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SHARPLY LOCALIZED POINTWISE AND W_∞^{-1} ESTIMATES FOR FINITE ELEMENT METHODS FOR QUASILINEAR PROBLEMS

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Abstract. We establish pointwise and W_∞^{-1} estimates for finite element methods for a class of second-order quasilinear elliptic problems defined on domains Ω in $\mathbb{R}^n$. These estimates are localized in that they indicate that the pointwise dependence of the error on global norms of the solution is of higher order. Our pointwise estimates are similar to and rely on results and analysis techniques of Schatz for linear problems. We also extend estimates of Schatz and Wahlbin for pointwise differences $e(x_1) - e(x_2)$ in pointwise errors to quasilinear problems. Finally, we establish estimates for the error in $W_\infty^{-1}(D)$, where $D \subset \Omega$ is a subdomain. These negative norm estimates are novel for linear as well as for nonlinear problems. Our analysis heavily exploits the fact that Galerkin error relationships for quasilinear problems may be viewed as perturbed linear error relationships, thus allowing easy application of properly formulated results for linear problems.

1. Introduction

We consider finite element approximations to second-order quasilinear Neumann problems of the form

$$(1.1)\qquad \begin{aligned} -\sum_{i=1}^n \frac{\partial}{\partial x_i} F_i(x,u,\nabla u) + F_0(x,u,\nabla u) &= f \text{ in } \Omega,\\ \sum_{i=1}^n F_i(x,u,\nabla u)\cdot n_i &= 0 \text{ on } \partial\Omega. \end{aligned}$$

Here $\Omega \subset \mathbb{R}^n$ is a bounded domain having smooth boundary $\partial\Omega$ and boundary normal $\vec{n}$, and $\{F_i\}_{0\le i\le n}$ are sufficiently smooth coefficients satisfying (not necessarily uniform) ellipticity and monotonicity conditions. Precise assumptions are given in §2. We also assume that (1.1) has a solution u with uniformly bounded derivatives of sufficiently high order. Two important examples are equations with a linear elliptic leading term of the form $-\sum_{i=1}^n \frac{\partial}{\partial x_i} F_i(x,u,\nabla u) = -\nabla\cdot(A\nabla u)$ and equations of prescribed mean curvature type with $-\sum_{i=1}^n \frac{\partial}{\partial x_i} F_i(x,u,\nabla u) = -\nabla\cdot \frac{\nabla u}{\sqrt{1+|\nabla u|^2}}$.

Our goals in this paper are to extend the localized pointwise error estimates from the works [Sch98] and [SW03] to quasilinear problems and to establish localized negative norm (W_∞^{-1}) estimates. The latter estimates are new for linear as well as for nonlinear problems. In order to describe our results, we state here a

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localized estimate for errors in point values. We summarize our assumptions as follows; precise statements are given in §2. Assume that $u_h \in S_h^r$ is the Galerkin approximation to u, where S_h^r is a finite element space consisting of continuous piecewise polynomials of degree $r-1$ defined on a quasi-uniform decomposition of Ω composed of simplices of diameter at most h. For an arbitrary but fixed point $x \in \Omega$ define the weighted norm

$$\|u\|_{W_p^k(\Omega),x,h,s} = \sum_{|\alpha| \le k} \|\sigma_{x,h}^s D^\alpha u\|_{L_p(\Omega)},$$

where $\sigma_{x,h}(y) = \frac{h}{|x-y|+h}$. Also let $0 \le s \le r-2$. Then

$$\begin{aligned} |(u-u_h)(x)| \;\; &\le \;\; C(u)[h\ell_{h,s}^{r-2} \inf_{\chi \in S_h^r} \|u-\chi\|_{W_\infty^1(\Omega),x,h,s} \\ &\quad + C_F \ell_h \inf_{\chi \in S_h^r} \|u-\chi\|_{W_\infty^1(\Omega)}^2], \end{aligned} \tag{1.2}$$

where $\ell_h = \ln \frac{1}{h}$, and $\ell_{h,s}^{r-2} = \ell_h$ if $s = r-2$ and $\ell_{h,s}^{r-2} = 1$ otherwise. Also, $C(u)$ is a generic constant depending on the coeffiecients F_i and sufficiently strong global Sobolev norms of the solution u. When (1.1) is linear, $C(u)$ depends only on the coefficients of the differential equation (1.1) and not on u and $C_F = 0$, so (1.2) reduces precisely to Theorem 2.1 of [Sch98]. Note also that we may recover standard quasi-optimal $L_\infty(\Omega)$ estimates from (1.2) by taking $s = 0$.

If the parameter $s > 0$, (1.2) indicates that the finite element error is "localized" in that the pointwise error $(u-u_h)(x)$ depends only weakly on norms of u away from the point x. A more precise statement may be gained by considering so-called pointwise error expansion inequalities as introduced in [Sch98]. Letting $\overline{s}$ be the least integer satisfying $\overline{s} \ge s$, (1.2) implies that

$$\begin{aligned} |(u-u_h)(x)| \;\; &\le \;\; C(u)\ell_{h,s}^{r-2}[h^r \textstyle\sum_{|\alpha|=r} |D^\alpha u(x)| + \dots \\ &\quad \dots + h^{r+\overline{s}-1} \textstyle\sum_{|\alpha|=r+\overline{s}-1} |D^\alpha u(x)| + h^{r+s}\|u\|_{W_\infty^{r+\overline{s}}(\Omega)}]. \end{aligned} \tag{1.3}$$

Thus the dependence of the error $|(u-u_h)(x)|$ on global norms of u occurs only via the term $h^{r+s}\|u\|_{W_\infty^{r+\overline{s}}(\Omega)}$, which is in turn of higher order so long as we may take $s > 0$. Note that in the practically important piecewise linear case $r = 2$, the parameter s must be 0 and no localization is indicated by (1.2) or (1.3). The fact that no localization occurs when piecewise linear elements are used was confirmed by a simple one-dimensional counterexample in [De04a]. It should also be noted that (1.3) has the same essential form for both linear and quasilinear problems, the only difference being that in nonlinear problems the constant $C(u)$ in (1.3) depends on the unknown solution u.

In addition to analyzing the pointwise error $(u-u_h)(x)$, we shall prove localized estimates for three other error notions. In total we shall consider the following four error notions:

1. Pointwise errors $|(u-u_h)(x)|$ as in (1.2) and (1.3).
2. Pointwise gradient errors $|\nabla(u-u_h)(x)|$.
3. *Differences* $|(u-u_h)(x_1) - (u-u_h)(x_2)|$ in pointwise errors.
4. Local W_∞^{-1} norms of errors, that is, $\|u-u_h\|_{W_\infty^{-1}(D)}$, where $D \subset \Omega$ is a subdomain of diameter $H \ge h$.

For linear problems, the first two types of estimates are contained in [Sch98]. In contrast to pointwise errors in function values, estimates of type 2 above for pointwise gradient errors indicate localization for all orders of finite element space $r \ge 2$. Estimates of type 3 for differences in pointwise errors were introduced in [SW03] for piecewise linear elements $r = 2$, and we extend this analysis to quasilinear problems. Such estimates are useful in the case of piecewise linear elements because of the lack of localization in the pointwise error $(u - u_h)(x)$. The W_∞^{-1} estimates we prove here are new even in the case of linear problems. As for pointwise estimates of type 1, our negative norm estimates indicate localization only when $r \geq 3$, thus excluding the piecewise linear case. In all four cases, we shall also give pointwise error expansion inequalities as in (1.3). In addition to yielding additional insight into error behavior, these expansions are more convenient for applications. Besides having inherent theoretical interest, localized estimates are a key tool in the analysis of a class of asymptotically exact a posteriori estimators for pointwise gradient errors and pointwise errors in function values. In [HSWW01] and [SW04] it was shown that a class of averaging or gradient recovery-type a posteriori estimators for pointwise gradient (W_∞^1) errors is asymptotically exact and yields highly local error control for linear problems. In the case of nonlinear problems, it is trivial to use our estimates of types 1, 2, and 3 above to show that the results of [HSWW01] and [SW04] hold almost verbatim for the class of problems analyzed here. The chief difference between the linear and nonlinear case is that more of the constants in the analysis depend on the unknown solution u in the nonlinear case.

A chief application of the W_∞^{-1} estimates which we introduce here is the analysis of averaging-type a posteriori estimators for pointwise errors in function values, as opposed to the estimators for pointwise gradient (W_∞^1) errors considered by previous authors. In particular, it is possible to show that a certain class of averaging estimators for the L_∞ norm behaves in a highly local fashion and is asymptotically exact when quadratic ($r = 3$) and higher order elements are used. Such estimators have not previously been extensively considered in the literature, and we plan to investigate their properties in a forthcoming work.

Next we survey other literature relevant to the present work. In [Fr78] quasi-optimal L_∞ bounds for a class of quasilinear elliptic equations similar to (1.1) were proved. Our estimates improve on these earlier results by removing unnecessary logarithmic factors, providing localized estimates, and explicitly analyzing the pointwise gradient error. We also treat homogeneous Neumann boundary conditions instead of Dirichlet conditions as in [Fr78]. After being introduced in [Sch98] and [Sch00], localized pointwise estimates have been extended to a number of other contexts. These include mixed ([De04b]) and discontinuous Galerkin ([Ch05], [Gu05]) methods for elliptic problems, parabolic problems ([Ley04]), and residual-based a posteriori estimates for quasilinear problems ([De06]). In addition to being applied to the analysis of asymptotically exact gradient recovery estimators as mentioned above, local versions of these estimates have also been used to establish improved superconvergence estimates on locally symmetric meshes ([Sch05]). Finally, in the course of preparing this work we became aware of the announcement by Schatz ([Sch06]) of W_∞^{-1} results which are slightly sharper than ours, but restricted to linear problems.

To conclude our introduction we comment on an essential feature of the proofs in this work. We strongly exploit the fact that we may treat (1.1) as a perturbed linear problem. In particular, [Sch98] contains estimates which assume that u_h satisfies a perturbed Galerkin orthogonality relationship of the form

$$B(u - u_h, \chi) = G(\chi),\ \chi \in S_h^r,$$

with G a linear functional. Analysis of the finite element method for quasilinear problems typically is carried out by treating nonlinear problems as perturbed linear problems, and we show that the perturbation term which arises in the Galerkin error relationship may be expressed as a linear functional G as above. This allows us to directly apply the results of [Sch98] instead of repeating most of the technicalities necessary to prove linear results as was essentially done for example in [Fr78].

The paper is organized as follows. In §2 we give preliminaries. In §3 we state and prove pointwise error estimates for function values and gradients. In §4 we give estimates for pointwise differences in the finite element error, and in §5 we establish W_∞^{-1} estimates.

2. Preliminaries

In this section we give a number of assumptions and definitions concerning u and its finite element approximation u_h. We note again that our proofs will directly employ results from [Sch98] for linear problems. Many of our assumptions will thus be tailored to ensure that we may use these results in the current situation, and a familiarity with this work would serve the reader well, as we shall refer to it often.

2.1. **Assumptions on the continuous problem.** It is assumed that (1.1) has a unique solution u which lies in $W_\infty^k(\Omega)$ for k sufficiently large. Here k depends on r and the order s of the weight in (1.2); we do not attempt to give a more precise dependence. In addition, the coefficients $F_i(x,z,\vec{p})$ (where $(x,z,\vec{p}) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$) are assumed to be sufficiently smooth in all of their arguments. For $0 \le i \le n$ and $1 \le j \le n$ let $F_{ij} = \frac{\partial}{\partial p_j} F_i(x,z,\vec{p})$, and also define $F_{i0} = \frac{\partial}{\partial z} F_i(x,z,\vec{p})$. The nonuniform ellipticity condition

$$\sum_{1 \le i,j \le n} F_{ij}(x,z,\vec{p})\xi_i\xi_j > 0 \ \forall \ (x,z,\vec{p}) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n, \xi \in \mathbb{R}^n \setminus \{0\} \tag{2.1}$$

is also assumed. Let

$$\langle Tu, v\rangle = \sum_{i=0}^{n} \int_\Omega F_i(x,u,\nabla u) v_{x_i} \, \mathrm{d}x,$$

where $v_{x_0} = v$. We shall require the following (nonuniform) monotonicity condition. For any $X > 0$, there exists $C_X > 0$ such that

$$\langle Tu - Tv, u - v\rangle \ge C_X \|u - v\|^2_{H^1(\Omega)} \tag{2.2}$$

for all $u, v \in W_\infty^1(\Omega)$ with $\|u\|_{W_\infty^1(\Omega)} \le X$ and $\|v\|_{W_\infty^1(\Omega)} \le X$.

For $v, w \in H^1(\Omega)$ we next define the bilinear form

$$A(v,w) = \int_\Omega \sum_{0 \le i,j \le n} F_{ij}(x,u,\nabla u) v_{x_j} w_{x_i} \, \mathrm{d}x.$$

We require that A be coercive and continuous, that is, that there exist constants K_1, K_2 such that for $v, w \in H^1(\Omega)$,

$$\begin{aligned} K_1 \|v\|^2_{H^1(\Omega)} &\le A(v,v), \\ A(v,w) &\le K_2 \|v\|_{H^1(\Omega)} \|w\|_{H^1(\Omega)}. \end{aligned}$$

Note that the ellipticity of A is guaranteed by (2.1) and our assumption that u has uniformly bounded derivatives of sufficiently high order, and in particular, that $\|\nabla u\|_{L_\infty(\Omega)}$ is finite.

In our arguments the bilinear form A above will fill the role of the bilinear form A defined in equation (0.1) of [Sch98]. A as defined here may have a term $\int_\Omega \sum_{i=1}^n \tilde{b}_i u v_{x_i}\, \mathrm{d}x$ which is not directly included in the corresponding definition in [Sch98]. This presents no difficulty, as the presence of such a term in no way affects the results or proofs of [Sch98].

2.2. **The finite element solution and assumptions on the finite element space.** We seek a finite element approximation $u_h \in S_h^r$ to u which satisfies

$$\langle Tu_h, v_h \rangle = (f, v) \; \forall \; v_h \in S_h^r, \tag{2.3}$$

where $(\cdot,\cdot)$ is the L_2-inner product. The existence of u_h will be proved in the course of establishing pointwise estimates.

In order to avoid technicalities involving finite element approximation of curved boundaries, we assume that the mesh $\mathcal{T}_h$ is an exact decomposition of Ω, so that elements having one curved face on $\partial\Omega$ are admitted. The finite element space S_h^r is assumed to have standard approximation properties. Let $D \subset \Omega$, and let $D_M = \{x \in \Omega : dist(x, D) < M\}$. We assume that there exists $\gamma > 0$ such that for $1 \le p \le \infty$, $0 \le k \le 1$, and $k \le m \le r$,

$$\|u - u_h\|_{W_p^k(D)} \le C h^{m-k} \|u\|_{W_p^m(D_{\gamma h})}. \tag{2.4}$$

We additionally require standard inverse and superapproximation properties and a scaling property as in §1.B of [Sch98], which we refer to for details. All necessary properties are satisfied by standard Lagrange finite element spaces of degree $r-1$ defined on quasi-uniform, but not necessarily structured, grids (i.e., grids with shape-regular elements whose diameters are equivalent).

2.3. **Weighted norms and logarithmic factors.** In our presentation we employ weighted norms and logarithmic factors which depend on a number of parameters. We thus introduce a standard shorthand notation for both concepts which we shall consistently use throughout the paper.

First we define a logarithmic factor. Given $a > 0$, let

$$\ell_a = 1 + |\ln a|.$$

For $s, t \ge 0$, let

$$\ell_{a,s}^t = \begin{cases} 1, & s < t, \\ \ell_a, & s = t. \end{cases}$$

Next we introduce a weight with standardized notation and also state a few of its properties. Let $D \subset \Omega$, and let $a > 0$. Then we define

$$\sigma_{D,a}(y) = \frac{\max(h, a)}{dist(D, y) + \max(h, a)}.$$

The weight $\sigma_{D,a}$ has several important properties. First, for fixed $\lambda \ge 0$ the mapping $a \mapsto \frac{a}{\lambda + a}$ is nondecreasing. In particular, if $0 < a_1 \le a_2$, then

$$\sigma_{D,a_1}(y) \le \sigma_{D,a_2}(y). \tag{2.5}$$

Secondly, $\sigma_{D,a}$ fulfills a multiplicative inequality. If $D \subset \Omega$ and $x, y \in \Omega$, then $\sigma_{D,a}(y)\sigma_{y,a}(x) \le C\sigma_{D,a}(x)$. Combining this estimate with (2.5) leads to the following statement. Assume that $D \subset \Omega$, $x, y \in \Omega$, and $0 < a_1, a_2$. Then

$$\sigma_{D,a_1}(y)\sigma_{y,a_2}(x) \le C\sigma_{D,\max(a_1,a_2)}(x). \tag{2.6}$$

Finally, for fixed $K > 0$,

$$\sigma_{B_{Ka}(x_0),a}(y) \le C(K)\sigma_{x_0,a}(y).$$

The above elementary properties are stated and used in previous works (see e.g. [Sch98], or equation (3.2) of [SW03]), and we do not give proofs.

Finally we define weighted norms and seminorms. For $D, \tilde{\Omega} \subset \Omega$, $a > 0$, and $s \ge 0$, we let

$$|u|_{W_p^k(\tilde{\Omega}),D,a,s} = \sum_{|\alpha|=k} \|\sigma_{D,a}^s D^\alpha u\|_{L_p(\tilde{\Omega})}$$

and

$$\|u\|_{W_p^k(\tilde{\Omega}),D,a,s} = \sum_{0 \le j \le k} |u|_{W_p^j(\tilde{\Omega}),D,a,s}.$$

2.4. **Discrete auxiliary operators.** Assume that $u_h^\theta \in S_h^r$, $0 \le \theta \le 1$. We define a family A_h^θ of auxiliary operators which serve as discrete counterparts to A. For $0 \le i \le n$, let

$$a_{ij,\theta}^h(x) = \int_0^1 F_{ij}(x, u_h^\theta + s(u - u_h^\theta), \nabla u_h^\theta + s\nabla(u - u_h^\theta))\,\mathrm{d}s.$$

We then define

$$A_h^\theta(v, w) = \int_\Omega \sum_{i,j=0}^n a_{ij,\theta}^h v_{x_j} w_{x_i}\,\mathrm{d}x.$$

Note that for $\chi \in S_h^r$,

$$A_h^\theta(u - u_h^\theta, \chi) = \langle Tu - Tu_h^\theta, \chi\rangle. \tag{2.7}$$

Since

$$|F_{ij}(x, u, \nabla u) - a_{ij,\theta}^h(x)| \le C(F, \|u\|_{W_\infty^1(\Omega)}, \|u_h^\theta\|_{W_\infty^1(\Omega)})\|u - u_h^\theta\|_{W_\infty^1(\Omega)},$$

we also have

(2.8)
$$(A - A_h^\theta)(u - u_h^\theta, v) \le C(\{F_i\}, \|u\|_{W_\infty^1(\Omega)}, \|u_h^\theta\|_{W_\infty^1(\Omega)})\|u - u_h^\theta\|_{W_\infty^1(\Omega)}^2 \|v\|_{W_1^1(\Omega)}.$$

2.5. **Pointwise estimates for perturbed forms.** The following lemma will play a fundamental role in our analysis. It is essentially Theorem 2.2 and Theorem 3.2 of [Sch98].

Lemma 2.1. *Let B be an $H^1(\Omega)$-coercive and continuous bilinear form with smooth coefficients and let L be a bounded linear functional on $W_1^1(\Omega)$. Assume that u is sufficiently smooth and that $u_h \in S_h^r$ satisfies*

$$B(u - u_h, \chi) = L(\chi),\ \chi \in S_h^r.$$

Then for $0 \le s \le r - 2$, $0 \le t \le r - 1$, and $x \in \Omega$,

(2.9)
$$\begin{aligned} |(u - u_h)(x)| &\le C(B)[h\ell_{h,s}^{r-2} \inf_{\chi \in S_h^r} \|u - \chi\|_{W_\infty^1(\Omega),x,h,s} + h\ell_{h,s}^{r-2}|||L|||_{-1,x,h,s} \\ &\quad + \ell_h|||L|||_{-2}] \\ &\le C(B)[h\ell_{h,s}^{r-2} \inf_{\chi \in S_h^r} \|u - \chi\|_{W_\infty^1(\Omega),x,h,s} + \ell_h|||L|||_{-1}] \end{aligned}$$

and

$$\|u - u_h\|_{W_\infty^1(B_h(x))} \le C(B)[\ell_{h,t}^{r-1} \inf_{\chi \in S_h^r} \|u - \chi\|_{W_\infty^1(\Omega),x,t} + \ell_h|||L|||_{-1}]. \tag{2.10}$$

Here

$$|||L|||_{-k} = \sup_{\phi \in W_1^k(\Omega), \|\phi\|_{W_1^k(\Omega)}=1} |L(\phi)|$$

and

$$|||L|||_{-1,x,h,s} = \sup_{\phi \in W_1^1(\Omega), \|\phi\|_{W_1^1(\Omega),x,h,-s}=1} |L(\phi)|.$$

2.6. **Pointwise bounds for the Green's function.** We shall need the following pointwise bounds for derivatives of the Green's function; cf. [Kr69] for a proof.

Lemma 2.2. *Let B be an $H^1(\Omega)$-coercive and continuous bilinear form with smooth coefficients, and let $G^x(y)$ be the Green's function with singularity at the point x for the problem $B(v,u) = (f,v)$ for all $v \in H^1(\Omega)$. Then if $x, y \in \overline{\Omega}$,*

$$|D_x^\alpha D_y^\beta G^x(y)| \le C(B)|x-y|^{2-n-|\alpha+\beta|} \text{ for } |\alpha+\beta| > 0.$$

3. Localized pointwise estimates

In this section we state and prove localized pointwise estimates.

3.1. **Statement of results.** In this subsection we state pointwise and pointwise gradient estimates.

Theorem 3.1. *Let the conditions of §2.1 and §2.2 concerning the continuous and discrete problems be met. Also let $0 \le s \le r-2$ and $0 \le t \le r-1$. Then there exists h_0 such that for $h \le h_0$, $u_h \in S_h^r$ satisfying (2.3) exists, and there hold for any $x \in \Omega$,*

$$|(u-u_h)(x)| \le C(u)[h\ell_{h,s}^{r-2} \inf_{\chi \in S_h^r} \|u-\chi\|_{W_\infty^1(\Omega),x,h,s} + \ell_h C_F \inf_{\chi \in S_h^r} \|u-\chi\|^2_{W_\infty^1(\Omega)}] \tag{3.1}$$

and

$$\|u-u_h\|_{W_\infty^1(B_h(x))} \le C(u)[\ell_{h,t}^{r-1} \inf_{\chi \in S_h^r} \|u-\chi\|_{W_\infty^1(\Omega),x,h,t} + \ell_h C_F \inf_{\chi \in S_h^r} \|u-\chi\|^2_{W_\infty^1(\Omega)}]. \tag{3.2}$$

Here h_0, $C(u)$, and C_F in general depend on the continuous solution u.

The following so-called asymptotic error expansion inequalities give additional insight into the behavior of pointwise errors and are also useful for applications.

Corollary 3.2. *Suppose that the conditions of Theorem 3.1 are met, and in addition that the strengthened approximation assumption*

$$\|u-u_h\|_{W_\infty^1(D)} \le Ch^{r-1}|u|_{W_\infty^r(D_{\rho h})} \tag{3.3}$$

holds. Then for integers $\overline{s}, \overline{t}$ with $r+1 \le \overline{s} \le 2r-2$ and $r \le \overline{t} \le 2r-2$ and for any point $x \in \Omega$,

$$\begin{aligned} |(u-u_h)(x)| \le C(u)\ell_{h,\overline{s}}^{2r-2}(h^r \sum_{|\alpha|=r} |D^\alpha u(x)| + \cdots \\ + h^{\overline{s}-1} \sum_{|\alpha|=\overline{s}-1} |D^\alpha u(x)| + h^{\overline{s}} \|u\|_{W_\infty^{\overline{s}}(\Omega)}) \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} |\nabla(u-u_h)(x)| \le \; & C(u)\ell_{h,\overline{t}}^{2r-2}(h^{r-1} \textstyle\sum_{|\alpha|=r} |D^\alpha u(x)| + \cdots \\ & + h^{\overline{t}-1} \textstyle\sum_{|\alpha|=\overline{t}-1} |D^\alpha u(x)| + h^{\overline{t}} \|u\|_{W_\infty^{\overline{t}}(\Omega)}). \end{aligned} \tag{3.5}$$

Here $C(u)$ is a generic constant depending on u and the coefficients F_i which may be different than in Theorem 3.1*.*

Remark 3.3. There are three main differences between the results of Theorem 3.1 above and the corresponding results of Theorem 2.1 and Theorem 3.1 of [Sch98] for linear problems. First, the constant $C(u)$ appearing in our results in general depends on the unknown solution u, whereas the corresponding constant for linear problems depends only on the domain and problem coefficients. Secondly, our results require that $h \le h_0$, with h_0 depending on u. Finally, the term $\ell_h \inf_{\chi \in S_h^r} \|u - \chi\|^2_{W^1_\infty(\Omega)}$ is a result of the nonlinearity. This nonlinear perturbation term has no effect on the form of the resulting error expansion inequalities in Corollary 3.2 since it is of order $\ell_h h^{2r-2}$, which in turn is the highest order which the global term in the expansion inequalities (3.4) and (3.5) may take on.

The only difference between the asymptotic expansion inequalities of Corollary 3.2 and the corresponding results of Theorem 4.1 and Theorem 4.2 of [Sch98] for linear problems is that in the quasilinear case we consider here, the constant $C(u)$ generally depends on the unknown solution u.

3.2. **Proof of Theorem 3.1.**

Lemma 3.4. *Let the assumptions of* §*2.1 and* §*2.2 concerning the continuous and discrete problems be satisfied. Then there exist constants h_0 and $C(u)$, both possibly depending on the continuous solution u, such that for $0 < h \le h_0$*

$$\|u - u_h\|_{W^1_\infty(\Omega)} \le C(u) \inf_{\chi \in S_h^r} \|u - \chi\|_{W^1_\infty(\Omega)}. \tag{3.6}$$

Proof of Lemma 3.4. For $\theta \in [0,1]$ we seek a finite element solution u_h^θ to the deformed problem

$$(1-\theta)A(u - u_h^\theta, v_h) + \theta\langle Tu - Tu_h^\theta, v_h\rangle = 0 \text{ for all } v_h \in S_h^r. \tag{3.7}$$

Using (2.7) and rearranging, we obtain the alternate error equation

$$A(u - u_h^\theta, v_h) = \theta(A - A_h^\theta)(u - u_h^\theta, v_h) \text{ for all } v_h \in S_h^r.$$

Note that u_h^0 solves the *linear* finite element problem $A(u - u_h^0, v_h) = 0$, while $u_h^1 = u_h$ is the fully nonlinear approximation to u which we wish to analyze. We comment that our deformation technique is more similar to that used in [Do80] to treat parabolic problems with quasilinear elliptic part than that used in [Fr78] in that here we only deform the discrete problem and not the continuous problem.

Note that $\theta(A - A_h^\theta)(u - u_h^\theta, v)$ is a linear functional with respect to the argument v, and by (2.8)

$$|||\theta(A - A_h^\theta)(u - u_h, \cdot)|||_{-1} \le \theta C(\{F_i\}, \|u\|_{W^1_\infty(\Omega)}, \|u_h^\theta\|_{W^1_\infty(\Omega)})\|u - u_h^\theta\|^2_{W^1_\infty(\Omega)}. \tag{3.8}$$

Assuming that u_h^θ exists, taking the maximum over $x \in \Omega$ of (2.10) with $t = 0$ then yields

$$\begin{aligned}\|u - u_h^\theta\|_{W^1_\infty(\Omega)} \le{}& \tilde{C}(u) \inf_{\chi \in S_h^r} \|u - \chi\|_{W^1_\infty(\Omega)} \\ &+ \theta \ell_h C(\{F_i\}, \|u\|_{W^1_\infty(\Omega)}, \|u_h^\theta\|_{W^1_\infty(\Omega)})\|u - u_h^\theta\|^2_{W^1_\infty(\Omega)}.\end{aligned} \tag{3.9}$$

Note that $\tilde{C}(u)$ depends only on the coefficients $F_{ij}(x, u, \nabla u)$ and the domain Ω and is thus fixed with respect to θ.

Next we define

$$C_F = C(\{F_i\}, \|u\|_{W^1_\infty(\Omega)}, \|u\|_{W^1_\infty(\Omega)} + 1). \tag{3.10}$$

We let h_0 be small enough that for $h \le h_0$,

$$2\tilde{C}(u) \inf_{\chi \in S_h^r} \|u - \chi\|_{W^1_\infty(\Omega)} < 1 \tag{3.11}$$

and

$$2C_F \tilde{C}(u) \ell_h \inf_{\chi \in S_h^r} \|u - \chi\|_{W^1_\infty(\Omega)} \le C^* < \frac{1}{2}. \tag{3.12}$$

Since u is assumed to be sufficiently smooth, such an h_0 exists by (2.4).

Next we define

$$\begin{aligned} \Xi \;\; = \;\; & \{\theta \in [0,1] : (3.7) \text{ has a unique solution } u_h^\theta \text{ satisfying} \\ & \|u - u_h^\theta\|_{W^1_\infty(\Omega)} < 2\tilde{C}(u) \inf\nolimits_{\chi \in S_h^r} \|u - \chi\|_{W^1_\infty(\Omega)}\}. \end{aligned}$$

We shall prove (3.6) by proving that Ξ is nonempty and both open and closed in the interval $[0,1]$ and is thus equal to $[0,1]$. In the structure of our argument we closely follow [Fr78], p.422.

(1) Ξ *is nonempty.* u_h^0 solves a finite element problem for a coercive linear operator with coefficients $F_{ij}(x, u, \nabla u)$. Existence is thus guaranteed, and from (3.9) we have

$$\|u - u_h^0\|_{W^1_\infty(\Omega)} \le \tilde{C}(u) \inf_{\chi \in S_h^r} \|u - \chi\|_{W^1_\infty(\Omega)}.$$

Thus we have established that $0 \in \Xi$.

(2) Ξ *is open.* If $\theta_0 \in \Xi$, then we obtain the unique solvability of (3.7) on a neighborhood of θ_0 by noting the monotonicity of the associated operator and applying the implicit function theorem. In particular, the uniqueness of solutions u_h^θ to (3.7) follows trivially from (2.2) and the coerciveness of A. To establish solvability in a neighborhood of $\theta_0 \in \Xi$, let $\{\phi_i\}_{i=1,\dots,M}$ be a basis for S_h^r, define $\langle T_\theta u, v\rangle := (1-\theta)A(u,v) + \theta\langle Tu, v\rangle$, and for any $v_h \in S_h^r$ let

$$\Gamma(v_h, \theta) = \begin{bmatrix} \langle T_\theta u - T_\theta v_h, \phi_1 \rangle \\ \cdot \\ \cdot \\ \langle T_\theta u - T_\theta v_h, \phi_M \rangle \end{bmatrix}. \tag{3.13}$$

We then have by assumption that $\Gamma(u_h^{\theta_0}, \theta_0) = 0$. Given $v_h \in S_h^r$, let V_h be the associated vector of coefficients. Then for any $v_h \in S_h^r$ with $\|v_h\|_{H^1(\Omega)} = 1$, the monotonicity of T_{θ_0} implies that

$$\begin{aligned} C_{u_h^{\theta_0}} \|v_h\|^2_{H^1(\Omega)} & \le \lim_{\epsilon \to 0} \tfrac{\langle T_{\theta_0}(u_h^{\theta_0} + \epsilon v_h) - T_{\theta_0} u_h^{\theta_0}, v_h \rangle}{\epsilon} \\ & = -\tfrac{\partial}{\partial v_h} \langle T_{\theta_0} u_h^{\theta_0}, v_h \rangle \\ & = -(\nabla_{v_h} \Gamma(u_h^{\theta_0}, \theta_0) V_h, V_h). \end{aligned} \tag{3.14}$$

Here $\nabla v_h \Gamma$ represents the gradient of Γ with respect to its first argument. Thus $\nabla_{v_h} \Gamma(u_h^{\theta_0}, \theta_0)$ is negative definite, and is in particular nonsingular. The implicit function theorem then guarantees the existence of u_h^θ and also its continuous dependence on θ in a neighborhood of θ_0, so that the inequality $\|u - u_h^\theta\|_{W^1_\infty(\Omega)} < 2\tilde{C}(u) \inf_{\chi \in S_h^r} \|u - \chi\|_{W^1_\infty(\Omega)}$ is likewise guaranteed in a neighborhood of θ_0.

(3) Ξ *is closed.* Assume $\theta_j \in \Xi$ with $\theta_j \to \theta$. First note that $\theta_j \in \Xi$ and (3.11) imply that for $h \le h_0$,

$$\|u_h^{\theta_j}\|_{W^1_\infty(\Omega)} \le \|u\|_{W^1_\infty(\Omega)} + \|u - u_h^{\theta_j}\|_{W^1_\infty(\Omega)} < \|u\|_{W^1_\infty(\Omega)} + 1.$$

Thus $\{u_h^{\theta_j}\}$ is bounded in S_h^r and has a cluster point u_h^θ which uniquely solves (3.7). By the continuous dependence of u_h^θ on θ, we also have $\|u_h^\theta\|_{W^1_\infty(\Omega)} \le \|u\|_{W^1_\infty(\Omega)} + 1$ and

$$\|u - u_h^\theta\|_{W^1_\infty(\Omega)} \le 2\tilde{C}(u) \inf_{\chi \in S_h^r} \|u - \chi\|_{W^1_\infty(\Omega)}.$$

(3.9), (3.10), and (3.12) then combine to give

$$\begin{aligned}\|u - u_h^\theta\|_{W^1_\infty(\Omega)} &\le \tilde{C}(u) \inf_{\chi \in S_h^r} \|u - \chi\|_{W^1_\infty(\Omega)} + \theta \ell_h C_F \|u - u_h^\theta\|^2_{W^1_\infty(\Omega)} \\ &\le \tilde{C}(u) \inf_{\chi \in S_h^r} \|u - \chi\|_{W^1_\infty(\Omega)} \\ &\quad + 2\ell_h C_F \tilde{C}(u) \inf_{\chi \in S_h^r} \|u - \chi\|_{W^1_\infty(\Omega)} \|u - u_h^\theta\|_{W^1_\infty(\Omega)} \\ &\le \tilde{C}(u) \inf_{\chi \in S_h^r} \|u - \chi\|_{W^1_\infty(\Omega)} + C^* \|u - u_h^\theta\|_{W^1_\infty(\Omega)} \\ &< 2\tilde{C}(u) \inf_{\chi \in S_h^r} \|u - \chi\|_{W^1_\infty(\Omega)}.\end{aligned}$$

Thus Ξ is closed, which completes the proof of the lemma with $C(u) = 2\tilde{C}(u)$.

In order to complete the proof of Theorem 3.1, we shall apply Lemma 2.1 with

$$L(\chi) = (A - A_h^1)(u - u_h^1, \chi).$$

Recalling that $u_h^1 = u_h$, we apply (3.8), (3.10), and (3.11) and finally (3.6) to find that

$$|||L|||_{-1} \le C_F \|u - u_h\|^2_{W^1_\infty(\Omega)} \le C(u)^2 C_F \inf_{\chi \in S_h^r} \|u - \chi\|^2_{W^1_\infty(\Omega)}. \tag{3.15}$$

Combining (3.15) with (2.9) and (2.10) yields (3.1) and (3.2), respectively, with a different constant $C(u)$.

The proof of Corollary 3.2 follows essentially as in §4 of [Sch98]. Indeed, the initial weighted norm terms on the right hand sides of (3.1) and (3.2) may be treated exactly as in the linear case. In order to treat the perturbation term $\ell_h \inf_{\chi \in S_h^r} \|u - \chi\|^2_{W^1_\infty(\Omega)}$ in both (3.1) and (3.2), we note that

$$\inf_{\chi \in S_h^r} \|u - \chi\|^2_{W^1_\infty(\Omega)} \le C h^{2r-2} |u|^2_{W^r_\infty(\Omega)}.$$

We may then bound $C(u)\ell_h C_F \inf_{\chi \in S_h^r} \|u - \chi\|^2_{W^1_\infty(\Omega)}$ by $C(u)\ell_{h,\bar{s}}^{2r-2} h^{\bar{s}} \|u\|_{W^{\bar{s}}_\infty(\Omega)}$ in order to complete the proof of (3.4), or by $C(u)\ell_{h,\bar{t}}^{2r-2} h^{\bar{t}} \|u\|_{W^{\bar{t}}_\infty(\Omega)}$ in order to complete the proof of (3.5). □

4. Localized estimates for differences in errors

While it has been established that pointwise errors in the function values in piecewise linear finite element approximations do not possess localized behavior (cf. [De04a]), the *differences* in pointwise errors do exhibit localized behavior. The results we present here are an extension of those given in [SW03] for linear problems, and we shall refer to this paper several times in the course of our proof. A side product of our analysis is an estimate for perturbed bilinear forms, which were not considered in [SW03].

4.1. **Statement of results.** Before stating our results, we describe the setting. Let $x_1, x_2 \in \Omega$, and let $\overline{x} = \frac{x_1+x_2}{2}$. We next set $\rho = |x_1 - x_2|$ and remind the reader of the definitions of the logarithmic factors ℓ_ρ and $\ell_{h/\rho}$, the weight $\sigma_{\overline{x},\rho}$, and the weighted norm $\|\cdot\|_{W^1_\infty(\Omega),\overline{x},\rho,s}$ from §2.3. Also let $e(x) = (u - u_h)(x)$. We finally emphasize that in this section we shall assume that u_h is piecewise linear, that is, that $u_h \in S^2_h$.

Theorem 4.1. *Let the conditions of §2.1 and §2.2 concerning the continuous and discrete problems be met, with $u_h \in S^2_h$. Then there exist h_0 and $C(u)$, both possibly depending on u, and $k > 0$ such that whenever $h \le h_0$:*

(1) *If $\rho \le kh$ and $0 \le s \le 1$,*

$$|e(x_2) - e(x_1)| \le C(u)\rho[\ell^1_{h,s} \inf_{\chi \in S^2_h} \|u - \chi\|_{W^1_\infty(\Omega),\overline{x},\rho,s} + \ell_h \inf_{\chi\in S^2_h} \|u-\chi\|^2_{W^1_\infty(\Omega)}]. \tag{4.1}$$

(2) *If $\rho \ge kh$ and $0 \le s < 1$,*

$$|e(x_2) - e(x_1)| \le C(u)[h\ell_{h/\rho} \inf_{\chi \in S^2_h} \|u - \chi\|_{W^1_\infty(\Omega),\overline{x},\rho,s} + \rho\ell_\rho \inf_{\chi\in S^2_h} \|u-\chi\|^2_{W^1_\infty(\Omega)}]. \tag{4.2}$$

As in the previous section, we also state asymptotic error expansion inequalities.

Corollary 4.2. *Let the conditions of Theorem* 4.1 *be met, and in addition let the strengthened approximation assumption* (3.3) *hold. Let also $\epsilon > 0$ be arbitrary but fixed. Then:*

(1) *If $\rho \le kh$, then*

$$|e(x_2) - e(x_1)| \le C(u)\rho h(\sum_{|\alpha|=2} |D^\alpha(\overline{x})| + h^{1-\epsilon}\|u\|_{W^3_\infty(\Omega)}).$$

(2) *If $\rho \ge kh$, then*

$$|e(x_2) - e(x_1)| \le C(u)h^2\ell_{h/\rho}(\sum_{|\alpha|=2} |D^\alpha u(\overline{x})| + \rho^{1-\epsilon}\|u\|_{W^3_\infty(\Omega)}).$$

Here $C(u)$ may depend on u, the coefficients F_i, and ϵ.

4.2. **Proof.** The results for pointwise differences of the finite element error given in [SW03] do not account for perturbations of the bilinear form. Instead of directly proving results for nonlinear equations, we shall thus first state and prove a result for perturbed bilinear forms which is analogous to Lemma 2.1. Theorem 4.1 will follow directly from this lemma. Our proof cites that of [SW03] at several points, so a familiarity with that work would be helpful for the reader in following the development in this section.

Lemma 4.3. *Let B be an $H^1(\Omega)$-coercive, elliptic, and continuous bilinear form with smooth coefficients and let L be a linear functional. Let u be sufficiently smooth and let $u_h \in S^2_h$ satisfy*

$$B(u - u_h, \chi) = L(\chi), \ \chi \in S^2_h. \tag{4.3}$$

Then

(1) *If* $\rho \le kh$ *and* $0 \le s \le 1$,

$$|e(x_2) - e(x_1)| \le C(B)\rho(\ell_{h,s}^1 \inf_{\chi \in S_h^2} \|u-\chi\|_{W_\infty^1(\Omega),\overline{x},\rho,s} + \ell_h |||L|||_{-1}). \tag{4.4}$$

(2) *If* $\rho \ge kh$ *and* $0 \le s < 1$,

$$|e(x_2) - e(x_1)| \le C(B)(h\ell_{h/\rho} \inf_{\chi \in S_h^2} \|u-\chi\|_{W_\infty^1(\Omega),\overline{x},\rho,s} + \rho\ell_\rho |||L|||_{-1}). \tag{4.5}$$

Here $\ell_{h,s}$ *is as in Theorem* 4.1.

Proof of Lemma 4.3. In order to prove (4.4), we note that there exists $\hat{x}$ on the line joining x_1 and x_2 such that

$$|e(x_2) - e(x_1)| \le \rho|\nabla e(\hat{x})| \le C(B)\rho[\ell_{h,s}^1 \inf_{\chi \in S_h^2} \|u-\chi\|_{W_\infty^1(\Omega),\hat{x},h,s} + \ell_h |||L|||_{-1}].$$

Here we have employed (2.10) with $r = 2$. Proceeding as in equations (3.1) and following of [SW03] in order to obtain the bound $\inf_{\chi \in S_h^2} \|u-\chi\|_{W_\infty^1(\Omega),\hat{x},h,s} \le 2\inf_{\chi \in S_h^2} \|u-\chi\|_{W_\infty^1(\Omega),\overline{x},\rho,s}$ completes the proof of (4.4).

In order to prove (4.5), we let G^x be the Green's function for B with singularity at the point x. Then in light of (4.3), there holds for any $\psi \in S_h^2$

$$\begin{aligned} |e(x_2) - e(x_1)| &= |B(e, G^{x_2} - G^{x_1})| \\ &= |B(e, G^{x_2} - G^{x_1} - \psi) + L(\psi)| \\ &\le C(B)\|e\|_{W_\infty^1(\Omega),\overline{x},\rho,s}\|G^{x_2} - G^{x_1} - \psi\|_{W_1^1(\Omega),\overline{x},\rho,-s} \\ &\quad + |||L|||_{-1}\|\psi\|_{W_1^1(\Omega)}. \end{aligned} \tag{4.6}$$

Recalling (2.10), we first compute that for some $z \in \Omega$,

$$\begin{aligned} \|e\|_{W_\infty^1(\Omega),\overline{x},\rho,s} &\le \sigma_{\overline{x},\rho}^s(z)(|e(z)| + |\nabla e(z)|) \\ &\le C(B)\sigma_{\overline{x},\rho}^s(z)[\inf_{\chi \in S_h^2} \|u-\chi\|_{W_\infty^1(\omega),z,h,s} + \ell_h |||L|||_{-1}]. \end{aligned}$$

It is proven in §3 of [SW03] that for appropriately chosen $\psi \in S_h^2$,

$$\|G^{x_2} - G^{x_1} - \psi\|_{W_1^1(\Omega),\overline{x},-s} \le C(B)h\ell_{h/\rho}.$$

In addition, it easily follows from (2.6) that

$$\sigma_{\overline{x}}^s(z) \inf_{\chi \in S_h^2} \|u-\chi\|_{W_\infty^1(\Omega),z,h,s} \le C \inf_{\chi \in S_h^2} \|u-\chi\|_{W_\infty^1(\Omega),\overline{x},\rho,s}.$$

Noting that $h\ell_{h/\rho}\ell_h \le C\rho\ell_\rho$ while combining the above three inequalities completes the bound of the term in the third line of (4.6).

It thus remains to bound $\|\psi\|_{W_1^1(\Omega)}$. We employ a W_1^1-stable interpolant such as the Scott-Zhang operator so that $\|\psi\|_{W_1^1(\Omega)} \le \|G^{x_2} - G^{x_1}\|_{W_1^1(\Omega)}$. Using Lemma 2.2, we then compute

$$\begin{aligned} \|G^{x_2} - G^{x_1}\|_{W_1^1(B_{3\rho}(\overline{x}))} &\le \|G^{x_2}\|_{W_1^1(B_{3\rho}(\overline{x}))} + \|G^{x_1}\|_{W_1^1(B_{3\rho}(\overline{x}))} \\ &\le C(B)\int_0^{3\rho} r^{1-n}r^{n-1}\,\mathrm{d}r \le C(B)\rho. \end{aligned} \tag{4.7}$$

Next we note that for $y \in \Omega \setminus B_{3\rho}(\overline{x})$, there is a point $\hat{x}$ on the line joining x_1 and x_2 such that

$$|\nabla_y G^{x_2}(y) - \nabla_y G^{x_1}(y)| \le \rho|\nabla_x\nabla_y G^{\hat{x}}(y)| \le C\rho|y-\overline{x}|^{-n}.$$

Thus

$$\begin{aligned} \|G^{x_2} - G^{x_1}\|_{W_1^1(\Omega \setminus B_{3\rho}(\overline{x}))} &\le C(B)\rho\int_{3\rho}^{diam(\Omega)} r^{-n}r^{n-1}\,\mathrm{d}r \\ &\le C(B)\rho\ell_\rho. \end{aligned} \tag{4.8}$$

Combining (4.7) and (4.8) yields the desired bound for $\|G^{x_2} - G^{x_1}\|_{W_1^1(\Omega)}$.

In order to complete the proof of Theorem 4.1, we apply Lemma 3.4 and Lemma 4.3 with $L(\chi) = (A - A_h^1)(u - u_h, \chi)$ as in the proof of Theorem 3.1. Corollary 4.2 is proved by bounding the weighted norm terms on the right hand sides of (4.1) and (4.2) as in the proof of Corollary 1 of [SW03] and then absorbing the remaining perturbation terms into the resulting expansion inequalities as in the proof of Corollary 3.2 above. □

5. Localized W_∞^{-1} estimates

In this section we shall prove error estimates in the norm W_∞^{-1}. We remark that these estimates are new for linear as well as for nonlinear problems.

5.1. **Statement of results.** Before stating results, we make some definitions. We shall assume that $D \subset \Omega$ is a subdomain having diameter $H \geq h$. We define

$$\|u\|_{W_\infty^{-1}(D)} = \sup_{v \in \overset{\circ}{W}{}_1^1(D), \|v\|_{W_1^1(D)}=1} (u, v). \tag{5.1}$$

Also, recall the definitions of the logarithmic factor ℓ_H, the weight $\sigma_{D,H}$, and the weighted norm $\|\cdot\|_{W_p^k(\Omega),D,H,s}$ from §2.3.

We shall prove the following theorem.

Theorem 5.1. *Let D be a subdomain of Ω having diameter $H \geq h$. Also let the conditions of §2.1 and §2.2 concerning the continuous and discrete problems be met. Then there exists h_0 such that if $h \leq h_0$, there holds for $r = 2$*

$$\|u - u_h\|_{W_\infty^{-1}(D)} \leq C(u) H \ell_h [h \inf_{\chi \in S_h^r} \|u - \chi\|_{W_\infty^1(\Omega)} + C_F \inf_{\chi \in S_h^r} \|u - \chi\|_{W_\infty^1(\Omega)}^2].$$

For $r \geq 3$ and $0 \leq s \leq r - 2$,

$$\begin{aligned} \|u - u_h\|_{W_\infty^{-1}(D)} \leq C_\delta(u)((\frac{H}{h})^\delta + \ell_{h,s}^{r-2})[h^2 \inf_{\chi \in S_h^r} \|u - \chi\|_{W_\infty^1(\Omega),D,H,s} \\ + C_F H \ell_H \inf_{\chi \in S_h^r} \|u - \chi\|_{W_\infty^1(\Omega)}^2]. \end{aligned} \tag{5.2}$$

Here $\delta > 0$ is arbitrary but fixed and $C_\delta(u)$ depends on δ and in general on u. For linear problems $C_F = 0$, and $C(u)$ and $C_\delta(u)$ no longer depend on u.

Remark 5.2. The estimate (5.2) is suboptimal in two ways. First, when $D = \Omega$ (so that $H \approx 1$) the factor $(\frac{H}{h})^\delta$ yields an overall convergence rate of $h^{r+1-\delta}$, modulo logarithmic factors. The loss of the exponent δ in the convergence rate is due to the use of regularity estimates in $W_p^2(\Omega)$ and $W_p^3(\Omega)$ for the solutions of a dual problem. This loss could likely be reduced to a logarithmic factor if an optimal dependence of these regularity estimates on p could be established (we note that this dependence is well-known in the case of W_p^2, but we are not aware of a proof in the literature). Secondly, the quadratic term $\inf_{\chi \in S_h^r} \|u - \chi\|_{W_\infty^1(\Omega)}^2$ has two logarithmic factors multiplying it. This stems from the fact that we employ the estimates of Lemma 2.1. We could improve (5.2) by instead essentially redoing these estimates, which would result in a longer proof but only minor improvement in (5.2) and no improvement in the error expansion inequality of Corollary 5.3.

As in the previous sections, we shall also prove an asymptotic error expansion inequality which is convenient for applications. The estimate we state here does not employ the highest possible powers of the weight σ_D especially when $r \geq 4$, but is sufficient for our intended application.

Corollary 5.3. *Assume that the conditions of Theorem* 5.1 *and the strengthened approximation assumption* (3.3) *hold. Also let* $\epsilon > 0$ *be arbitrary but fixed. Then for any point* $x \subset D$ *and for* $r \geq 3$,

$$\|u-u_h\|_{W_\infty^{-1}(D)} \leq C_\delta(u) h^{r+1} \left(\frac{H}{h}\right)^\delta [\sum_{|\alpha|=r} |D^\alpha u(x)| + H^{1-\epsilon}\|u\|_{W_\infty^{r+1}(\Omega)}].$$

Here $C_\delta(u)$ *depends on* ϵ *as well as on* δ *and* u.

5.2. **Proof of W_∞^{-1} estimates.** As in the previous sections, we shall first establish an estimate for a perturbed bilinear form.

Lemma 5.4. *Let* B *be an* $H^1(\Omega)$*-coercive, elliptic, and continuous bilinear form with smooth coefficients and let* L *be a linear functional. Also let* D *be a subdomain of* Ω *with diameter* $H \geq h$. *Finally, let* u *be sufficiently smooth, and let* $u_h \in S_h^r$ *satisfy*

$$B(u-u_h, \chi) = L(\chi),\ \chi \in S_h^r.$$

Then for $r = 2$,

$$\|u-u_h\|_{W_\infty^{-1}(D)} \leq C(B) H \ell_h [h \inf_{\chi \in S_h^r} \|u-\chi\|_{W_\infty^1(\Omega)} + h|||L|||_{-1} + |||L|||_{-2}]. \tag{5.3}$$

For $r \geq 3$ *and* $0 \leq s \leq r-2$,

$$\begin{aligned}\|u-u_h\|_{W_\infty^{-1}(D)} \leq C_\delta(B)((\frac{H}{h})^\delta + \ell_{h,s}^{r-2})[h^2 \inf_{\chi \in S_h^r} \|u-\chi\|_{W_\infty^1(\Omega),D,H,s} \\ + h^2 \ell_h |||L|||_{-1} + H\ell_H |||L|||_{-2}].\end{aligned} \tag{5.4}$$

Here $\delta > 0$ *is arbitrary but fixed,* $C(B)$ *depends on the coefficients of* B, *and* $C_\delta(B)$ *depends on* δ *and the coefficients of* B.

Proof of Lemma 5.4. In order to prove (5.3), we recall the definition (5.1) and use a Poincaré inequality to find that for some $v \in \overset{\circ}{W}{}_1^1(D)$ with $\|v\|_{W_1^1(D)} = 1$

$$\begin{aligned}\|u-u_h\|_{W_\infty^{-1}(D)} &\leq (u-u_h, v) \\ &\leq \|u-u_h\|_{L_\infty(D)} \|v\|_{L_1(D)} \leq CH\|u-u_h\|_{L_\infty(\Omega)}.\end{aligned}$$

Applying (2.9) with $s = 0$ completes the proof of (5.3).

We begin the proof of (5.4) by letting I_h be a local, W_1^1-stable interpolant such as the Scott-Zhang operator. Recalling the definition (5.1), we let $v \in \overset{\circ}{W}{}_1^1(D)$ with $\|v\|_{W_1^1(D)} = 1$. Next we define $D_{KH} = \{x \in \Omega : dist(x, D) < KH\}$ for $K > 0$ and employ the local nature of I_h, the assumption $H \geq h$, and the approximation assumption (2.4) to compute

$$\begin{aligned}|(u-u_h, v)| &\leq |(u-u_h, v - I_h v)| + |(u-u_h, I_h v)| \\ &\leq C\|u-u_h\|_{L_\infty(D_{\gamma H})} \|v - I_h v\|_{L_1(D_{\gamma H})} + |(u-u_h, I_h v)| \\ &\leq Ch\|u-u_h\|_{L_\infty(D_{\gamma H})} + |(u-u_h, I_h v)|.\end{aligned} \tag{5.5}$$

Using (2.9) while taking a maximum over $x \in D_{\gamma H}$ and noting that $\sigma_{D_{\gamma H},h} \leq C\sigma_{D,H}$, we find that

$$\begin{aligned}h\|u-u_h\|_{L_\infty(D_{\gamma H})} &\leq C(B)[h^2 \ell_{h,s}^{r-2} \inf_{\chi \in S_h^r} \|u-\chi\|_{W_\infty^1(\Omega),D_{\gamma H},h,s} \\ &\quad + h^2 \ell_{h,s}^{r-2} |||L|||_{-1,D_{\gamma H},h,s} + H\ell_H |||L|||_{-2}] \\ &\leq C(B)[h^2 \ell_{h,s}^{r-2} \inf_{\chi \in S_h^r} \|u-\chi\|_{W_\infty^1(\Omega),D,H,s} \\ &\quad + h^2 \ell_h |||L|||_{-1} + h\ell_h |||L|||_{-2}].\end{aligned} \tag{5.6}$$

Noting that $h\ell_h \leq CH\ell_H$ for $H \geq h$, the term $Ch\|u-u_h\|_{L_\infty(D_{\gamma H})}$ in (5.5) is thus bounded as desired.

In order to bound the term $|(u-u_h, I_h v)|$ from (5.5), we let $z \in H^1(\Omega)$ solve

$$B(w,z) = (w, I_h v),\ \ w \in H^1(\Omega).$$

Then

$$(u-u_h, I_h v) = B(u-u_h, z) = B(u-u_h, z - I_h z) + L(I_h z). \tag{5.7}$$

With K sufficiently large, we compute

$$\begin{aligned} B(u - u_h, z - I_h z) &\leq \|u-u_h\|_{W^1_\infty(D_{KH})}\|z - I_h z\|_{W^1_1(D_{KH})} \\ &\quad + \|u-u_h\|_{W^1_\infty(\Omega\setminus D_{KH}),D,H,s}\|z - I_h z\|_{W^1_1(\Omega\setminus D_{KH}),D,H,-s} \\ &\leq Ch^2\|u-u_h\|_{W^1_\infty(D_{KH})}\|z\|_{W^3_1(D_{(K+\gamma)H})} \\ &\quad + Ch^{r-1}\|u-u_h\|_{W^1_\infty(\Omega\setminus D_{KH}),D,H,s}\|z\|_{W^r_1(\Omega\setminus D_{(K-\gamma)H}),D,H,-s}. \end{aligned} \tag{5.8}$$

Next we note that for $p > 1$ and $k = 0, 1$ (cf. [ADN59])

$$\|z\|_{W^{k+2}_p(\Omega)} \leq C_p(A)\|I_h v\|_{W^1_k(\Omega)}. \tag{5.9}$$

Employing (5.9) with $k = 1$, a standard inverse inequality, and the W^1_1 stability of I_h yields

$$\begin{aligned} \|z\|_{W^3_1(D_{2KH})} &\leq CH^{n(1-1/p)}\|z\|_{W^3_p(\Omega)} \\ &\leq C_p(A)H^{n(1-1/p)}\|I_h v\|_{W^1_p(\Omega)} \\ &\leq C_p(A)(\tfrac{H}{h})^{n(1-1/p)}\|I_h v\|_{W^1_1(\Omega)} \\ &\leq C_p(A)(\tfrac{H}{h})^{n(1-1/p)}. \end{aligned}$$

Choosing p so that $n(1-1/p) = \delta$ then yields

$$\|z\|_{W^3_1(D_{2KH})} \leq C_\delta(B)(\frac{H}{h})^\delta. \tag{5.10}$$

In order to bound $\|z\|_{W^r_1(\Omega\setminus D_{(K-\gamma)H}),D,H,-s}$, we use Lemma 2.2, the local nature of the operator I_h, the fact that we have assumed K is large enough, and a Poincaré inequality to find that for $x \in \Omega \setminus D_{(K-\gamma)H}$

$$\begin{aligned} D^r z(x) &= \textstyle\int_{supp(I_h v)} D^r G^x(y) I_h v(y)\,\mathrm{d}y \\ &\leq C(B)\sup_{y\in supp(I_h v)} dist(x,y)^{2-n-r}\|I_h v\|_{L_1(\Omega)} \\ &\leq C(B) dist(x,D)^{2-n-r}(\|I_h v - v\|_{L_1(\Omega)} + \|v\|_{L_1(D)}) \\ &\leq C(B)(h+H)\|v\|_{W^1_1(D)} dist(x,D)^{2-n-r} \\ &\leq C(B)H dist(x,D)^{2-n-r}. \end{aligned} \tag{5.11}$$

Thus for some $c > 0$,

$$\begin{aligned} \|z\|_{W^r_1(\Omega\setminus D_{(K-\gamma)H}),D,H,-s} &\leq C(B)H^{1-s}\textstyle\int_{\Omega\setminus D_{(K-\gamma)H}} dist(x,D)^{2-n-r+s}\,\mathrm{d}x \\ &\leq C(B)H^{1-s}\textstyle\int_{cH}^{diam(\Omega)} t^{2-n-r+s+n-1}\,\mathrm{d}t \\ &\leq C(B)H^{1-s}H^{2-r+s}\ell^{r-2}_{H,s} \leq CH^{3-r}\ell^{r-2}_{H,s}. \end{aligned} \tag{5.12}$$

Inserting (5.10) and (5.12) into (5.8) yields

$$\begin{aligned} B(u-u_h, z - I_h z) &\leq C_\delta(B)h^2[(\tfrac{H}{h})^\delta\|u-u_h\|_{W^1_\infty(D_{KH})} \\ &\quad + (\tfrac{h}{H})^{r-3}\ell^{r-2}_{H,s}\|u-u_h\|_{W^1_\infty(\Omega\setminus D_{KH}),D,H,s}] \\ &\leq C_\delta(B)h^2(\ell^{r-2}_{h,s} + (\tfrac{H}{h})^\delta)\|u-u_h\|_{W^1_\infty(\Omega),D,H,s}. \end{aligned}$$

Employing (2.10) while noting that $s < r-1$ and also using (2.6) then yields that for some $y \in \Omega$,

$$\begin{aligned}\|u \;\; - \;\; & u_h\|_{W^1_\infty(\Omega),D,H,s} \le \sigma_{D,H}(y)\|u-u_h\|_{W^1_\infty(B_h(y))} \\ \le \;\; & C(B)[\sigma_{D,H}(y)[\inf_{\chi\in S_h^r}\|u-\chi\|_{W^1_\infty(\Omega),y,h,s} + \ell_h|||L|||_{-1}]] \\ \le \;\; & C(B)[\inf_{\chi\in S_h^r}\|u-\chi\|_{W^1_\infty(\Omega),D,H,s} + \ell_h|||L|||_{-1}].\end{aligned}$$

Thus

$$\begin{aligned}B(u-u_h, z-I_hz) \;\; \le \;\; & C_\delta(B)h^2(\ell_{h,s}^{r-2} + (\tfrac{H}{h})^\delta) \\ & \cdot[\inf_{\chi\in S_h^r}\|u-\chi\|_{W^1_\infty(\Omega),D,H,s} + \ell_h|||L|||_{-1}].\end{aligned} \tag{5.13}$$

We finally turn to bounding the term $L(I_hz)$ from (5.7). We compute

$$\begin{aligned}L(I_hz) \;\; = \;\; & L(I_hz - z) + L(z) \\ \le \;\; & |||L|||_{-1}\|I_hz - z\|_{W^1_1(\Omega)} + |||L|||_{-2}\|z\|_{W^2_1(\Omega)}.\end{aligned}$$

Combining approximation estimates with (5.10) and (5.12) (with $s = 0$ in (5.12)) yields

$$\|I_hz - z\|_{W^1_1(\Omega)} \le Ch^2(\frac{H}{h})^\delta. \tag{5.14}$$

Next we again employ (5.9) while recalling that $h \le H$, and use a Poincaré inequality to find that

$$\begin{aligned}\|z\|_{W^2_1(D_{KH})} \;\; \le \;\; & CH^{n(1-1/p)}\|z\|_{W^2_p(D_{KH})} \\ \le \;\; & C_p(A)H^{n(1-1/p)}\|I_hv\|_{L_1(\Omega)} \\ \le \;\; & C_p(A)(\tfrac{H}{h})^{n(1-1/p)}(\|I_hv - v\|_{L_1(\Omega)} + \|v\|_{L_1(D_{KH})}) \\ \le \;\; & C_p(A)H(\tfrac{H}{h})^{n(1-1/p)}.\end{aligned}$$

Setting $\delta = n(1-1/p)$ yields

$$\|z\|_{W^2_1(D_{KH})} \le C_\delta(B)H(\frac{H}{h})^{n(1-1/p)}. \tag{5.15}$$

Finally, computing as in (5.11) yields that for $x \in \Omega \setminus D_{KH}$

$$|D^2z(x)| \le C(B)dist(x,D)^{-n}.$$

Thus

$$\|z\|_{W^2_1(\Omega\setminus D_{KH})} \le C(B)H\int_{cH}^{diam(\Omega)} t^{-1}\,\mathrm{d}t \le CH\ell_H. \tag{5.16}$$

Combining (5.15) and (5.16), we obtain

$$\|z\|_{W^2_1(\Omega)} \le C_\delta(B)H(\ell_H + (\frac{H}{h})^\delta),$$

so that

$$L(I_hz) \le C_\delta(B)(h^2(\frac{H}{h})^\delta|||L|||_{-1} + H(\ell_H + (\frac{H}{h})^\delta)|||L|||_{-2}). \tag{5.17}$$

Inserting (5.13) and (5.17) into (5.7) and combining the resulting inequality with (5.6) and (5.5) completes the proof of (5.4).

In order to complete the proof of Theorem 5.1, we note that $h^2\ell_h|||L|||_{-1} + H\ell_H|||L|||_{-2} \le H\ell_H|||L|||_{-1}$ and then apply (5.4) with $B = A$ and $L(\chi) = (A - A_h^1)(u - u_h, \chi)$. The proof of Corollary 5.3 follows in a fashion similar to that of Corollary 4.2 and Corollary 3.2. □

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