclotomic polynomials and a power of X. The Lehmer question is the following

Lehmer's question. *Does there exist a positive constant c such that, if $M(\alpha) > 1$, then $M(\alpha) \geq 1 + c$?*

We say that the polynomial P is reciprocal if $X^n P(1/X) = P(X)$, and that an algebraic number is reciprocal if its minimal polynomial is reciprocal. C.J. Smyth [SM71] has proved that, if the algebraic number $\alpha \neq 0, 1$ is nonreciprocal, then $M(\alpha) \geq \theta_0$, where $\theta_0 = 1.324717\cdots$ is the smallest Pisot number which is the real root of $X^3 - X - 1$.

P. Voutier [V] has proved that, if the algebraic number α of degree $n > 2$ is not a root of unity, then

$$M(\alpha) \geq 1 + \frac{1}{4}\left(\frac{\log \log n}{\log n}\right)^3.$$

The smallest value for $M(\alpha) > 1$ has been obtained by D.H. Lehmer [DHL] himself and is $M(\alpha) = 1.1762808\cdots$ given by the polynomial of degree 10:

$$L(X) = X^{10} + X^9 - X^7 - X^6 - X^5 - X^4 - X^3 + X + 1.$$

Exhaustive searches have been made by D.W. Boyd [BO80] for degrees 4 to 16. He wrote "The case $n = 18$ is feasible, but potentially expensive, so this has been deferred in the hope of future theoretical progress or else an improved algorithm." Nevertheless he [BO89] and M.J. Mossinghoff [MO] and [WM] succeeded to complete the search for degree 18 to 20 and 22 to 24, respectively, to find all the algebraic numbers α with $M(\alpha) < 1.3$. The search for degree 24 involved testing about 9.8×10^{12} polynomials. Here we will reduce this number to 1.3×10^6. We give the list [WE] of all algebraic integers of degree ≤ 36 with Mahler measure $< \theta_0$ and of degree 38 and 40 with measure < 1.31 (which is close to Mossinghoff's limit point $1.3090983806\cdots$) and then prove that Mossinghoff's table [WM] is complete up to degree 40.

Other extensive searches have been made by M.J. Mossinghoff [WM] with heuristic methods up to degree 180. G. Rhin and J.-M. Sac-Épée [RSE] used both a statistical method and a minimization method to get polynomials of large degree and small Mahler measure.

In this paper we give new bounds for the coefficients of the polynomials with small measure. For this, we use a large family of explicit auxiliary functions. These functions are related to a generalization of the integer transfinite diameter.

We follow Boyd's strategy [BO80]. For a fixed d and a fixed $M \leq \theta_0$ we put E_0 as the set of all irreducible polynomials P with $\deg(P) = 2d$ and $M(P) < M$. The first step is to get a large set E_1 of polynomials P containing E_0. Then, we eliminate from the set E_1 a large proportion of polynomials which do not succeed to some tests. We get a new set E_2. These computations are done with the language Pascal (in double precision). In the last step we use GP-Pari [PARI] to find the final set of desired polynomials E_0. For this, we use a modified Graeffe algorithm.

In Section 2, we give some notations and a brief scheme of the computations. In Section 3, we recall the method of explicit auxiliary functions as it was used to study the particular case of small Salem numbers. In Section 4, we define the generalized integer transfinite diameter closely related to the previous explicit auxiliary function of Section 3. In Section 5, we explain how we generalize the situation of Sections 3 and 4 to the case of polynomials with small Mahler measure and how Wu's algorithm [WU] and the semi-infinite linear programming method provide good explicit auxiliary functions. This gives bounds for s_k $(1 \leq k \leq 40)$, where s_k is the sum of the kth powers of the roots of P and then bounds for the coefficients of P. In Section 6 we give new bounds for s_{2k} depending on the values of s_k. Then we get the set E_1. In Section 7 we explain how to get the subset E_2. Section 8 is devoted to the final computations to get E_0.

2. The technique for the computation

2a. **Notations.** Because of Smyth's theorem, an algebraic integer with Mahler measure less than θ_0 is reciprocal. Its minimal polynomial P is of degree $2d$

$$P = X^{2d} + c_1 X^{2d-1} + \cdots + c_{2d-1} X + 1 = \prod_{i=1}^{2d} (X - \alpha_i).$$

We may suppose that $|\boldsymbol{\alpha}_i| \geq 1$ and $\boldsymbol{\alpha}_{d+i} = 1/\boldsymbol{\alpha}_i$ for $1 \leq i \leq d$. We define the polynomial Q, associated to the polynomial P by the formula $X^d Q(X + 1/X) = P(X)$. Therefore

$$Q = X^d + b_1 X^{d-1} + \cdots + b_{d-1} X + b_d$$

is a monic polynomial of degree d with integer coefficients whose roots are $\gamma_i = \boldsymbol{\alpha}_i + 1/\boldsymbol{\alpha}_i$ for $1 \leq i \leq d$.

We define $s_1 = \sum_{1 \leq i \leq d} \gamma_i = \sum_{1 \leq i \leq 2d} \boldsymbol{\alpha}_i$. More generally for $k > 1$ we define $\gamma_{i,k} = \boldsymbol{\alpha}_i^k + 1/\boldsymbol{\alpha}_i^k$ and $s_k = \sum_{1 \leq i \leq d} \gamma_{i,k} = \sum_{1 \leq i \leq 2d} \boldsymbol{\alpha}_i^k$. Let $1 < M \leq \theta_0$ be a fixed bound. If we suppose that $1 \leq |\boldsymbol{\alpha}_i| \leq M^a$ with $0 < a \leq 1$, then γ_i lies inside the ellipse

$$\mathcal{E}_a = \{z = x + iy \text{ such that } \left(\frac{x}{A}\right)^2 + \left(\frac{y}{B}\right)^2 \leq 1\},$$

where $A = M^a + M^{-a}$ and $B = M^a - M^{-a}$. We denote by $\mathcal{E}_0$ the interval $[-2, 2]$.

2b. **A more general set of polynomials.** The bounds for the coefficients of the polynomials P in the set E_0 are also valid for more general polynomials. Since the polynomials with $M(P) < 1.33$ and $2d < 16$ have been already computed in [BO80], we suppose now that $2d \geq 16$. For M fixed, we assume that the polynomials P satisfy the following conditions:

    1 P is a monic reciprocal polynomial of degree $2d$ with integer coefficients, and $M(P) < M$. Its roots are, as before, the numbers $\boldsymbol{\alpha}_i$ for $1 \leq i \leq 2d$ and Q, is the polynomial associated to P.

        Let k be a fixed integer. Let R_k^+ (respectively R_k^-) be the polynomial whose roots are the numbers $\boldsymbol{\alpha}_i^k$ (respectively $-\boldsymbol{\alpha}_i^k$) for $1 \leq i \leq 2d$. We say that *the polynomial P satisfies the property $\mathcal{C}_k$* if no factor of the polynomials Q associated to R_k^+ or R_k^- divides any of the polynomials Q_j which belong to a finite set S which will be defined later.

    2 For $1 \leq j \leq 40$, the polynomial P has the property $\mathcal{C}_j$.

2c. **The technique for computation of the set E_1.** The main task of the calculation is to compute bounds for the quantities s_k for $1 \leq k \leq 40$. This is done by using a family of explicit auxiliary functions. For small k, this method drastically improves the classical bounds. Then, using Newton's formula, $s_k c_0 + s_{k-1} c_1 + \cdots + s_1 c_{k-1} + k c_k = 0$, we obtain by induction bounds for the coefficients c_k for $1 \leq k \leq d$. At this point, all the coefficients of P are known because P is reciprocal. Then we get the set E_1 of polynomials P (and Q). There are two cases to examine.

2d. **The first case.** In the first case we suppose that P has a root $\boldsymbol{\alpha}_1$ of modulus greater than $M^{1/2}$. Since $\boldsymbol{\alpha}_1$ is real, we may suppose that $\boldsymbol{\alpha}_1$ is positive. We divide this case in 6 subcases: $1/2 + (j-1)/12 \leq \log|\boldsymbol{\alpha}_1|/\log M < 1/2 + j/12$ for $1 \leq j \leq 6$. For each j the numbers $\gamma_2, \ldots, \gamma_d$ belong to the ellipse $\mathcal{E}_{1/2-(j-1)/12}$, and, therefore, they are close to the real axis. Moreover, we will see that the worst situation occurs when $d - 2$ roots $\boldsymbol{\alpha}_i$ are of modulus 1, i.e., the corresponding numbers $\gamma_{i,k}$ belong to $\mathcal{E}_0$, and one root is of modulus $M^{1/2-(j-1)/12}$.

2e. **The second case.** In the second case all the roots $\boldsymbol{\alpha}_i$ of P for $1 \leq i \leq d$ are of modulus $\leq M^{1/2}$. We will see that, in this case, the worst situation occurs when $d - 2$ roots $\boldsymbol{\alpha}_i$ are of modulus 1 and two are of modulus $M^{1/2}$. We may assume that the polynomial Q has the following property: the first nonzero coefficient b_i with i odd is positive.

2f. **Conclusion of Section 2.** Then, in any case and in the worst situation, at least $d - 2$ roots of Q are in the interval $(-2, 2)$. So, it is of particular interest to study what happens when all the roots of Q are real > -2 and only one is > 2, that is to say, when P is the minimal polynomial of a Salem number. This situation is investigated in the next section.

3. An explicit auxiliary function in the case of Salem numbers

3a. **Definition.** A Salem number is a positive algebraic integer $\tau > 1$ whose one conjugate $(1/\tau)$ lies inside the open unit disk, and all the others have modulus 1. The minimal polynomial P of τ is reciprocal of degree $2d$. We define the polynomial Q as in Section 2. Q is a polynomial with d real roots, and all but $\gamma_1 = \tau + 1/\tau > 2$ are in the real interval $(-2, 2)$.

3b. **A lower bound for the trace of Salem numbers.** We define the explicit auxiliary function f by the formula

$$(3.1) \qquad f(x) = x - \sum_{1 \leq j \leq J} e_j \log |Q_j(x)|,$$

where the polynomials Q_j are nonzero elements of $\mathbb{Z}[X]$ and the numbers e_j are positive real numbers. We denote by m the minimum of the function $f(x)$ for $x > -2$ and $Q_j(x) \neq 0$ for $1 \leq j \leq J$. Such an auxiliary function was given by Smyth [SM84] with $m = -0.2281$. It was slightly improved in [FGR] to $m = -0.2264606$ and recently by Smyth [SM03] to $m = -0.2216214$. In this case $J = 17$ and for all j $\deg Q_j \leq 10$. We have $\sum_{1 \leq i \leq d} f(\gamma_i) \geq dm$ and

$$s_1 \geq dm + \sum_{1 \leq j \leq J} e_j \log |\prod_{1 \leq i \leq d} Q_j(\gamma_i)|.$$

We assume now that the polynomial P satisfies the conditions of Section 2 where the set S is the set of polynomials Q_j. Then $\prod_{1 \leq i \leq d} Q_j(\gamma_i)$ is a nonzero rational integer, because it is the resultant of Q and Q_j. Therefore

$$(3.2) \qquad s_1 \geq dm.$$

3c. **Upper and lower bounds for s_k in the case of Salem numbers.** For $k > 1$, τ^k is also a Salem number; then we get the same lower bound for s_k. We now replace $\gamma_{i,k}$ by $-\gamma_{i,k}$ in the previous summation. We now suppose that $\tau < \theta_0$. Since f is an increasing function in $(-\infty, -2)$, we get the following inequality:

$$(3.3) \qquad -s_k \geq (d-1)m + f(-\theta_0^k - \theta_0^{-k}).$$

3d. **Improvement of the lower bound for** s_k**.** Now, we want to get good lower bounds for s_k when τ belongs to a small interval, for example to the last interval of the first case of Section 2: $t_1 = \theta_0^{11/12} = 1.294036 \cdots \leq \tau < \theta_0$. The function f is increasing in the interval $[3.03, \infty)$ so, if $t_1^k + t_1^{-k} > 3.03$, i.e., for $k > 4$, the inequality (3.2) may be replaced by

$$(3.4) \qquad s_k \geq (d-1)m + f(t_1^k + t_1^{-k}).$$

3e. **A numerical example for the degree** 34**.** We give an example to show that the bounds for s_k are fairly good. The numbers $\tau_1 = 1.296210\cdots$ and $\tau_2 = 1.324667\cdots$ are two Salem numbers of degree 34 which belong to the interval $[t_1, \theta_0)$. Their minimal polynomials are

$$\begin{aligned} P_1 = {} & X^{34} - X^{33} - X^{31} + X^{29} + X^{27} - 2X^{26} + X^{23} + X^{22} - X^{21} \\ & - X^{20} - X^{19} + X^{18} + X^{17} + X^{16} - X^{15} - X^{14} - X^{13} + X^{12} \\ & + X^{11} - 2X^8 + X^7 + X^5 - X^3 - X + 1 \end{aligned}$$

and

$$P_2 = X^{34} - X^{32} - X^{29} - X^{26} - X^{23} - X^{20} - X^{17} - X^{14} - X^{11} - X^8 - X^5 - X^2 + 1,$$

respectively. The conditions on P_1 and P_2 are satisfied because the degrees of the polynomials Q_j are at most 10. We give in Table 1, for some values of k, the lower bound (min) for s_k given by (3.2) or (3.4), the values of $s_{k,1}$ and $s_{k,2}$, for P_1 and P_2 respectively, and the upper bound (max) for s_k given by (3.3). In the last line we give the value of the classical upper bound (MAX) for $|s_k|$, which is $M^k + 2d - 2 + M^{-k}$.

TABLE 1.

k	1	2	3	4	6	8	10	20	30	40
min	−3	−3	−3	−3	−3	−1	3	157	2259	30021
$s_{k,1}$	1	1	4	5	4	13	11	175	2399	32149
$s_{k,2}$	0	2	0	2	2	10	17	277	4607	76605
max	4	6	7	9	13	18	27	295	4635	76758
MAX	34	34	34	35	37	41	48	308	4642	76757

3f. **Conclusion of Section 3.** This explicit auxiliary function is sufficient to compute all Salem numbers τ which are less than a reasonable bound (say 1.3 as in [FGR]) and with $\deg(\tau) \leq 40$. We remark that there is no need for the second case because τ^2 would belong to the first case. On the one hand, since in the first case of small Mahler measure the nonreal roots of Q are not too far from the real axis, this auxiliary function is certainly not a bad one. On the other hand, in the second case, it is useful to get a better auxiliary function adapted to this particular situation.

Therefore, it is suitable to understand how to construct such functions in more general situations. For this purpose, we first show, in the next section, how these explicit auxiliary functions are connected to a generalization of the integer transfinite diameter.

4. Relations between explicit auxiliary functions and integer transfinite diameter

4a. A new form for $f(x)$. If, inside the auxiliary function of (3.1), we replace the real numbers e_j by rational numbers, we may write

$$f(x) = x - \frac{t}{h} \log |H(x)|,$$

where H is in $\mathbb{Z}[X]$ of degree h and t is a positive real number. We want to get a function f whose minimum m on $(-2, \infty)$ is as large as possible. That is to say, we seek a polynomial $H \in \mathbb{Z}[X]$ such that

$$\sup_{x > -2} |H(x)|^{t/h} e^{-x} \le e^{-m}.$$

Now, if we suppose that t is fixed, say $t = 1$, it is clear that we need to get an effective upper bound for the quantity

$$(4.1) \qquad t_{\mathbb{Z}, \boldsymbol{\varphi}}([-2, \infty)) = \liminf_{\substack{h \ge 1 \\ h \to \infty}} \inf_{\substack{H \in \mathbb{Z}[X] \\ e \deg H = h}} \sup_{x > -2} |H(x)|^{t/h} \boldsymbol{\varphi}(x)$$

in which we use the weight $\boldsymbol{\varphi}(x) = e^{-x}$. To get an upper bound for $t_{\mathbb{Z}, \boldsymbol{\varphi}}([-2, \infty))$, it is sufficient to get an explicit polynomial $H \in \mathbb{Z}[X]$ and then to use the sequence of the successive powers of H. Let us now define the usual integer transfinite diameter.

4b. The integer transfinite diameter. Let K be a compact subset of $\mathbb{C}$. If P is a polynomial in $\mathbb{C}[X]$ we put $|P|_{\infty, K} = \sup_{z \in K} |P(z)|$. We define the integer transfinite diameter of K by

$$t_{\mathbb{Z}}(K) = \liminf_{\substack{n \ge 1 \\ n \to \infty}} \min_{\substack{P \in \mathbb{Z}[X] \\ \deg P = n}} |P|_{\infty, K}^{1/n}.$$

For any $n \ge 1$ we say that a polynomial P_n (not always unique) is a *Chebyshev polynomial* if $|P_n|_{\infty, K} = \min_{P \in \mathbb{Z}[X], \deg(P) = n} |P|_{\infty, K}$. It is known that, if $K = [a, b]$ is a real interval of length $b - a \ge 4$, then $t_{\mathbb{Z}}(K) = \frac{b-a}{4}$. However, if $b - a < 4$, then $t_{\mathbb{Z}}(K) < 1$, but in this case the exact value of $t_{\mathbb{Z}}(K)$ is not known. For more details see [BE], [FRS] or [PR].

4c. How to construct an explicit auxiliary function. It can be seen that in (4.1), if t is fixed, we have a generalization of the classical integer transfinite diameter. Even though we have replaced the set K by an infinite interval, it is clear that, because of the weight $\boldsymbol{\varphi}$, the quantity $t_{\mathbb{Z}, \boldsymbol{\varphi}}([-2, \infty))$ is finite. In general, to get a good upper bound for $t_{\mathbb{Z}}$ (we recall that t is fixed), we need a polynomial H of large degree h (about 10^8), but it is not possible to compute a Chebyshev polynomial P_h of such a large degree h. So we will proceed as follows. For a fixed t, it is possible to use Wu's algorithm [WU] to compute Chebyshev polynomials, or at least polynomials whose norm is close to the minimal norm, of degree less than 40. We then use the irreducible factors of such a polynomial as polynomials Q_j in the auxiliary function. So, to deal with the general situation, i.e., for varying t, we first take an initial value of t (say $t_0 = 1$) and compute a polynomial H of small degree (if possible less than 5). We define Q_1 as an irreducible factor of H, and we take the best value for e_1 to get the best auxiliary function f_1. We deduce from this the value of $t = t_1$, and we compute a new polynomial H to get a new irreducible factor Q_2. Then we optimize the function f_2 with the two factors Q_1

and Q_2. We continue this process until the degree of H becomes too large. For the optimization of the auxiliary function, we use a refinement of the semi-infinite linear programming method introduced into number theory by Smyth [SM84].

5. Bounds for s_k for polynomials with small Mahler measure

In the rest of the paper we assume that the polynomials P satisfy the conditions of Section 2, where S is the set of polynomials listed in Table 3 (at the end of this section).

Throughout this section we will use an explicit auxiliary function f of the following type:

$$(5.1) \qquad f(z) = \text{Re}(z) - \sum_{1 \leq j \leq J} e_j \log |Q_j(z)|.$$

Here z is a complex number, the e_j are positive real numbers and the Q_j are those given in Table 3. Note that the numbers e_j are always chosen to get the best auxiliary function. For the degree 34 we use 1111 different auxiliary functions.

For $a \geq 0$ we define the function g as

$$(5.2) \qquad g(a) = \min_{z \in \mathcal{E}_a} f(z).$$

We need the following lemma.

Lemma 1. 1. *Let $n \geq 1$ be an integer and let $a_1, \ldots, a_n, b$ be $n + 1$ real numbers satisfying the conditions $b > 0$, $a_i \in [0, b]$ for $1 \leq i \leq n$ and $\sum_{1 \leq i \leq n} a_i \leq b$. Let h be a linear decreasing real function on $[0, b]$. Then*

$$(5.3) \qquad \sum_{1 \leq i \leq n} h(a_i) \geq (n - 1)h(0) + h(b).$$

2. *We now suppose that $n > 1$ and all the numbers $a_i \in [0, b/2]$. Then*

$$(5.4) \qquad \sum_{1 \leq i \leq n} h(a_i) \geq (n - 2)h(0) + 2h(b/2).$$

Proof. 1. We have

$$(5.5) \qquad \sum_{1 \leq i \leq n} h(a_i) \geq \sum_{1 \leq i \leq n-1} h(a_i) + h\Big(b - \sum_{1 \leq i \leq n-1} a_i\Big),$$

and the right term is constant and equal to $(n - 1)h(0) + h(b)$.

2. If $\sum_{1 \leq i \leq n-1} a_i$ is not greater than $b/2$, then we apply (5.3) to this partial sum (with b replaced by $b/2$) and take the trivial lower bound $h(b/2)$ for the last term. Otherwise we proceed as in case 1 and get, as in (5.5), a constant term which is equal to the right term of (5.4).

a. A lower bound for s_1 in the second case.

We first examine the second case which is the simpler one, i.e., when the roots $\boldsymbol{\alpha}_i$ satisfy $1 \leq |\boldsymbol{\alpha}_i| \leq M^{1/2}$ for $1 \leq i \leq d$. Then all the roots γ_i of Q belong to the ellipse $\mathcal{E}_{1/2}$. Since $\gamma_i \in \mathcal{E}_{a_i}$ with $a_i = \log |\boldsymbol{\alpha}_i| / \log M$, we have

$$\sum_{1 \leq i \leq d} f(\gamma_i) \geq \sum_{1 \leq i \leq d} g(a_i)$$

and, since Q does not divide any Q_j,

$$(5.6) \qquad s_1 \geq \sum_{1 \leq i \leq d} g(a_i).$$

If we have a linear decreasing function h such that, for all $x \in [0, 1/2]$, $h(x) \leq g(x)$ (we write now $h \leq g$), then, using case 2 of Lemma 1, we get from (5.6)

$$s_1 \geq (d-2)h(0) + 2h(1/2).$$

For a fixed ε (say $\varepsilon = 1/10$), we want to find a function $h \leq g$ such that

$$(d-2)h(0) + 2h(1/2) \geq (d-2)g(0) + 2g(1/2) - \varepsilon.$$

Now we explain how to choose a suitable function h in a special case. For $2d = 34$, let f be the auxiliary function which is used to get a lower bound for s_1. The polynomials Q_j are the following 11 polynomials given in Table 3:

$$Q_1, Q_2, Q_4, Q_8, Q_9, Q_{11}, Q_{13}, Q_{14}, Q_{17}, Q_{20}, Q_{21},$$

and the coefficients e_j are, respectively, equal to

$$0.\,14768307, \ 0.\,49117397, \ 0.\,56247516, \ 0.\,03015563, \ 0.\,22586214, \ 0.\,05159352,$$

$$0.\,15415127, \ 0.\,02119227, \ 0.\,05373559, \ 0.\,04126272, \ 0.\,04828564.$$

We have

$$g(0) = -0.\,20529481\cdots, \qquad g(1/2) = -0.\,67503341\cdots,$$

and

$$(d-2)g(0) + 2g(1/2) = -4.\,42948908\cdots.$$

Put

$$\varepsilon_0 = \frac{\varepsilon}{2(d-2)}, \qquad \varepsilon_1 = \frac{\varepsilon}{4},$$

$$m_0 = g(0) - \varepsilon_0, \qquad m_1 = g(1/2) - \varepsilon_1.$$

The function h is defined by $h(0) = m_0$ and $h(1/2) = m_1$. We find a staircase function s such that $h \leq s \leq g$. The sequence $0 < x_t < \cdots < x_2 < x_1 = 1/2$ of points where s is not continuous is defined by induction as follows. We begin at $x_1 = 1/2$, where $s(x_1) = g(x_1)$. We define x_2, which is $< x_1$, by $h(x_2) = g(x_1)$. Since g and h are decreasing we have, for any $x \in (x_2, x_1)$, $h(x) < s(x) = g(x_1) < g(x)$, and we continue until $g(x_t) \geq m_0$. The last stair is $(0, x_t)$, where s has the value $g(x_t)$. In our special case $t = 13$ and $(d-2)h(0) + 2h(1/2) \geq -4.\,52948908$, so $s_1 \geq -4$.

Now we have

$$(5.7) \qquad s_1 \geq (d-2) \min_{x \in \mathcal{E}_0} f(x) + 2 \min_{z \in \mathcal{E}_{1/2}} f(z) - \varepsilon,$$

and we proceed as in Section 3. We replace the numbers e_j by rational numbers, and we take the exponential. Then the right side of inequality (5.7) becomes

$$(5.8) \qquad \Big(\sup_{x \in \mathcal{E}_0} |H(x)|^{t/h} e^{-x} \Big)^{d-2} \Big(\sup_{z \in \mathcal{E}_{1/2}} |H(z)|^{t/h} e^{-\mathrm{Re}(z)} \Big)^2,$$

where $H \in \mathbb{Z}[X]$. Since, for d large and M sufficiently close to 1, the first factor in (5.8) is the larger one, we apply Wu's algorithm to $\sup_{x \in \mathcal{E}_0} |H(x)|^{t/h} e^{-x}$ to get the polynomials Q_j. Once we have obtained a list of polynomials Q_j, it remains to get the best real numbers e_j.

b. The semi-infinite linear programming method.

Since f is a harmonic function, the minimum of $f(z)$ in the ellipse $\mathcal{E}_{1/2}$ is taken on its boundary $\mathcal{B}(\mathcal{E}_{1/2})$. It is clear that, if in (5.7) we take the minimum on a finite set of points of $[-2,2] \cup \mathcal{B}(\mathcal{E}_{1/2})$, we obtain a value which is greater than or equal to the right side of (5.7). Moreover, if we replace $\mathrm{Re}(z)$ by $e_0\mathrm{Re}(z)$ with the condition $e_0 = 1$, we may find the best numbers e_j by linear programming. We define by induction a sequence of finite sets X_n, $n \geq 0$, as $X_n = Y_n \cup Z_n$ with $Y_n \subset \mathcal{E}_0$ and $Z_n \subset \mathcal{B}(\mathcal{E}_{1/2})$. We start with an arbitrary set of points X_0 of cardinal greater than J. At each step $n \geq 0$ we compute the best numbers e_j by linear programming with the set of points X_n. We get a function f_n. This gives a minimum

$$\mu_n = (d-2) \min_{x \in Y_n} f_n(x) + 2 \min_{z \in Z_n} f_n(z).$$

This quantity is greater than

$$m_n = (d-2) \min_{x \in \mathcal{E}_0} f_n(x) + 2 \min_{z \in \mathcal{B}(\mathcal{E}_{1/2})} f_n(z).$$

Let $m_{n,0} = \min_{x \in \mathcal{E}_0} f_n(x)$ and $m_{n,1} = \min_{z \in \mathcal{B}(\mathcal{E}_{1/2})} f_n(z)$. We add to the set Y_n (resp. Z_n) the points of $\mathcal{E}_0$ (resp. $\mathcal{B}(\mathcal{E}_{1/2})$), where f_n has a local minimum which is smaller than $m_{n,0} + c_{n,0}$ (resp. $m_{n,1} + c_{n,1}$). $c_{n,0}$ and $c_{n,1}$ are decreasing sequences of positive numbers tending to 0 when n is increasing. These sequences are chosen such that the set X_n does not increase too quickly. After a small number of steps, the integer parts of μ_n and m_n are equal, and we stop. This gives a lower bound for s_1.

c. The bounds for s_k in the second case.

To obtain an upper bound for s_k it is sufficient to replace M by M^k and to replace $\boldsymbol{\alpha}_i$ by $\boldsymbol{\alpha}_i^k$. Since the numbers $-\boldsymbol{\alpha}_i$ satisfy the same conditions as $\boldsymbol{\alpha}_i$, $-s_k$ has the same upper bound as s_k.

d. A numerical example in the second case.

The polynomial $P_3 = X^{38} + X^{37} - X^{29} - X^{19} - X^9 + X + 1$ has a Mahler measure equal to $1.308128 \cdots < 1.31$, and it has only two complex conjugate roots of modulus > 1. The classical bound for s_k, when we are in the second case, is $|s_k| \leq 2M^{k/2} + 2d - 4 + 2M^{-k/2}$. We give in Table 2, for some values of k, the value of s_k for P_3, the bound that we obtain (max) and the value of the classical bound (MAX).

<div align="center">TABLE 2.</div>

k	1	2	3	4	10	19	28	38	39	40
s_k	-1	1	-1	1	-9	18	99	-246	324	-409
max	4	6	7	8	17	41	106	360	409	465
MAX	38	38	38	38	42	60	121	372	421	477

e. Bounds for s_k in the first case.

Now we examine the case where Q has a positive real root $\boldsymbol{\alpha}_1 > M^{1/2}$ so $\boldsymbol{\alpha}_1$ belongs to one of the 6 intervals $[t_1(j), t_2(j)]$ defined in Section 2. Then all the other roots $\gamma_2, \ldots, \gamma_d$ of Q belong to the ellipse $\mathcal{E}_{1/2-(j-1)/12}$. We get

$$(5.9) \qquad \sum_{1 \leq i \leq d} f(\gamma_i) \geq \sum_{2 \leq i \leq d} g(a_i) + \min_{t_1 \leq x \leq t_2} f(x),$$

where $a_i = \log|\boldsymbol{\alpha}_i|/\log M$.

As in item a above we get a suitable linear decreasing function h defined on the interval $[0, 1/2 - (j-1)/12]$ such that $g(a) \geq h(a)$ for $a \in [0, 1/2 - (j-1)/12]$.

We use the case 1 of Lemma 1 and get from (5.9)

$$(5.10) \qquad s_1 \geq (d-2)h(0) + h(1/2 - (j-1)/12) + \min_{t_1 \leq x \leq t_2} f(x)$$

and, as in item a,

$$(5.11) \qquad s_1 \geq (d-2)\min_{x \in \mathcal{E}_0} f(x) + \min_{z \in \mathcal{E}_{1/2-(j-1)/12}} f(z) + \min_{t_1 \leq x \leq t_2} f(x) - \varepsilon.$$

We use the list of polynomials Q_j of Table 3, and for the semi-infinite linear programming we get a sequence of finite sets of points in $[-2, 2] \cup \mathcal{B}(\mathcal{E}_{1/2-(j-1)/12}) \cup [t_1, t_2]$.

For the upper bound of s_1 we use the numbers $-\boldsymbol{\alpha}_i$ and, similar to (5.11), we get

$$(5.12) \qquad -s_1 \geq (d-2)\min_{x \in \mathcal{E}_0} f(x) + \min_{z \in \mathcal{E}_{1/2-(j-1)/12}} f(z) + \min_{-t_2 \leq x \leq -t_1} f(x) - \varepsilon.$$

For $k > 1$ we replace M by M^k and we obtain two inequalities of the same kind as (5.11) and (5.12).

Table 3 gives the list of polynomials Q_j of the set S which are used in the auxiliary functions of Sections 5 and 6. $d = \deg Q_j$ and the coefficients of Q_j are written from degree d to 0.

TABLE 3.

Q	d	Coefficients of Q				
Q_1	1	1	0			
Q_2	1	1	1			
Q_3	1	1	-1			
Q_4	1	1	2			
Q_5	1	1	-2			
Q_6	1	2	3			
Q_7	1	3	5			
Q_8	2	1	0	-2		
Q_9	2	1	1	-1		
Q_{10}	2	1	-1	-1		
Q_{11}	2	1	0	-3		
Q_{12}	2	2	4	1		
Q_{13}	3	1	1	-2	-1	
Q_{14}	3	1	0	-3	1	
Q_{15}	3	3	9	6	-1	
Q_{16}	3	2	-3	-2	1	
Q_{17}	4	1	1	-4	-4	1
Q_{18}	4	1	1	-5	-7	-1
Q_{19}	4	4	15	16	3	-1
Q_{20}	5	1	1	-4	-3	3	1
Q_{21}	5	1	1	-5	-5	4	3

6. A REFINEMENT OF THE METHOD FOR THE BOUNDS FOR s_k

In order to get a smaller set E_1 we need more information on the numbers s_k. In this section we consider only the second case, because it gives the most important contribution to E_1. Our heuristic idea is that "s_k and s_{2k} cannot be too close to their bounds simultaneously." Here the classical inequality [BO80] may be written as

$$(6.1) \qquad s_{2k} \geq \frac{1}{d}(s_k)^2 - 2M^k - (2d - 4) - 2M^{-k}.$$

We consider in this section an explicit auxiliary function f of the following type:

$$(6.2) \qquad f(z) = \mathrm{Re}(z^2 - 2) + e_0 \mathrm{Re}(z) - \sum_{1 \leq j \leq J} e_j \log |Q_j(z)|$$

with the same conditions as before for the numbers e_j for $0 \leq j \leq J$ and for the polynomials Q_j which belong to the set S defined in Section 5. Doing the summation on the values of f at the numbers γ_i we get

$$\sum_{1 \leq i \leq d} f(\gamma_i) \geq (d - 2) \min_{x \in \mathcal{E}_0} f(x) + 2 \min_{z \in \mathcal{E}_{1/2}} f(z) - \varepsilon = m - \varepsilon.$$

Since $\gamma_{i,2} = \boldsymbol{\alpha}_i^2 + 1/\boldsymbol{\alpha}_i^2 = (\boldsymbol{\alpha}_i + 1/\boldsymbol{\alpha}_i)^2 - 2 = \gamma_i^2 - 2$ we get $s_2 + e_0 s_1 \geq m - \varepsilon$. If we assume that s_1 has the value $\boldsymbol{\sigma}$, then

$$(6.3) \qquad s_2 \geq m - e_0 \boldsymbol{\sigma} - \varepsilon.$$

So, we want to get the largest value for the right-side of (6.3) which is linear in the numbers e_j. We maximize $m - e_0 \boldsymbol{\sigma}$. Therefore we get a lower bound for s_2 depending on the value $\boldsymbol{\sigma}$ of s_1. When the value $\boldsymbol{\sigma}$ increases from the lower bound of s_1 that has been computed in Section 5, say $-B_1$, the bound for s_2 decreases. We stop when this lower bound is less than $-B_2$, which is the lower bound of s_2 that has been computed in Section 5. Of course we may replace the numbers $\boldsymbol{\alpha}_i$ by the numbers $-\boldsymbol{\alpha}_i$. If we replace $+e_0 \mathrm{Re}(z)$ by $-e_0 \mathrm{Re}(z)$ in (6.2), we get the same lower bound for s_2 when s_1 takes the value $-\boldsymbol{\sigma}$. We may also replace $\mathrm{Re}(z^2 - 2)$ by $-\mathrm{Re}(z^2 - 2)$ in (6.2). Then we get an upper bound for s_2 depending on the value of $|s_1|$. Then we replace the numbers $\boldsymbol{\alpha}_i$ by the numbers $\pm\boldsymbol{\alpha}_i^k$ and M by M^k for $1 \leq k \leq 20$, and we get lower and upper bounds for s_{2k} depending on the value of $|s_k|$.

7. DECREASING THE SET E_1

Since we know all the coefficients of P, it is possible to compute all the numbers s_k for $d + 1 \leq k \leq 40$. Now, as D. Boyd in [BO85], we eliminate any polynomial P of E_1 for which s_k is not inside its bounds (those computed in Section 5) for any $k \leq 40$, and for even $k = 2l$ we verify that, for this value for s_l, s_{2l} is between the bounds that have been computed in Section 6.

In the first case, since P has only one root $\boldsymbol{\alpha}_1$ for $x > t_1 = t_1(j)$, the function $P(x)$ is convex in this range. Therefore the line joining the points $(t_1, P(t_1))$ and $(t_2, P(t_2))$ cuts the real axis at a value $t_3 < \boldsymbol{\alpha}_1$. Then all the other roots of P are in the disk of radius $M_1 = M/t_3$.

In the second case all the roots of P lie in the disk of radius $M^{1/2}$. In the two cases we compute the number of roots of P which lie outside a sequence of disks of decreasing radius by the Schur-Cohn algorithm [MA]. This will give a lower bound

of $M(P)$ which has to be $< M$. After the elimination of the polynomials which have cyclotomic factors of small degree, i.e., the degree of the associated polynomial is ≤ 3, we get the new set E_2 of polynomials P. Then we use GP-Pari [PARI] to get the set E_0 of all the irreducible primitive polynomials with Mahler measure less than M.

8. The final computation

We first eliminate the polynomials which have a factor associated to one of the 13 cyclotomic polynomials of degrees 8 to 12. Then we use a modified Graeffe algorithm. For this purpose we need

Lemma 2. *Let P be a nonzero polynomial in $\mathbb{C}[X]$ of degree n. Let $L(P)$ be the length of P; that is, the sum of the modulus of the coefficients of P. Then we have*

$$(8.1) \qquad\qquad M(P) \leq L(P) \leq 2^n M(P).$$

We now suppose that P is a reciprocal polynomial of degree $2d$ and that Q is the polynomial of degree d associated to P. Then we have

$$(8.2) \qquad\qquad \omega^{-d} M(P) \leq L(Q) \leq 3^d M(P),$$

where ω is the golden ratio $\frac{1+\sqrt{5}}{2} = 1.618033\cdots$.

Proof. Let $\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_d$ be d roots of P of modulus ≥ 1 and $1/\boldsymbol{\alpha}_1, \ldots, 1/\boldsymbol{\alpha}_d$ the last d other roots. We have

$$P(X) = c_0 \prod_{1 \leq i \leq 2d} (X - \boldsymbol{\alpha}_i) \quad \text{and} \quad Q(X) = c_0 \prod_{1 \leq i \leq d} (X - (\boldsymbol{\alpha}_i + 1/\boldsymbol{\alpha}_i)).$$

The inequalities in (8.1) are well known and are used in Graeffe's algorithm (see [BO80]). It is plain that

$$L(Q) \leq |c_0| \prod_{1 \leq i \leq d} (1 + |\boldsymbol{\alpha}_i| + 1/|\boldsymbol{\alpha}_i|)$$

and that for any $1 \leq i \leq d$

$$1 + |\boldsymbol{\alpha}_i| + 1/|\boldsymbol{\alpha}_i| \leq 3 \max(1, |\boldsymbol{\alpha}_i|).$$

So we get the right-hand inequality in (8.2). By (8.1) we have $M(Q) \leq L(Q)$ and

$$M(Q) = |c_0| \prod_{1 \leq i \leq d} \max(1, |\boldsymbol{\alpha}_i + 1/\boldsymbol{\alpha}_i|).$$

If $|\boldsymbol{\alpha}_i| > \omega$, then

$$|\boldsymbol{\alpha}_i + 1/\boldsymbol{\alpha}_i| \geq |\boldsymbol{\alpha}_i|(1 - 1/|\boldsymbol{\alpha}_i|^2) \geq |\boldsymbol{\alpha}_i|(1 - 1/\omega^2) = |\boldsymbol{\alpha}_i|/\omega.$$

If $1 \leq |\boldsymbol{\alpha}_i| \leq \omega$, then $1 \geq |\boldsymbol{\alpha}_i|/\omega$. Since $M(P) = |c_0| \prod_{1 \leq i \leq d} |\boldsymbol{\alpha}_i|$, this proves the left-hand inequality of (8.2).

Remark. The inequalities in (8.1) and the right-hand one in (8.2) are the best possible. For (8.1) take the polynomials X^n and $(X + 1)^n$ for the left- and right-hand side, respectively. For the right-hand side of (8.2) take $P = (X + 1)^{2d}$, then $M(P) = 1$ and $Q = (X + 2)^d$, which gives $L(Q) = 3^d$. The constant ω^{-d} in the left-hand side of (8.2) is perhaps not the best one, but it cannot be replaced by 1. If we take $P = (X^4 + 3X^2 + 1)^n$, then $d = 2n$, $M(P) = \omega^d$ and $Q = (X^2 + 1)^n$, so $L(Q) = 2^n = (\sqrt{2})^d$, $M(P) = \omega^d M(Q)$ and $M(P) = L(Q)(1.144\cdots)^d$.

The inequalities in (8.1) are the basic tools for the Graeffe algorithm. If P_1 is the polynomial whose roots are the squares of the roots of P, then $L(P_1) \leq 2^{\deg(P)} M(P_1) = 2^{\deg(P)} M(P)^2$. Iterating this process, we get a sequence of polynomials P_m for $m > 0$, and we have

$$(8.3) \qquad M(P) = M(P_m)^{2^{-m}} \leq L(P_m)^{2^{-m}} \leq 2^{\deg(P)/2^m} M(P).$$

This sequence of inequalities shows that, if $M(P) > M$, after a finite number of root-squaring we will have $L(P_m) > 2^{\deg(P)} M^{2^m}$. Since the constants in (8.2) are smaller than in (8.1) we will use the inequalities (8.2) for a modified Graeffe algorithm. We have

$$X - \boldsymbol{\alpha}_i^2 - 1/\boldsymbol{\alpha}_i^2 = X + 2 - (\boldsymbol{\alpha}_i + 1/\boldsymbol{\alpha}_i)^2,$$

therefore the polynomial Q_1 associated to P_1 may be defined by the following formulae. Let g and h be the two polynomials such that

$$Q(X) = g(X^2) + Xh(X^2);$$

then

$$Q_1(X) = g(X+2)^2 - (X+2)h(X+2)^2.$$

Let us give an example. Let P be the polynomial

$$X^{40} + X^{39} + X^{34} + X^{33} + X^{28} - X^{26} - X^{23} - X^{21} - 3X^{20}$$

$$-X^{19} - X^{17} - X^{14} + X^{12} + X^7 + X^6 + X + 1.$$

Then $M(P) = 1.395928\cdots > 1.31$ and

$$L(P_8) = 5.29\cdots 10^{39} < 2^{40} 1.31^{2^8} = 1.15\cdots 10^{42},$$

but

$$L(Q_8) = 7.44\cdots 10^{39} > 3^{20} 1.31^{2^8} = 3.66\cdots 10^{39}.$$

We test the polynomials Q_m until $m = 9$, and we keep only those which survive all the tests. Then, for the polynomials Q which are irreducible, we compute their roots and the measure of P.

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UMR CNRS 7122, Département de Mathématiques, UFR MIM, Université de Metz, Ile du Saulcy, 57045 Metz Cedex 01, France
  E-mail address: `flammang@poncelet.univ-metz.fr`

UMR CNRS 7122, Département de Mathématiques, UFR MIM, Université de Metz, Ile du Saulcy, 57045 Metz Cedex 01, France
  E-mail address: `rhin@poncelet.univ-metz.fr`

UMR CNRS 7122, Département de Mathématiques, UFR MIM, Université de Metz, Ile du Saulcy, 57045 Metz Cedex 01, France
  E-mail address: `jmse@poncelet.univ-metz.fr`