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TWO LOWER ORDER NONCONFORMING RECTANGULAR ELEMENTS FOR THE REISSNER-MINDLIN PLATE

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Abstract. In this paper, we propose two lower order nonconforming rectangular elements for the Reissner-Mindlin plate. The first one uses the conforming bilinear element to approximate both components of the rotation, and the modified nonconforming rotated Q_1 element to approximate the displacement, whereas the second one uses the modified nonconforming rotated Q_1 element to approximate both the rotation and the displacement. Both elements employ a projection operator to overcome the shear force locking. We prove that both methods converge at optimal rates uniformly in the plate thickness t in both the H^1- and L^2-norms, and consequently they are locking free.

1. Introduction

One benchmark problem in computational science is the Reissner-Mindlin plate (R-M hereinafter) problem. For this problem, the straight-forward approach using lower order conforming finite elements in the primal formulation faces with the locking phenomenon. This occurs when the thickness t of the plate tends to zero and the problem enforces a constraint (namely, the Kirchhoff constraint). For the discrete problem, this constraint, especially for lower order elements, cannot be fully satisfied. Various methods have been proposed to weaken or overcome the locking effect since the nineties of the last century, and most of them can be regarded as reduced integration methods. Recently, the discontinuous Galerkin method [1] has also been used to design finite element methods for the R-M plate problem (see, for instance, [2, 11, 16]). One common favorable feature of these discontinuous Galerkin R-M plate elements is that all the variables share the same nodes and consequently can also be extended to shell problems.

In this paper, we propose and analyze two lower order nonconforming rectangular elements for the R-M plate model. In the first element, the usual conforming bilinear element space is chosen as the rotation space and the modified nonconforming rotated element Q_1 space (NRQ_1 element hereinafter) [13, 22, 15] as the displacement space, and the reduced integration method is used to overcome the shear force locking. The second element differs from the first one only in the approximation of the rotation; which employs the modified NRQ_1 element for both components of the rotation, consequently all the variables in this method share the same nodes. Furthermore, the second finite element method enjoys the same

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promising features as the lower order triangular elements proposed in [2, 11, 16] and therefore is possible to be generalized to the shell problems.

We conclude this introduction with a list of some basic notations used in the sequel. In Section 2, we recall the Reissner-Mindlin plate model and its mixed formulation by Brezzi and Fortin [4], and Section 3 presents our elements for the R-M plate. The equivalent formulation of the discrete problem will be given and proven in Section 4. In Section 5, we show the well-posedness of the discrete problems. This paper ends with Section 6, which is devoted to error analysis.

In the sequel, $D(\Omega)$ is the linear space of an infinitely differentiable function with compact support on Ω. We use the standard notation and definition for the Sobolev spaces $(H^s(\Omega))^2$ and $(H^s(\partial\Omega))^2$ for $s \geq 0$; the standard associated inner products are denoted by $(\cdot,\cdot)_s$ and $(\cdot,\cdot)_{s,\partial\Omega}$, and their respective norms by $\|\cdot\|_s$ and $\|\cdot\|_{s,\partial\Omega}$. For $s = 0$, $(H^s(\Omega))^2$ coincides with $(L^2(\Omega))^2$. In this case, the inner product is denoted by $(\cdot,\cdot)$. As usual, $H_0^s(\Omega)$ is a closure of $D(\Omega)$ with respect to the norm $\|\cdot\|_s$.

Define

$$\hat{H}^1(\Omega) = \{v \in H^1(\Omega) : \int_\Omega v dxdy = 0\}.$$

Finally, we use the standard differential operators:

$$\nabla r = \begin{pmatrix} \partial r/\partial x \\ \partial r/\partial y \end{pmatrix}, \quad \operatorname{curl} p = \begin{pmatrix} -\partial p/\partial y \\ \partial p/\partial x \end{pmatrix},$$
$$\operatorname{div} \boldsymbol{\psi} = \frac{\partial \psi_1}{\partial x} + \frac{\partial \psi_2}{\partial y}, \quad \operatorname{rot} \boldsymbol{\psi} = \frac{\partial \psi_2}{\partial x} - \frac{\partial \psi_1}{\partial y}.$$

Throughout this paper, the generic constant C is assumed to be independent of the plate thickness t and the mesh size h.

2. Reissner-Mindlin plate model

In this section, we recall the widely used Reissner-Mindlin plate equations. Let Ω be the region occupied by the plate, and ω and $\boldsymbol{\phi} = (\phi_1, \phi_2)$ denote the transverse dispacenent of mid-section and the rotation of the fibers normal to mid-section, repectively. The Reissner-Mindlin plate model determines ω and $\boldsymbol{\phi}$ as the solution to the following variational problem,

Problem 2.1. Find $(\omega, \boldsymbol{\phi}) \in H_0^1(\Omega) \times (H_0^1(\Omega))^2$ such that

$$(1) \quad a(\boldsymbol{\phi}, \boldsymbol{\psi}) + \lambda t^{-2}(\nabla\omega - \boldsymbol{\phi}, \nabla v - \boldsymbol{\psi}) = (g, v), \quad \forall (v, \boldsymbol{\psi}) \in H_0^1(\Omega) \times (H_0^1(\Omega))^2,$$

where g is the scaled transverse loading function, t the plate thickness,

$$\lambda = Ek/2(1+\nu)$$

the shear modulus with E Young's modulus, ν the Poisson ratio, and κ the shear correction factor. The bilinear form $a(\cdot,\cdot)$ is defined as

$$a(\boldsymbol{\phi}, \boldsymbol{\psi}) = \frac{E}{12(1-\nu^2)} \int_\Omega [(1-\nu)\mathcal{E}(\boldsymbol{\phi}) : \mathcal{E}(\boldsymbol{\psi}) + \nu \nabla\cdot\boldsymbol{\phi} \nabla\cdot\boldsymbol{\psi}] dxdy,$$

where $\mathcal{E}(\boldsymbol{\phi}) = 1/2[\nabla\boldsymbol{\phi} + \nabla\boldsymbol{\phi}^T]$ and $-1 < \nu < 1/2$.

In our analysis we shall make use of a mixed formulation of the Reissner-Mindlin plate equations proposed by Brezzi and Fortin in [4] based on the following Helmholtz decomposition of the shear force vector:

$$\lambda t^{-2}(\nabla\omega - \boldsymbol{\phi}) = \nabla r + \operatorname{curl} p \tag{2}$$

with $(r,p) \in H_0^1(\Omega) \times \hat{H}^1(\Omega)$. With this decomposition, Problem 2.1 can be written as the following Brezzi-Fortin mixed formulation

Problem 2.2. Find $(r, \boldsymbol{\phi}, p, \omega) \in H_0^1(\Omega) \times (H_0^1(\Omega))^2 \times \hat{H}^1(\Omega) \times H_0^1(\Omega)$ such that

$$(\nabla r, \nabla\mu) = (g, \mu), \quad \forall \mu \in H_0^1(\Omega), \tag{3}$$

$$a(\boldsymbol{\phi}, \boldsymbol{\psi}) - (\operatorname{curl} p, \boldsymbol{\psi}) = (\nabla r, \boldsymbol{\psi}), \quad \forall \boldsymbol{\psi} \in (H_0^1(\Omega))^2, \tag{4}$$

$$-(\boldsymbol{\phi}, \operatorname{curl} q) - \lambda^{-1}t^2(\operatorname{curl} p, \operatorname{curl} q) = 0, \quad \forall q \in \hat{H}^1(\Omega), \tag{5}$$

$$(\nabla\omega, \nabla s) = (\boldsymbol{\phi} + \lambda^{-1}t^2\nabla r, \nabla s), \quad \forall s \in H_0^1(\Omega). \tag{6}$$

The following result concerning the existence and uniqueness of solutions to Problem 2.2 and the regularity can be found in [4, 3].

Lemma 2.3. *Let Ω be a convex polygon or smoothly bounded domain in the plane. For any $t \in (0,1]$ and any $g \in H^{-1}(\Omega)$, there exists a unique quadruple $(r, \boldsymbol{\phi}, p, \omega) \in H_0^1(\Omega) \times (H_0^1(\Omega))^2 \times \hat{H}^1(\Omega) \times H_0^1(\Omega)$ solving Problem 2.2. Moreover, $\boldsymbol{\phi} \in H^2(\Omega)$ and there exists a constant C independent of t and g, such that*

$$\|r\|_1 + \|\boldsymbol{\phi}\|_2 + \|p\|_1 + t\|p\|_2 + \|\omega\|_1 \le C\|g\|_{-1}. \tag{7}$$

If $g \in L^2(\Omega)$, then $r, \omega \in H^2(\Omega)$ and

$$\|r\|_2 + \|\omega\|_2 \le C\|g\|_0. \tag{8}$$

3. Finite element method for the R-M plate

For approximating Problem 2.1 by the finite element method, we introduce a rectangular mesh J^h of the rectangular domain Ω. The regularity of the mesh J^h is assumed in the sense of Ciarlet [12] such that $\bigcup_{K\in J^h} K = \bar{\Omega}$, the two distinct elements K and K' in J^h are either disjoint, or share the common edge e, or a common vertex. Let F denote the set of all edges in J^h with F' the set of interior edges. Given any edge $e \in F$ we assign a unit normal $\boldsymbol{n}_e$. In relation to $\boldsymbol{n}_e$ one can define the element $K^+ \in J^h$ and the element $K^- \in J^h$, with $e = K^+ \cap K^-$. Let ∂K denote the boundary of K.

For each $K \in J^h$, we introduce the following affine invertible transformation:

$$F_K : \hat{K} \to K, x = \frac{h_{x,K}}{2}\xi + x_{0,K}, \quad y = \frac{h_{y,K}}{2}\eta + y_{0,K}$$

with $(x_{0,K}, y_{0,K})$ the center and $h_{x,K}$ and $h_{y,K}$ the horizontal and vertical edge length of K, respectively, and $\hat{K} = [-1,1]^2$ the reference element. Let $Q_1(\hat{K})$ denote the usual bilinear function space on the reference elment $\hat{K}$, and set

$$b(\xi,\eta) = (1+\xi+\eta)(1-\xi^2)(1-\eta^2).$$

Obviously, $b(\xi,\eta)$ is a bubble function and can be condensed out on the element level.

Remark 3.1. The factor $1+\xi+\eta$ is added so that the space pair $(\boldsymbol{V}_{1,h}, Q_h)$ satisfies the discrete B-B condition; cf. Lemma 5.1 below. Using the bubble $(1-\xi^2)(1-\eta^2)$ instead, one can show by a similar argument of Lemma 5.1 that the space N_M defined in Lemma 5.1 is not one-dimensional. This in turn implies that the discrete B-B condition is not valid.

Define

$$V_h^c := \{v \in H_0^1(\Omega) : v|_K \circ F_K \in Q_1(\hat{K}) \oplus \operatorname{span}(b), \forall K \in J^h\}.$$

Denote by $\mathcal{Q}_1(\hat{K})$ the modified nonconforming rotated Q_1 element space defined by

$$\mathcal{Q}_1(\hat{K}) = \operatorname{span}\{1, \xi, \eta, \xi^2 - \eta^2, 1 - \frac{3}{4}(\xi^2 + \eta^2)\}. \tag{9}$$

Note that the nonconforming bubble function $1 - \frac{3}{4}(\xi^2 + \eta^2)$ in (9) can also be condensed out on the element level with small effort.

For any $v \in H^1(K)$, define the following edge functional:

$$\mathcal{F}_e(v) = \frac{1}{h_e}\int_e v\, ds$$

with $e \subset \partial K$ and h_e the length of the edge e. The modified nonconforming rotated Q_1 element space V_h^{nc} is then defined as [13, 22, 15]

$$\begin{aligned} V_h^{nc} := \ & \big\{v \in L^2(\Omega) : v|_K \circ F_K \in \mathcal{Q}_1(\hat{K}) \text{ for each } K \in J^h, v \text{ continuous} \\ & \text{with respect to } \mathcal{F}_e \text{ for all } e \in F',\ \text{and } \mathcal{F}_e(v) = 0 \text{ for all } e \text{ on } \partial\Omega\big\}. \end{aligned}$$

Define

$$\boldsymbol{V}_{1,h} = V_h^c \times V_h^c \text{ and } \boldsymbol{V}_{2,h} = V_h^{nc} \times V_h^{nc}$$

as the approximation space of the rotation.

Define the discrete norm and semi-norm on V_h^{nc} by

$$\|v\|_{1,h}^2 = \sum_{K\in J^h} \|v\|_{1,K}^2, \quad |\, v\,|_{1,h}^2 = \sum_{K\in J^h} |\, v\,|_{1,K}^2 .$$

By Poincare's inequality [22], we have $|\cdot|_{1,h}$ as a norm on V_h^{nc}. The same rule is applicable to functions in $\boldsymbol{V}_{2,h}$.

To deal with the discontinuity of $\boldsymbol{V}_{2,h}$, we follow the idea in [11, 16, 18] and define for any vector-valued function $\boldsymbol{\psi} \in \boldsymbol{\Pi}_{K\in J^h} H^1(K)$ the jump across the edge $e \in F'$ as

$$[\boldsymbol{\psi}] = (\boldsymbol{\psi}^+ \otimes \boldsymbol{n}^+)_S + (\boldsymbol{\psi}^- \otimes \boldsymbol{n}^-)_S,$$

where $(\boldsymbol{\psi} \otimes \boldsymbol{n})_S$ denotes the symmetric part of the tensor, and $\boldsymbol{n}^+$ (resp. $\boldsymbol{n}^-$) is the outward unit normal to $e \subset \partial K^+$ (resp. $e \subset \partial K^-$). On the boundary edge, we define the jump as $[\boldsymbol{\psi}] = (\boldsymbol{\psi} \otimes \boldsymbol{n})_S$ with $\boldsymbol{n}$ the outward unit normal to $\partial\Omega$.

Moreover, we introduce the following discrete bilinear form with a penalty term:

(10)

$$\begin{aligned} a_h(\boldsymbol{\phi}_h, \boldsymbol{\psi}_h) = \ & \frac{E}{12(1-\nu^2)} \sum_{K\in J^h} \int_K [(1-\nu)\mathcal{E}(\boldsymbol{\phi}_h) : \mathcal{E}(\boldsymbol{\psi}_h) + \nu \nabla\cdot\boldsymbol{\phi}_h \nabla\cdot\boldsymbol{\psi}_h]\, dxdy \\ & + \sum_{e\in F} \frac{\gamma_e}{h_e} \int_e [\boldsymbol{\phi}_h] : [\boldsymbol{\psi}_h]\, ds \end{aligned}$$

with γ_e some constant. For the analysis, we need to define the following auxiliary pressure finite element space and the discrete shear force space, respectively,

$$Q_h = \{q \in \hat{H}^1(\Omega) : q\mid_K \circ F_K \in Q_1(\hat{K}), \forall K \in J^h\},$$
$$\boldsymbol{\Gamma}_h = \left\{ \boldsymbol{v} \in (L^2(\Omega))^2 : \boldsymbol{v}\mid_K = \begin{pmatrix} b+dx \\ c+ey \end{pmatrix}, \forall K \in J^h \right\}.$$

Remark 3.2. The following shear force space is used in [28]:

$$M = \left\{ \boldsymbol{v} \in (L^2(\Omega))^2 : \boldsymbol{v}\mid_K = \begin{pmatrix} b+dx \\ c-dy \end{pmatrix}, \forall K \in J^h \right\}.$$

However, a close observation finds that $\nabla_h M_1 \notin M$ (in the notation of [28]) for general rectangular meshes unless $h_{x,K} = h_{y,K}$, therefore the analysis therein is only valid for square meshes.

Let $\boldsymbol{R}_h : (L^2(\Omega))^2 \to \boldsymbol{\Gamma}_h$ denote the usual L^2 projection operator; then our finite element methods for the R-M plate problem can be stated as

Problem 3.3. Find $(\omega_h, \boldsymbol{\phi}_h) \in V_h^{nc} \times \boldsymbol{V}_{i,h}$, such that

$$a_h(\boldsymbol{\phi}_h, \boldsymbol{\psi}) + \lambda t^{-2}(\nabla_h \omega_h - \boldsymbol{R}_h\boldsymbol{\phi}_h, \nabla_h v - \boldsymbol{\psi}) = (g, v), \quad \forall (v, \boldsymbol{\psi}) \in V_h^{nc} \times \boldsymbol{V}_{i,h}, \tag{11}$$

with $i = 1, 2$.

Remark 3.4. Notice that the penalty term in (10) vanishes for the space $\boldsymbol{V}_{1,h}$; we keep it there only for the convenience of the presentation and the simplicity of the notation.

4. Discrete Helmholtz Decomposition and equivalent formulations of discrete problems

In this section we prove the discrete Helmholtz Decomposition and present the equivalent formulation of the discrete problem.

Denote

$$C_h = \{\boldsymbol{v} \mid \boldsymbol{v} = \operatorname{curl} q, q \in Q_h\} \text{ and } G_h = \{\boldsymbol{v} \mid \boldsymbol{v} = \nabla_h w, w \in V_h^{nc}\}.$$

Note that

$$C_h \subset \boldsymbol{\Gamma}_h \text{ and } G_h \subset \boldsymbol{\Gamma}_h. \tag{12}$$

Furthermore, a counting argument gives

$$\dim G_h + \dim C_h = \dim \boldsymbol{\Gamma}_h. \tag{13}$$

Lemma 4.1. *There holds*

$$C_h \perp G_h.$$

Proof. Let $w \in V_h^{nc}$ and $q \in Q_h$, and

$$(\nabla_h w, \operatorname{curl} q) = \sum_{K \in J^h} \int_K \nabla w \cdot \operatorname{curl} q dxdy = -\sum_{K \in J^h} \int_{\partial K} w \frac{\partial q}{\partial \boldsymbol{t}} ds,$$

where $\boldsymbol{t}$ is the counterclockwise unit tangential vector to ∂K. Since $\frac{\partial q}{\partial t}$ is continuous constant on each edge of the element K, we have

$$(\nabla_h w, \operatorname{curl} q) = -\sum_{e \in F'} \int_e [w] \frac{\partial q}{\partial \boldsymbol{t}} ds - \sum_{e \in F \setminus F'} \int_e w \frac{\partial q}{\partial \boldsymbol{t}} ds = 0$$

with $[w]$ the jump across e. $\square$

Thanks to Lemma 4.1 and (13), we have the following discrete Helmholtz Decomposition:

$$\boldsymbol{\Gamma}_h = C_h \oplus G_h. \tag{14}$$

We now introduce the following auxiliary discrete problem

Problem 4.2. Find $(r_h, \boldsymbol{\phi}_h, p_h, \omega_h) \in V_h^{nc} \times \boldsymbol{V}_{i,h} \times Q_h \times V_h^{nc}$ such that

$$(\nabla_h r_h, \nabla_h \mu) = (g, \mu), \quad \forall \mu \in V_h^{nc}, \tag{15}$$

$$a_h(\boldsymbol{\phi}_h, \boldsymbol{\psi}) - (\operatorname{curl} p_h, \boldsymbol{\psi}) = (\nabla_h r_h, \boldsymbol{\psi}), \quad \forall \boldsymbol{\psi} \in \boldsymbol{V}_{i,h}, \tag{16}$$

$$-(\boldsymbol{\phi}_h, \operatorname{curl} q) - \lambda^{-1} t^2 (\operatorname{curl} p_h, \operatorname{curl} q) = 0, \quad \forall q \in Q_h, \tag{17}$$

$$(\nabla_h \omega_h, \nabla_h s) = (\boldsymbol{\phi}_h + \lambda^{-1} t^2 \nabla_h r_h, \nabla_h s), \quad \forall s \in V_h^{nc}, \tag{18}$$

with $i = 1, 2$.

Theorem 4.3. *For any $g \in L^2(\Omega)$ and $t \in (0, 1]$ there exists a unique solution $(r_h, \boldsymbol{\phi}_h, p_h, \omega_h)$ to Problem 4.2. Moreover, the pair $(\omega_h, \boldsymbol{\phi}_h)$ is the unique solution of Problem 3.3 and*

$$\lambda t^{-2} (\nabla_h \omega_h - \boldsymbol{R}_h \boldsymbol{\phi}_h) = \nabla_h r_h + \operatorname{curl} p_h. \tag{19}$$

Proof. The existence and uniqueness of the solution to Problem 4.2 follows immediately from the discrete inf-sup condition (see Lemma 5.3 and Lemma 5.4 below) and the Korn inequality (see Lemma 5.5 below) and Lemma 5.7 (see the next section for details).

Now we prove that $(\omega_h, \boldsymbol{\phi}_h)$ is the unique solution to Problem 3.3 and that (19) holds. We use the orthogonality and definition of $\boldsymbol{R}_h$, (17) and (18) to get

$$(\nabla_h \omega_h - \boldsymbol{R}_h \boldsymbol{\phi}_h, \operatorname{curl} q) = \lambda^{-1} t^2 (\nabla_h r_h + \operatorname{curl} p_h, \operatorname{curl} q) = 0, \quad \forall q \in Q_h, \tag{20}$$

$$(\nabla_h \omega_h - \boldsymbol{R}_h \boldsymbol{\phi}_h, \nabla_h s) = \lambda^{-1} t^2 (\nabla_h r_h + \operatorname{curl} p_h, \nabla_h s), \quad \forall s \in V_h^{nc}, \tag{21}$$

which imply that

$$\nabla_h \omega_h - \boldsymbol{R}_h \boldsymbol{\phi}_h = \lambda^{-1} t^2 (\nabla_h r_h + \operatorname{curl} p_h). \tag{22}$$

Thanks to (22), (16) and (18) can be written as, respectively

$$a_h(\boldsymbol{\phi}_h, \boldsymbol{\psi}) - \lambda t^{-2} (\nabla_h \omega_h - \boldsymbol{R}_h \boldsymbol{\phi}_h, \boldsymbol{\psi}) = 0, \quad \forall \boldsymbol{\psi} \in \boldsymbol{V}_{i,h}, i = 1, 2, \tag{23}$$

$$\lambda t^{-2} (\boldsymbol{R}_h \boldsymbol{\phi}_h - \nabla_h \omega_h, \nabla_h s) = -(\nabla_h r_h, \nabla_h s), \quad \forall s \in V_h^{nc}. \tag{24}$$

We obtain from (23) and (24) that

$$a_h(\boldsymbol{\phi}_h, \boldsymbol{\psi}) + \lambda t^{-2} (\nabla_h \omega_h - \boldsymbol{R}_h \boldsymbol{\phi}_h, \nabla_h s - \boldsymbol{\psi}) = (\nabla_h r_h, \nabla_h s). \tag{25}$$

By virtue of (15) and (25), we come to

$$a_h(\boldsymbol{\phi}_h, \boldsymbol{\psi}) + \lambda t^{-2} (\nabla_h \omega_h - \boldsymbol{R}_h \boldsymbol{\phi}_h, \nabla_h s - \boldsymbol{\psi}) = (g, s), \tag{26}$$

which ends the proof. □

5. The well-posedness of the discrete problems

In this section we shall show the well-posedness of the discrete Problem 4.2. Because (15) and (18) are elliptic problems which are decoupled from the system, and their well-posedness follows immediately from Lemma 5.7, we only need to show the well-posedness for Stokes-like problem (16)-(17), which hangs on the discrete inf-sup condition, namely the B-B condition and the continuity and coercivity of a_h. We first prove the discrete inf-sup condition for the pairs $(\boldsymbol{V}_{i,h}, Q_h)$, $i = 1, 2$. To this end, we shall use the macroelement trick from [26, 9].

For any interior node P_i, we define the associated macroelement by

$$M(P_i) = \{K \mid K \cap P_i \neq \emptyset, K \in J^h\}.$$

Lemma 5.1. *There exists a positive constant β independent of h, such that*

$$\sup_{\boldsymbol{\psi} \in \boldsymbol{V}_{1,h}} \frac{(\operatorname{div} \boldsymbol{\psi}, q)}{\|\boldsymbol{\psi}\|_1} \geq \beta \|q\|_0, \quad \forall q \in Q_h. \tag{27}$$

Proof. Let M be a macroelement with nodes $P_i(x_i, y_i)$, $i = 1, \cdots, 9$, and elements K_i, $i = 1, \cdots, 4$ (see Figure 1). Define

$$S_{2,M} := \{\boldsymbol{\psi} : \boldsymbol{\psi} \in (H_0^1(M))^2 \cap \boldsymbol{V}_{1,h}\},$$
$$S_{3,M} = \{q : q \in H^1(M) \cap Q_h\},$$
$$N_M = \{q \in S_{3,M} : (\operatorname{div} \boldsymbol{\psi}, q) = 0, \forall \boldsymbol{\psi} \in S_{2,M}\}.$$

By a theory from [26], we only need to show that N_M is one-dimensional.

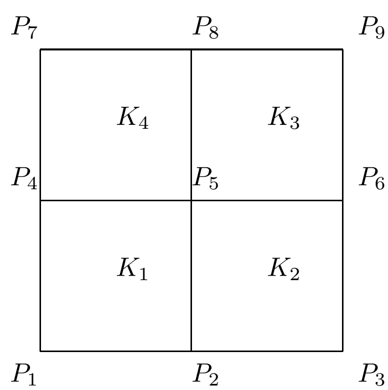

Figure 1. Macroelement

Denote the bilinear basis on nodes $P_1, \cdots, P_9$ by $\phi_1, \cdots, \phi_9$, and the bubble basis function of V_h^c on the element $K_1, \cdots, K_4$ by $\varphi_1, \cdots, \varphi_4$ respectively. We have for any $\boldsymbol{\psi} \in S_{2,M}$ and $q \in S_{3,M}$ the following expressions:

$$\boldsymbol{\psi} = \sum_{i=1}^{i=4} \begin{pmatrix} v_{1,i} \\ v_{2,i} \end{pmatrix} \varphi_i + \begin{pmatrix} v_{1,5} \\ v_{2,5} \end{pmatrix} \phi_5 \text{ and } q = \sum_{i=1}^{9} a_i \phi_i. \tag{28}$$

Integrating by parts, we have

$$(\operatorname{div} \boldsymbol{\psi}, q) = -(\boldsymbol{\psi}, \nabla q) = -\sum_{i=1}^{4} (\boldsymbol{\psi}, \nabla q)_{K_i}$$

for any $\boldsymbol{\psi} \in S_{2,M}$ and $q \in S_{3,M}$. Take $v_{2,i} = 0$, $i = 1, \cdots, 4$, $v_{1,5} = v_{2,5} = 0$, $v_{1,i} = 0$, $i = 2, 3, 4$, and $v_{1,1} = 1$ and set

$$0 = (\operatorname{div} \boldsymbol{\psi}, q) = -(\boldsymbol{\psi}, \nabla q).$$

This yields

$$\begin{aligned}
0 &= \int_{K_1} \varphi_1 \frac{\partial(a_1\phi_1 + a_2\phi_2 + a_4\phi_4 + a_5\phi_5)}{\partial x} dxdy \\
&= \frac{h_{y,K_1}}{2} \int_{\hat{K}} (1+\xi+\eta)(1-\xi^2)(1-\eta^2)(-a_1 + a_2 - a_4 + a_5) d\xi d\eta \\
&\quad + \frac{h_{y,K_1}}{2} \int_{\hat{K}} (1+\xi+\eta)(1-\xi^2)(1-\eta^2)(a_1 - a_2 - a_4 + a_5)\eta d\xi d\eta.
\end{aligned}$$

Then a direct calculation gives

$$2a_1 - 2a_2 + 3a_4 - 3a_5 = 0.$$

Similarly, let $v_{2,i} = 0$ with $i = 2, \cdots, 4$, $v_{1,5} = v_{2,5} = 0$, $v_{1,i} = 0$, $i = 1, 2, 3, 4$, and $v_{2,1} = 1$. We have

$$-2a_1 - 3a_2 + 2a_4 + 3a_5 = 0.$$

Now let one of degrees $v_{j,i}$, $i = 2, 3, 4$, $j = 1, 2$, be 1 and the others be zero successively. We can get a system of equations with respect to $(a_1, \cdots, a_9)$, which reads

$$\left\{\begin{array}{l}
2a_1 - 2a_2 + 3a_4 - 3a_5 = 0, \\
-2a_1 - 3a_2 + 2a_4 + 3a_5 = 0, \\
2a_2 - 2a_3 + 3a_5 - 3a_6 = 0, \\
-2a_2 - 3a_3 + 2a_5 + 3a_6 = 0, \\
2a_4 - 2a_5 + 3a_7 - 3a_8 = 0, \\
-2a_4 - 3a_5 + 2a_7 + 3a_8 = 0, \\
2a_5 - 2a_6 + 3a_8 - 3a_9 = 0, \\
-2a_5 - 3a_6 + 2a_8 + 3a_9 = 0.
\end{array}\right.$$

We solve this system to get

$$\left\{\begin{array}{lll}
a_1 = \frac{13a-9b}{4} & a_2 = \frac{9b-7a}{2} & a_3 = a, \\
a_4 = \frac{9b-7a}{2} & a_5 = a & a_6 = 3b - 2a, \\
a_7 = a & a_8 = 3b - 2a & a_9 = b,
\end{array}\right.$$

such that a and b are two free parameters. Finally we take $v_{2,5} = 1$, and $v_{1,i} = 0$, $i = 1, \cdots, 4$, $v_{2,i} = 0$, $i = 1, \cdots, 4$, $v_{1,5} = 0$, to obtain $a = b$; therefore

$$a_1 = \cdots = a_9.$$

This is to say N_M is one dimensional. $\square$

Remark 5.2. Note from the above calculation that we can only obtain the following system:

$$\left\{\begin{array}{l}
-a_1 + a_2 - a_4 + a_5 = 0, \\
-a_1 - a_2 + a_4 + a_5 = 0, \\
-a_2 + a_3 - a_5 + a_6 = 0, \\
-a_2 - a_3 + a_5 + a_6 = 0, \\
-a_4 + a_5 - a_7 + a_8 = 0, \\
-a_4 - a_5 + a_7 + a_8 = 0, \\
-a_5 + a_6 - a_8 + a_9 = 0, \\
-a_5 - a_6 + a_8 + a_9 = 0,
\end{array}\right.$$

if we employ $(1-\xi^2)(1-\eta^2)$ instead. This leads to

$$a_1 = a_3 = a_5 = a_7 = a_9, \quad a_2 = a_4 = a_6 = a_7 = a_8.$$

With these identities, we cannot prove $a_1 = a_2$ by taking $v_{1,5} = 1$ or $v_{2,5} = 1$, and letting the remaining degrees of freedom be zero. Therefore N_M is two-dimensional in this case.

Lemma 5.3. *There holds that*

$$\sup_{\boldsymbol{\psi}\in \boldsymbol{V}_{1,h}} \frac{(\operatorname{rot}\boldsymbol{\phi}, q)}{\|\boldsymbol{\phi}\|_1} = \sup_{\boldsymbol{\psi}\in \boldsymbol{V}_{1,h}} \frac{(\boldsymbol{\phi}, \operatorname{curl} q)}{\|\boldsymbol{\phi}\|_1} \geq \beta\|q\|_0, \quad \forall q \in Q_h. \tag{29}$$

Proof. Let $\psi_2 = -\phi_1$ and $\psi_1 = \phi_2$ in Lemma 5.1. We obtain with $\boldsymbol{\phi} = (\phi_1, \phi_2)$ and $\boldsymbol{\psi} = (\psi_1, \psi_2)$,

$$\begin{aligned}\sup_{\boldsymbol{\phi}\in V_{1,h}} \frac{(\operatorname{rot}\boldsymbol{\phi}, q)}{\|\boldsymbol{\phi}\|_1} &= \sup_{\boldsymbol{\phi}\in V_{1,h}} \frac{(\boldsymbol{\phi}, \operatorname{curl} q)}{\|\boldsymbol{\phi}\|_1} \\ &= \sup_{\boldsymbol{\psi}\in V_{1,h}} \frac{(\operatorname{div}\boldsymbol{\psi}, q)}{\|\boldsymbol{\psi}\|_1} \geq \beta\|q\|_0 \quad \forall q \in Q_h.\end{aligned}$$

This ends the proof. □

By the same argument of Lemma 5.1, we have

Lemma 5.4. *There exists a positive constant β such that*

$$\sup_{\boldsymbol{\psi}\in \boldsymbol{V}_{2,h}} \frac{(\boldsymbol{\psi}, \operatorname{curl} q)}{\|\boldsymbol{\psi}\|_{1,h}} \geq \beta\|q\|_0, \quad \forall q \in Q_h. \tag{30}$$

To prove the well-posedness of the discrete problem, we remain to show the continuity and coercivity of a_h, i.e.,

Lemma 5.5. *There exist two positive constants C_1 and C_2 independent of h and t such that*

$$C_1\|\boldsymbol{\psi}\|_{1,h}^2 \leq a_h(\boldsymbol{\psi}, \boldsymbol{\psi}), \quad \forall \boldsymbol{\psi} \in \boldsymbol{V}_{i,h}, i = 1, 2, \tag{31}$$

$$|a_h(\boldsymbol{\phi}, \boldsymbol{\psi})| \leq C_2\|\boldsymbol{\phi}\|_{1,h}\|\boldsymbol{\psi}\|_{1,h}, \quad \forall \boldsymbol{\phi}, \boldsymbol{\psi} \in \boldsymbol{V}_{i,h}, i = 1, 2. \tag{32}$$

Proof. Lemma 5.5 holds obviously for the space $\boldsymbol{V}_{1,h}$, and it is also easy to show (32) for $\boldsymbol{V}_{2,h}$. The proof of (31) for $\boldsymbol{V}_{2,h}$ follows immediately from Lemma 5.6 and Lemma 5.7 below. □

Lemma 5.6. *There exists a positive constant C independent of h such that*

$$C|\boldsymbol{\psi}|_{1,h} \leq \|\mathcal{E}_h(\boldsymbol{\psi})\|_0 + \Big(\sum_{e\in F} \frac{1}{h_e}\int_e [\boldsymbol{\psi}]^2\, ds\Big)^{1/2}, \quad \forall \boldsymbol{\psi} \in \boldsymbol{V}_{2,h}. \tag{33}$$

Proof. The proof can be found, for instance, in [8]. □

Lemma 5.7. *There exists a positive constant C independent of h such that*

$$C\|v\|_0 \leq |v|_{1,h}, \quad \forall v \in V_h^{nc}. \tag{34}$$

Proof. The proof can be found, for instance, in [7, 22, 15]. □

6. Error estimate

We show error estimates in this section. We first derive some approximation results. For the modified nonconforming rotated Q_1 element, we define the interpolation operator $\pi_h : H_0^1(n) \to V_h^{nc}$ by

$$\begin{aligned} &\int_e \pi_h v ds = \int_e v ds, \quad \forall v \in H_0^1(n), \text{ for any } e \in F\ , \\ &\int_K \pi_h v dxdy = \int_K v dxdy, \text{ for any } K \in J^h. \end{aligned} \tag{35}$$

We have the following result

Lemma 6.1.

$$| \sum_K \int_{\partial K} v\psi \cdot n ds | \leq Ch\|v\|_{1,h}\|\psi\|_1, \quad \forall v \in V_h^{nc}, \quad \forall \psi \in (H^1(\Omega))^2, \tag{36}$$

$$\|v - \pi_h v\| + h \mid v - \pi_h v \mid_{1,h} \leq Ch^2 \mid v \mid_2, \quad \forall v \in H^2(n) \cap H_0^1(n), \tag{37}$$

Proof. The proof can be found in [13, 22, 15]. □

Remark 6.2. Note that (36) is obviously satisfied by $V_h^c \in H_0^1(\Omega)$, consequently we shall not differ $\boldsymbol{V}_{1,h}$ from $\boldsymbol{V}_{2,h}$ when the consistency error is concerned.

For the projection operator $\boldsymbol{R}_h$, we have the following approximation property,

Lemma 6.3. *There exists a constant C, for any $\boldsymbol{u} \in (H^1(\Omega))^2$, such that*

$$\|\boldsymbol{R}_h\boldsymbol{u} - \boldsymbol{u}\|_0 \leq Ch\|\boldsymbol{u}\|_1. \tag{38}$$

Proof. The proof is elementary. □

Lemma 6.4. *Let $G \in L^2(\Omega)$ and $F \in (H^1(\Omega))^2$, u be the weak solution to the following boundary value problem*

$$-\triangle u = G - \nabla \cdot F \ \textit{in}\ \Omega, \tag{39}$$

$$u \mid_{\partial\Omega} = 0, \tag{40}$$

and $u_h \in V_h^{nc}$ be the solution to the discrete problem

$$(\nabla_h u_h, \nabla_h v) = (G, v) + (F, \nabla_h v), \quad \forall v \in V_h^{nc}. \tag{41}$$

Then there exists a constant C independent of h, G and F such that

$$\|u - u_h\|_{1,h} \leq Ch(\|G\|_0 + \|F\|_1), \tag{42}$$

$$\|u - u_h\|_0 \leq Ch^2(\|G\|_0 + \|F\|_1). \tag{43}$$

Proof. Using Lemma 6.1, we can obtain (42)-(43) by standard arguments from nonconforming finite element methods for the second order elliptic problems [24, 25, 6]. For the brevity, we omit the details. □

Theorem 6.5. *Let $(r, \boldsymbol{\phi}, p, \omega)$ and $(r_h, \boldsymbol{\phi}_h, p_h, \omega_h)$ be the solution to Problem 2.2 and 4.2, respectively. For any $g \in L^2(\Omega)$ and $t \in (0, 1]$, there exists a constant C independent of h, g and t, such that*

$$\begin{aligned} &\|r - r_h\|_{1,h} + \|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_{1,h} + \|p - p_h\|_0 + t\|\operatorname{curl}(p - p_h)\|_0 \\ &\quad + \|\omega - \omega_h\|_{1,h} \leq Ch\|g\|_0. \end{aligned} \tag{44}$$

Proof. Throughout the proof, $i = 1, 2$. Owing to (3), (15) and Lemma 6.4, we have

$$\|\nabla_h(r - r_h)\|_0 \le Ch\|g\|_0.$$

We have by (4) and (16) for any $\boldsymbol{\psi} \in \boldsymbol{V}_{i,h}$, $i = 1, 2$, that

$$\begin{aligned} a_h(\boldsymbol{\phi}_h - \boldsymbol{\psi}, \boldsymbol{\phi}_h - \boldsymbol{\psi}) &= a_h(\boldsymbol{\phi} - \boldsymbol{\psi}, \boldsymbol{\phi}_h - \boldsymbol{\psi}) \\ &+ (\operatorname{curl}(p_h - p), \boldsymbol{\phi}_h - \boldsymbol{\psi}) + (\nabla_h r_h - \nabla r, \boldsymbol{\phi}_h - \boldsymbol{\psi}) \\ &- a_h(\boldsymbol{\phi}, \boldsymbol{\phi}_h - \boldsymbol{\psi}) + (\operatorname{curl} p, \boldsymbol{\phi}_h - \boldsymbol{\psi}) + (\nabla r, \boldsymbol{\phi}_h - \boldsymbol{\psi}). \end{aligned}$$

By (5) and (17), we obtain for any $\boldsymbol{\psi} \in \boldsymbol{V}_{i,h}$ and $q \in Q_h$ that

$$\begin{aligned} \lambda^{-1}t^2\|\operatorname{curl}(p_h - q)\|_0^2 &= \lambda^{-1}t^2(\operatorname{curl}(p - q), \operatorname{curl}(p_h - q)) - (\boldsymbol{\phi}_h - \boldsymbol{\phi}, \operatorname{curl}(p_h - q)) \\ &= \lambda^{-1}t^2(\operatorname{curl}(p - q), \operatorname{curl}(p_h - q)) - (\boldsymbol{\phi}_h - \boldsymbol{\psi}, \operatorname{curl}(p_h - q)) \\ &\quad + (\boldsymbol{\phi} - \boldsymbol{\psi}, \operatorname{curl}(p_h - q)). \end{aligned}$$

Taking these two equations together,

(45)
$$\begin{aligned} a_h(\boldsymbol{\phi}_h - \boldsymbol{\psi}, \boldsymbol{\phi}_h - \boldsymbol{\psi}) &+ \lambda^{-1}t^2\|\operatorname{curl}(p_h - q)\|_0^2 \\ &= a_h(\boldsymbol{\phi} - \boldsymbol{\psi}, \boldsymbol{\phi}_h - \boldsymbol{\psi}) \\ &\quad + (\nabla_h r_h - \nabla r, \boldsymbol{\phi}_h - \boldsymbol{\psi}) + \lambda^{-1}t^2(\operatorname{curl}(p - q), \operatorname{curl}(p_h - q)) \\ &\quad - (\operatorname{curl}(p - q), \boldsymbol{\phi}_h - \boldsymbol{\psi}) + (\boldsymbol{\phi} - \boldsymbol{\psi}, \operatorname{curl}(p_h - q)) \\ &\quad - a_h(\boldsymbol{\phi}, \boldsymbol{\phi}_h - \boldsymbol{\psi}) + (\operatorname{curl} p, \boldsymbol{\phi}_h - \boldsymbol{\psi}) + (\nabla r, \boldsymbol{\phi}_h - \boldsymbol{\psi}). \end{aligned}$$

It follows from (36) (resp. Remark 6.2) and (4) that

$$|a_h(\boldsymbol{\phi}, \boldsymbol{\phi}_h - \boldsymbol{\psi}) + (\operatorname{curl} p, \boldsymbol{\phi}_h - \boldsymbol{\psi}) + (\nabla r, \boldsymbol{\phi}_h - \boldsymbol{\psi})| \le Ch\|\boldsymbol{\phi}\|_2|\boldsymbol{\phi}_h - \boldsymbol{\psi}|_{1,h}. \tag{46}$$

For any $\boldsymbol{\psi} \in (H_0^1(n))^2 \cup \boldsymbol{V}_{i,h}$ and $q \in H^1(\Omega)$, we need to bound the term $(\boldsymbol{\psi}, \operatorname{curl} q)$. Integrating by parts and using (36), we derive it as

$$\begin{aligned} (\boldsymbol{\psi}, \operatorname{curl} q) &= -\sum_{K \in J^h} \int_K \operatorname{rot} \boldsymbol{\psi} q \, dx + \sum_{e \in E} \int_e [\boldsymbol{\psi}] \cdot \boldsymbol{t} q \, ds \\ &\le C(\|\boldsymbol{\psi}\|_{1,h}\|q\|_0 + h\|\boldsymbol{\psi}\|_{1,h}\|q\|_1). \end{aligned} \tag{47}$$

Thanks to Lemma 5.3, Lemma 5.4 and Lemma 5.5, we get by (4), (16), (46) and (47) that

(48)
$$\begin{aligned} \beta\|p_h - q\|_0 &\le \sup_{\boldsymbol{\psi} \in \boldsymbol{V}_{i,h}} \frac{(\boldsymbol{\psi}, \operatorname{curl}(p_h - q))}{\|\boldsymbol{\psi}\|_{1,h}} \\ &\le C(\|\boldsymbol{\phi}_h - \boldsymbol{\phi}\|_{1,h} + \|\nabla_h r_h - \nabla r\|_0 + h\|\boldsymbol{\phi}\|_2) + \sup_{\boldsymbol{\psi} \in \boldsymbol{V}_{i,h}} \frac{(\boldsymbol{\psi}, \operatorname{curl}(p - q))}{\|\boldsymbol{\psi}\|_{1,h}} \\ &\le C(\|\boldsymbol{\phi}_h - \boldsymbol{\psi}\|_{1,h} + \|\boldsymbol{\phi} - \boldsymbol{\psi}\|_{1,h} + \|\nabla_h r_h - \nabla r\|_0 + h\|\boldsymbol{\phi}\|_2) \\ &\quad + C(h\|p - q\|_1 + \|p - q\|_0). \end{aligned}$$

Substituting (46), (48) and (47) into (45), and using Lemma 5.5, Lemma 5.7 and the inverse estimate, we proceed as

$$\begin{aligned} \|\boldsymbol{\phi}_h - \boldsymbol{\psi}\|_{1,h} &+ t\|\operatorname{curl}(p_h - q)\|_0 \\ &\le C(\|\boldsymbol{\phi} - \boldsymbol{\psi}\|_{1,h} + t\|p - q\|_1 + \|\nabla_h r_h - \nabla r\|_0 + h\|\boldsymbol{\phi}\|_2) \\ &\quad + C(h\|p - q\|_1 + \|p - q\|_0). \end{aligned} \tag{49}$$

We now use the triangle inequality to obtain

$$\begin{aligned}\|\boldsymbol{\phi}_h-\boldsymbol{\phi}\|_{1,h} &+ t\|\operatorname{curl}(p_h-p)\|_0\\ &\leq C(\|\boldsymbol{\phi}-\boldsymbol{\psi}\|_{1,h}+t\|p-q\|_1+h\|p-q\|_1\\ &+ \|p-q\|_0+\|\nabla_h r_h-\nabla r\|_0+h\|\boldsymbol{\phi}\|_2).\end{aligned}$$

It follows from (48) and (49) that

$$\begin{aligned}\|p_h-p\|_0\leq C(&\|\boldsymbol{\phi}-\boldsymbol{\psi}\|_{1,h}+t\|p-q\|_1+h\|p-q\|_1\\ &+\|p-q\|_0+\|\nabla_h r_h-\nabla r\|_0+h\|\boldsymbol{\phi}\|_2).\end{aligned}\tag{50}$$

We take $q=\pi_1^h p$ with π_1^h the bilinear Clement interpolation operator [23] which admits the following approximation property:

$$\|p-\pi_1^h p\|_0+h\|p-\pi_1^h p\|_1\leq Ch\|p\|_1 \text{ and } \|p-\pi_1^h p\|_1\leq Ch\|p\|_2.\tag{51}$$

Applying Lemma 2.3, Lemma 6.3 and Lemma 6.4, we finally obtain

$$\|\boldsymbol{\phi}-\boldsymbol{\phi}\|_{1,h}+\|p-p_h\|_0+t\|\operatorname{curl}(p_h-p)\|_0\leq Ch\|g\|_0,$$

since $\boldsymbol{\psi}\in\boldsymbol{V}_{i,h}$ is arbitrary.

Now, we remain to bound $\|\omega-\omega_h\|_{1,h}$. Let $\bar{\omega}_h\in V_h^{nc}$ be the solution to the following problem:

$$(\nabla_h\bar{\omega}_h,\nabla_h s)=(\boldsymbol{\phi}+\lambda^{-1}t^2\nabla r,\nabla_h s),\quad \forall s\in V_h^{nc}.$$

Taking into account Lemma 6.4, we deduce

$$\|\nabla_h(\omega-\bar{\omega}_h)\|_0\leq Ch\|\boldsymbol{\phi}+\lambda^{-1}t^2\nabla r\|_1\leq Ch\|g\|_0.\tag{52}$$

It follows from (6) and (18) that

$$(\nabla_h(\omega_h-\bar{\omega}_h),\nabla_h s)=(\boldsymbol{\phi}_h-\boldsymbol{\phi}+\lambda^{-1}t^2(\nabla_h r_h-\nabla r),\nabla_h s).\tag{53}$$

Let $s=\omega_h-\bar{\omega}_h$ in (53); we have

$$\|\nabla_h(\omega_h-\bar{\omega}_h)\|_0\leq C(\|\boldsymbol{\phi}_h-\boldsymbol{\phi}\|_0+t^2\|\nabla_h r_h-\nabla r\|_0)\leq Ch\|g\|_0,$$

which, together with (52), implies

$$\|\omega_h-\omega\|_{1,h}\leq Ch\|g\|_0,$$

which completes the proof. □

In order to analyse the L^2 error, we need to introduce the following dual problem

Problem 6.6. Find $(\boldsymbol{\phi}_d,p_d)\in(H_0^1(n))^2\times\hat{H}^1(\Omega)$, such that

$$a(\boldsymbol{\phi}_d,\boldsymbol{\psi})-(\boldsymbol{\psi},\operatorname{curl}p_d)=(d,\boldsymbol{\psi}),\quad \forall\boldsymbol{\psi}\in(H_0^1(\Omega))^2,\tag{54}$$

$$-(\boldsymbol{\phi}_d,\operatorname{curl}q)-\lambda^{-1}t^2(\operatorname{curl}p_d,\operatorname{curl}q)=0,\quad\forall q\in\hat{H}^1(\Omega).\tag{55}$$

The solution to Problem 6.6 admits the following regularity:

$$\|\boldsymbol{\phi}_d\|_2+\|p_d\|_1+t\|p_d\|_2\leq C\|d\|_0.\tag{56}$$

Define the following interpolation:

$$\Pi_h\boldsymbol{\psi}=\begin{cases}(\pi_1^h\psi_1,\pi_1^h\psi_2) \text{ when } \boldsymbol{V}_{1,h} \text{ is used,}\\ (\pi_h\psi_1,\pi_h\psi_2) \text{ when } \boldsymbol{V}_{2,h} \text{ is used}\end{cases}$$

for any $(\psi_1,\psi_2)=\boldsymbol{\psi}\in(H_0^1(\Omega))^2$. We have the following estimates

$$\|\Pi_h\boldsymbol{\psi}-\boldsymbol{\psi}\|_0+h\|\Pi_h\boldsymbol{\psi}-\boldsymbol{\psi}\|_{1,h}\leq Ch\|\boldsymbol{\psi}\|_1 \text{ and } \|\Pi_h\boldsymbol{\psi}-\boldsymbol{\psi}\|_{1,h}\leq Ch\|\boldsymbol{\psi}\|_2.\tag{57}$$

Theorem 6.7. *Let $(r, \boldsymbol{\phi}, p, \omega)$ and $(r_h, \boldsymbol{\phi}_h, p_h, \omega_h)$ be the solution to Problem 2.2 and 4.2, respectively. For any $g \in L^2(\Omega)$ and $t \in (0, 1]$, there exists a constant C independent of h, g and t, such that*

$$\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_0 + \|\omega_h - \omega\|_0 \le Ch^2\|g\|_0. \tag{58}$$

Proof. First, it follows from Lemma 6.4 that

$$\|r - r_h\|_0 \le Ch^2\|g\|_0. \tag{59}$$

Applying (16) and (17) as well as (54) and (55), we derive it as

$$\begin{aligned}
(d, \boldsymbol{\phi} - \boldsymbol{\phi}_h) &= (d, \boldsymbol{\phi} - \boldsymbol{\phi}_h) - a_h(\boldsymbol{\phi}_d, \boldsymbol{\phi} - \boldsymbol{\phi}_h) + (\boldsymbol{\phi} - \boldsymbol{\phi}_h, \operatorname{curl} p_d) \\
&\quad + a(\boldsymbol{\phi}_d, \boldsymbol{\phi} - \boldsymbol{\phi}_h) - (\boldsymbol{\phi} - \boldsymbol{\phi}_h, \operatorname{curl} p_d) \\
&= (d, \boldsymbol{\phi} - \boldsymbol{\phi}_h) - a_h(\boldsymbol{\phi}_d, \boldsymbol{\phi} - \boldsymbol{\phi}_h) + (\boldsymbol{\phi} - \boldsymbol{\phi}_h, \operatorname{curl} p_d) \\
&\quad + a_h(\Pi_h\boldsymbol{\phi}_d - \boldsymbol{\phi}_d, \boldsymbol{\phi}) - (\Pi_h\boldsymbol{\phi}_d - \boldsymbol{\phi}_d, \operatorname{curl} p) - (\nabla r, \Pi_h\boldsymbol{\phi}_d - \boldsymbol{\phi}_d) \\
&\quad + a_h(\boldsymbol{\phi}_d - \Pi_h\boldsymbol{\phi}_d, \boldsymbol{\phi} - \boldsymbol{\phi}_h) \\
&\quad - (\boldsymbol{\phi} - \boldsymbol{\phi}_h, \operatorname{curl}(p_d - \pi_1^h p_d)) \\
&\quad + (\Pi_h\boldsymbol{\phi}_d - \boldsymbol{\phi}_d, \operatorname{curl}(p - p_h)) \\
&\quad + (\nabla r - \nabla_h r_h, \Pi_h\boldsymbol{\phi}_d) \\
&\quad + \lambda^{-1}t^2(\operatorname{curl}(p - p_h), \operatorname{curl}(\pi_1^h p_d - p_d)) = I_1 + \cdots + I_7.
\end{aligned} \tag{60}$$

I_1 and I_2 are consistency error terms, which can be estimated by a classic argument

$$|I_1| \le Ch\|\boldsymbol{\phi}_d\|_2\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_{1,h} \text{ and } |I_2| \le Ch\|\boldsymbol{\phi}\|_2\|\Pi_h\boldsymbol{\phi}_d - \boldsymbol{\phi}_d\|_{1,h}. \tag{61}$$

Owing to Lemma 5.5

$$|I_3| \le C\|\Pi_h\boldsymbol{\phi}_d - \boldsymbol{\phi}_d\|_{1,h}\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_{1,h}. \tag{62}$$

Thanks to (47),

$$|I_4| \le C(\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_{1,h}\|p_d - \pi_1^h p_d\|_0 + h\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_{1,h}\|p_d - \pi_1^h p_d\|_1). \tag{63}$$

Using (47) again, we have by the inverse and triangle inequality

$$\begin{aligned}
I_5 &= (\Pi_h\boldsymbol{\phi}_d - \boldsymbol{\phi}_d, \operatorname{curl}(p - \pi_1^h p)) + (\Pi_h\boldsymbol{\phi}_d - \boldsymbol{\phi}_d, \operatorname{curl}(\pi_1^h p - p_h)) \\
&\le C\|\Pi_h\boldsymbol{\phi}_d - \boldsymbol{\phi}_d\|_{1,h}(\|p - \pi_1^h p\|_0 + \|\pi_1^h p - p_h\|_0 + h\|p - \pi_1^h p\|_1) \\
&\le C\|\Pi_h\boldsymbol{\phi}_d - \boldsymbol{\phi}_d\|_{1,h}(\|p - \pi_1^h p\|_0 + h\|p - \pi_1^h p\|_1 + \|p - p_h\|_0).
\end{aligned} \tag{64}$$

We have the following decomposition for the sixth term I_6:

$$I_6 = (\nabla r - \nabla_h r_h, \Pi_h\boldsymbol{\phi}_d - \boldsymbol{\phi}_d) + (\nabla r - \nabla_h r_h, \boldsymbol{\phi}_d). \tag{65}$$

By virtue of Theorem 6.5, we bound the first term in (65) as

$$(\nabla r - \nabla_h r_h, \Pi_h\boldsymbol{\phi}_d - \boldsymbol{\phi}_d) \le Ch\|g\|_0\|\Pi_h\boldsymbol{\phi}_d - \boldsymbol{\phi}_d\|_{1,h}. \tag{66}$$

Integrating by parts and applying (59), the second term in (65) can be bounded as

$$\begin{aligned}
(\nabla r - \nabla_h r_h, \boldsymbol{\phi}_d) &= -(r - r_h, \operatorname{div} \boldsymbol{\phi}_d) + \sum_{e \in E'} \int_e [r - r_h]\boldsymbol{\phi}_d \cdot \boldsymbol{n}_e \, ds \\
&\le Ch(h\|g\|_0 + \|r - r_h\|_{1,h})\|\boldsymbol{\phi}_d\|_1.
\end{aligned} \tag{67}$$

Finally, we have the following estimate for the last term I_7:

$$|I_7| \le Ct^2\|p - p_h\|_1\|p_d - \pi_1^h p_d\|_1. \tag{68}$$

Substituting inequalities (61)-(68) into (60), using the regularity (56) and the interpolation error estimate (57) and (51) with Theorem 6.5, we obtain

$$(d, \boldsymbol{\phi} - \boldsymbol{\phi}_h) \le Ch^2\|d\|_0\|g\|_0, \tag{69}$$

which gives us

$$\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_0 \le Ch^2\|g\|_0.$$

Now, we turn to bound $\|\omega - \omega_h\|_0$. We first introduce the following problem: Find $\theta \in H_0^1(\Omega)$ such that

$$(\nabla\theta, \nabla s) = (\boldsymbol{\phi}, \nabla s), \quad \forall s \in H_0^1(\Omega).$$

Let $\bar{\theta}_h \in V_h^{nc}$ be the solution of the discrete problem

$$(\nabla_h\bar{\theta}_h, \nabla_h s) = (\boldsymbol{\phi}, \nabla_h s), \quad \forall s \in V_h^{nc}. \tag{70}$$

It follows from Lemma 6.4 that

$$\|\bar{\theta}_h - \theta\|_0 \le Ch^2\|\boldsymbol{\phi}\|_1 \le ch^2\|g\|_0. \tag{71}$$

From (6), $\theta = \omega - \lambda^{-1}t^2 r$. Denote $\theta_h = \omega_h - \lambda^{-1}t^2 r_h$; by (18) and (70), we have

$$(\nabla_h(\theta_h - \bar{\theta}_h), \nabla_h s) = (\boldsymbol{\phi}_h - \boldsymbol{\phi}, \nabla_h s), \quad \forall s \in V_h^{nc}.$$

Setting $s = \theta_h - \bar{\theta}_h$,

$$\|\nabla_h(\theta_h - \bar{\theta}_h)\|_0 \le C\|\boldsymbol{\phi}_h - \boldsymbol{\phi}\|_0 \le Ch^2\|g\|_0. \tag{72}$$

It follows from (59), (71) and (72) that

$$\begin{aligned}
\|\omega - \omega\|_0 &\le \|\theta - \theta_h\|_0 + \lambda^{-1}t^2\|r - r_h\|_0 \\
&\le \|\theta - \bar{\theta}_h\|_0 + \|\theta_h - \bar{\theta}_h\|_0 + \lambda^{-1}t^2\|r - r_h\|_0 \\
&\le Ch^2\|g\|_0.
\end{aligned}$$

□

Remark 6.8. We can extend our analysis to the element of [16] and obtain its optimal L^2 error estimate, which is missing in the literature.

Remark 6.9. To simplify the notation and fix the main idea, we present the analysis on the rectangular mesh. Obviously, both elements can be generalized to the general quadrilateral mesh. In addition, a similar argument herein shows that the energy error estimate is of order h^α, and that the L^2 norm error estimate is of order $h^{2\alpha}$, provided that the mesh satisfies the $(1+\alpha)$ section condition with $0 \le \alpha \le 1$ [17]. This implies that the convergence rates in both norms depend on the mesh parameter α. As a consequence, optimal error estimates hold only for mildly distorted meshes with $\alpha = 1$. In [18], two nonconforming quadrilateral elements are proposed with optimal error estimates uniformly in α with respect to both energy norm and L^2 norm.

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