AN EXPONENTIALLY CONVERGENT ALGORITHM
FOR NONLINEAR DIFFERENTIAL EQUATIONS
IN BANACH SPACES

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ABSTRACT. An exponentially convergent approximation to the solution of a
nonlinear first order differential equation with an operator coefficient in Ba-
nach space is proposed. The algorithm is based on an equivalent Volterra
integral equation including the operator exponential generated by the opera-
tor coefficient. The operator exponential is represented by a Dunford-Cauchy
integral along a hyperbola enveloping the spectrum of the operator coefficient,
and then the integrals involved are approximated using the Chebyshev inter-
polation and an appropriate Sinc quadrature. Numerical examples are given
which confirm theoretical results.

1. Introduction

We consider the problem

$$(1.1) \qquad \begin{aligned} \frac{\partial u(t)}{\partial t} + Au(t) &= f(t, u(t)), \quad t \in (0, 1], \\ u(0) &= u_0, \end{aligned}$$

where $u(t)$ is an unknown vector valued function with values in a Banach space X,
$u_0 \in X$ is a given vector, $f(t, u) : (\mathbb{R}_+ \times X) \to X$ is a given function (nonlinear
operator) and A is a linear densely defined closed operator with the domain $D(A)$
acting in X. The abstract setting (1.1) covers many applied problems such as non-
linear heat conduction or diffusion in porous media, the flow of electrons and holes
in semiconductors, nerve axon equations, chemically reacting systems, equations of
the population genetics theory, dynamics of nuclear reactors, Navier-Stokes equa-
tions of the viscous flow, etc. (see e.g. [26] and the references therein). This fact
together with theoretical interest are important reasons to study efficient discrete
approximations of problem (1.1).

Given a discretization parameter N we are interesting in approximations pos-
sessing an exponential convergence rate with respect to $N \to \infty$ which for a given
tolerance ε provide algorithms of optimal or low complexity [14, 13]. Exponentially
convergent algorithms were proposed recently for various linear problems.

Such algorithms for linear homogeneous parabolic problems of the type (1.1)
were proposed probably first in [17, 19] and later in [11, 14, 20, 35, 40]. These
algorithms are based on a representation of the operator exponential $T(t) = e^{-At}$

by the improper Dunford-Cauchy integral along a path enveloping the spectrum of A where a hyperbola containing the spectrum of A or a parabola as the integration path were used. The methods from [20, 17, 11, 23] use Sinc-quadratures [32, 37] and possess an exponential convergence rate. An exponential convergence rate for all $t \geq 0$ was proven in [12, 41] under assumptions that the initial function u_0 belongs to the domain of $D(A^\sigma)$ for some $\sigma > 1$, where the preliminary computation of $A^\sigma u_0$ is needed. Note that all these algorithms cannot be directly applied to inhomogeneous problems due to the inefficiency of computation of the operator exponential at $t = 0$. In [20] a hyperbola as the integration path and a proper modification of the resolvent were used which allows one to get the uniform and numerically stable exponential convergence rate with respect to $t \geq 0$ without preliminary computation of $A^\sigma u_0$. An exponentially convergent algorithm for the case of an operator family $A(t)$ depending on the parameter t was proposed in [16]. This algorithm uses an exponentially convergent algorithm for the operator exponential generating by a constant operator. Moreover, these algorithms inherit two levels of parallelism (with respect to various time points and with respect to the treatment of the summands in the quadrature sum) which was perhaps first observed in the paper [34] and independently for the exponentially convergent algorithms in [17, 19, 11]. A parallel method for the numerical solution of an integro-differential equation with positive memory was described in [28].

In [14, 12, 13, 10] exponentially convergent algorithms based on the Dunford-Cauchy integral representation and on the Sinc-quadratures were proposed for parabolic and elliptic solution operators, for the Silvester operator equation, for the inverse of an elliptic operator and other operator-valued functions. Moreover, the paper [13] combines exponentially convergent algorithms with tensor product approximations in order to obtain algorithms of almost linear complexity (with respect to the discretization parameter in one dimension only) for high dimensional problems. The paper [6] deals with exponentially convergent algorithms for parabolic PDE's based on the Runge-Kutta methods. In [29, 30] exponentially convergent algorithms for inverting Laplace transforms were proposed and justified. The paper [4] deals with exponentially convergent algorithms for the first order differential equations with an operator coefficient possessing a variable domain. But the exponentially convergent methods for nonlinear parabolic problems in an abstract framework are missing in the literature.

In the present paper which represents a development of the technical report [22] we construct exponentially convergent approximations to the solution of nonlinear problem (1.1). To this end we use an equivalent Volterra integral equation including the operator exponential and represent the operator exponential by a Dunford-Cauchy integral along a hyperbola enveloping the spectrum of the operator coefficient. Then we approximate the integrals involved using the Chebyshev interpolation and an appropriate Sinc quadrature.

Problem (1.1) is equivalent to the nonlinear Volterra integral equation

$$(1.2) \qquad\qquad u(t) = u_h(t) + u_{nl}(t),$$

where

$$(1.3) \qquad\qquad u_h(t) = T(t)u_0,$$

$T(t) = e^{-At}$ is the operator exponential (the semi-group) generated by A and the nonlinear term is given by

$$(1.4) \qquad u_{nl}(t) = \int_0^t e^{-A(t-s)} f(s, u(s)) ds.$$

We suppose that the solution $u(t)$ and the function $f(t, u(t))$ can be analytically extended (with respect to t) into a domain which we will describe below.

The aims of this paper are to propose a suitable abstract framework covering typical applied nonlinear problems relative to (1.1) and allowing exponentially convergent (for certain holomorphic right-hand sides), parallelizable approximations. In this sense the present paper extends the problem classes from the papers cited above for which it is possible to construct efficient algorithms with exponential accuracy.

The paper is organized as follows. In Section 2 using results from [20] we represent the operator exponential by the Dunford-Cauchy integral along a hyperbola enveloping the spectral angle of the operator A which implies that the Volterra integral equation (1.2) contains two integrals: the first one over a variable but finite time interval and the second one over the real axis. The first integral is then approximated by the Gauss-Chebyshev quadrature and the second one by a Sinc quadrature rule. This procedure leads to a system of nonlinear algebraic equations which can be solved by an iteration method. Section 4 is devoted to the error analysis of this algorithm in the case of a small Lipschitz constant. Under some natural assumption on the spectral properties of the unbounded operator A and on the nonlinear term $f(t, u)$ we show that our algorithm possesses a uniform with respect to t exponential convergence rate. As an auxiliary result which we could not find in the literature on approximation theory we proved the following estimates for the classical Lebesque functions $\Lambda_j^{(1)}(\xi) = \sum_{k=1}^N \left| \int_{-1}^\xi \chi_j(\eta) L_{k,N-1}(\eta) d\eta \right| \leq \kappa_j \sqrt{\pi(\xi+1)}, \quad \Lambda_j^{(2)}(\xi) = \sum_{k=1}^N \left| \int_\xi^{x_{j,N}} \chi_j(\eta) L_{k,N-1}(\eta) d\eta \right| \leq \kappa_j \sqrt{\pi(x_{j,N} - \xi)}, \; \xi \in (-1, x_{j,N}), \; j = 1, ..., N,$ where $L_{k,N-1}(\eta)$ are the Lagrange fundamental polynomials subject to the Chebyshev nodes $x_{j,N}$ and $|\chi_j(\eta)| \leq \kappa_j \; \forall \eta \in [-1, 1], \; j = 1, ..., N,$ are some bounded functions.

Section 5 is concerned with a modification and with the analysis of the numerical algorithm from Section 4 for the case of an arbitrary Lipschitz constant.

In Section 6 we discuss the implementation of our algorithm. Several numerical and analytical examples are given throughout the text in order to make clear or to confirm the theoretical results.

2. Exponentially convergent approximation to the operator exponential

Let A be a densely defined strongly positive operator in a Banach space X with the domain $D(A)$, i.e. its spectrum $\Sigma(A)$ lies in the sector

$$(2.1) \qquad \Sigma = \{z = a_0 + re^{i\theta} : r \in [0, \infty), \; |\theta| < \varphi < \frac{\pi}{2}\}$$

and on its boundary Γ_Σ, and outside of the sector the following estimate for the resolvent holds true:

$$(2.2) \qquad \|(zI - A)^{-1}\| \leq \frac{M}{1 + |z|}$$

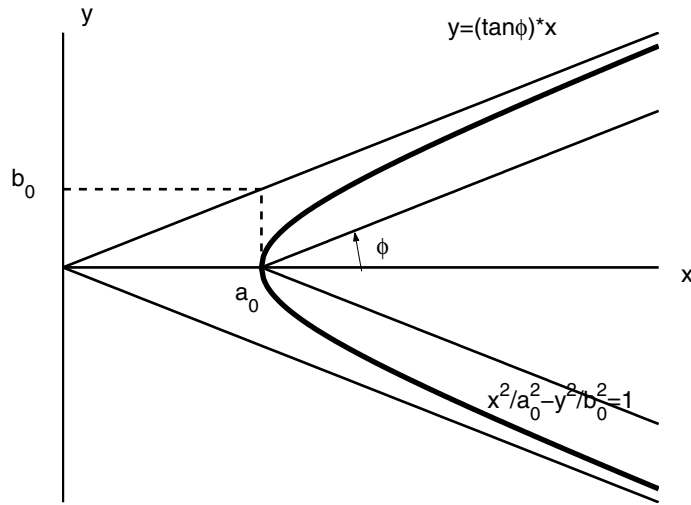FIGURE 1. Spectral characteristics of the operator A.

with some positive constant M (compare with [15, 27, 33, 36]). The angle φ is called the spectral angle of the operator A. A practically important example of strongly positive operators represents a strongly elliptic partial differential operator [7, 15, 16, 33] where the parameters a_0, φ of the sector Σ are defined by its coefficients. We call the hyperbola

$$(2.3) \qquad \Gamma_0 = \{z(\xi) = a_0 \cosh \xi - ib_0 \sinh \xi : \ \xi \in (-\infty, \infty), \ b_0 = a_0 \tan \varphi\}$$

the spectral hyperbola, which pathes through the vertex $(a_0, 0)$ of the spectral angle and possesses asymptotes which are parallel to the rays of the spectral angle Σ. We choose the following hyperbola as an integration path [20]:

$$(2.4) \qquad \Gamma_I = \{z(\xi) = a_I \cosh \xi - ib_I \sinh \xi : \ \xi \in (-\infty, \infty)\},$$

where

$$(2.5) \qquad \begin{aligned} a_I &= a_0 \cos\left(\frac{\pi}{4} - \frac{\varphi}{2}\right) - b_0 \sin\left(\frac{\pi}{4} - \frac{\varphi}{2}\right) \\ &= \sqrt{a_0^2 + b_0^2} \cos\left(\frac{\pi}{4} + \frac{\varphi}{2}\right) = a_0 \frac{\cos\left(\frac{\pi}{4} + \frac{\varphi}{2}\right)}{\cos \varphi}, \\ b_I &= a_0 \sin\left(\frac{\pi}{4} - \frac{\varphi}{2}\right) + b_0 \cos\left(\frac{\pi}{4} - \frac{\varphi}{2}\right) \\ &= \sqrt{a_0^2 + b_0^2} \sin\left(\frac{\pi}{4} + \frac{\varphi}{2}\right) = a_0 \frac{\sin\left(\frac{\pi}{4} + \frac{\varphi}{2}\right)}{\cos \varphi}. \end{aligned}$$

Obviously this integration path envelops the spectral hyperbola but does not intersect it. Since the operator A is strongly positive it holds on the integration path and outside of it:

$$(2.6) \qquad \begin{aligned} \|(zI - A)^{-1}w\| &\leq \frac{M}{1 + |z|}\|w\|, \\ \|A(zI - A)^{-1}w\| &\leq (1 + M)\|w\|. \end{aligned}$$

Now, Theorem 4 from [27] yields

$$
\|A^{1-\alpha}(zI-A)^{-1}w\| \le K\|A(zI-A)^{-1}w\|^{1-\alpha}\|(zI-A)^{-1}w\|^{\alpha},
$$
(2.7)
$$
\forall w \in X, \ \alpha \in [0,1],
$$

where the constant K depends on α and M only.

We consider the following representation of the operator exponential on an element u_0:

$$
\text{(2.8)} \qquad u_h(t) = \frac{1}{2\pi i} \int_{\Gamma_I} e^{-zt}(zI-A)^{-1}u_0 dz.
$$

One can also represent

$$
\text{(2.9)} \qquad u_h(t) = \frac{1}{2\pi i} \int_{\Gamma_I} e^{-zt}\left[(zI-A)^{-1} - \frac{1}{z}I\right]u_0 dz
$$

instead of (2.8) (for $t>0$ the integral from the second summand is equal to zero due to the analyticity of the integrand inside of the integration path), and this integral represents the solution of the homogeneous problem (1.1) for $u_0 \in D(A^{\alpha})$, $\alpha > 0$. For the modified resolvent the following estimate holds true [20]:

$$
\| \left[(zI-A)^{-1} - \sum_{k=1}^{m+1} \frac{A^{k-1}}{z^k}\right] u_0\| = \|\frac{1}{z^{m+1}}(zI-A)^{-1}A^{m+1}u_0\|
$$
(2.10)
$$
= \frac{1}{|z|^{m+1}}\|A^{1-\alpha}(zI-A)^{-1}A^{m+\alpha}u_0\| \le \frac{1}{|z|^{m+1}}\frac{(1+M)K}{(1+|z|)^{\alpha}}\|A^{m+\alpha}u_0\|,
$$
$$
\forall \alpha \in [0,1],
$$

provided that $u_0 \in D(A^{m+\alpha})$.

After parametrizing integral (2.9) by (2.4) we get

$$
\text{(2.11)} \qquad u_h(t) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \mathcal{F}(t,\xi)d\xi
$$

with

$$
\mathcal{F}(t,\xi) = F_A(t,\xi)u_0,
$$
(2.12)
$$
F_A(t,\xi) = e^{-z(\xi)t}(a_I \sinh \xi - ib_I \cosh \xi)\left[(z(\xi)I-A)^{-1} - \frac{1}{z(\xi)}I\right].
$$

It was shown in [20] that

$$
\text{(2.13)} \qquad \|\mathcal{F}(t,\xi)\| \le C(\varphi,\alpha)e^{-\alpha|\xi|}\|A^{\alpha}u_0\|, \ \xi \in \mathbb{R}, \ t \ge 0,
$$

with $C(\varphi,\alpha) = (1+M)K\tan\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)\left(\frac{2\cos\varphi}{a_0\cos\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)}\right)^{\alpha}$. The change of $\xi \in \mathbb{R}$ to $w = \xi + i\nu \in \mathbb{C}$ implies that the integration hyperbola will be translated into the parametric set of hyperbolas

$$
\Gamma(\nu) = \{z(w) = a_I \cosh(\xi+i\nu) - ib_I \sinh(\xi+i\nu): \ \xi \in (-\infty,\infty)\}
$$
(2.14)
$$
= \{z(w) = a(\nu)\cosh\xi - ib(\nu)\sinh\xi: \ \xi \in (-\infty,\infty)\}
$$

so that $\Gamma(0) = \Gamma_I$, where [20]

$$a(\nu) = a_I \cos \nu + b_I \sin \nu = \frac{a_0 \sin\left(\nu + \pi/4 - \varphi/2\right)}{\cos \varphi}$$

(2.15)
$$= \frac{a_0 \cos\left(\pi/4 + \varphi/2 - \nu\right)}{\cos \varphi},$$

$$b(\nu) = b_I \cos \nu - a_I \sin \nu = \frac{a_0 \sin\left(\pi/4 + \varphi/2 - \nu\right)}{\cos \varphi}.$$

The vector valued function $\mathcal{F}(t, w)$ is analytic with respect to $w = \xi + i\nu$ in the strip

(2.16) $$D_{d_1} = \{ w = \xi + i\nu : \ \xi \in (-\infty, \infty), \ |\nu| < d_1/2 \},$$

where $d_1 = \pi/2 - \varphi$ for all $t \geq 0$. Moreover, for an arbitrarily small positive δ in the strip D_d with $d = d_1 - \delta$ there holds

$$\|\mathcal{F}(t, w)\|$$

(2.17) $$\leq (1 + M)K \tan\left(\frac{\pi}{4} + \frac{\varphi}{2} - \nu\right) \left(\frac{2 \cos \varphi}{a_0 \cos\left(\pi/4 + \varphi/2 - \nu\right)}\right)^\alpha e^{-\alpha|\xi|} \|A^\alpha u_0\|,$$

$$\forall w \in D_d.$$

It was also shown that

$$|z'(w)/z(w)| \leq b(\nu)/a(\nu) = \tan\left(\pi/4 + \varphi/2 - \nu\right),$$

(2.18)
$$|z'(w)|/|z(w)|^{1+\alpha} \leq \tan\left(\frac{\pi}{4} + \frac{\varphi}{2} - \nu\right) \left(\frac{2 \cos \varphi}{a_0 \cos\left(\frac{\pi}{4} + \frac{\varphi}{2} - \nu\right)}\right)^\alpha,$$

which implies

(2.19) $$\|\mathcal{F}(t, \cdot)\|_{H^1(D_d)} \leq C(\varphi, \alpha, \delta)\|A^\alpha u_0\|,$$

where

(2.20)
$$C(\varphi, \alpha, \delta) = \frac{2}{\alpha}[C_+(\varphi, \alpha, \delta) + C_-(\varphi, \alpha, \delta)],$$

$$C_\pm(\varphi, \alpha, \delta) = (1 + M)K(\cos \varphi)^\alpha \tan\left(\frac{\pi}{4} + \frac{\varphi}{2} \pm \frac{d}{2}\right) \left(\frac{2}{a_0 \cos\left(\frac{\pi}{4} + \frac{\varphi}{2} \pm \frac{d}{2}\right)}\right)^\alpha$$

and the constant $C(\varphi, \alpha, \delta)$ tends to ∞ if $\alpha \to 0$ or $\delta \to 0$, $\varphi \to \pi/2$.

We approximate integral (2.11) by the following Sinc-quadrature:

(2.21) $$u_N(t) = e_N^{-At} u_0 = \frac{h}{2\pi i} \sum_{k=-N}^{N} \mathcal{F}(t, z(kh)),$$

where

(2.22) $$e_N^{-At} = \frac{h}{2\pi i} \sum_{k=-N}^{N} F_A(t, z(kh))$$

is an approximation to the operator exponential. For the error we have the estimate (see [20])

$$\|\eta_N(\mathcal{F}, h)\| = \|u(t) - u_N(t)\|$$

$$(2.23) \qquad \leq \frac{c\|A^\alpha u_0\|}{\alpha} \left\{ \frac{e^{-\pi d/h}}{\sinh(\pi d/h)} + exp[-a_I t \cosh((N+1)h) - \alpha(N+1)h] \right\},$$

where the constant c does not depend on h, N, t.

Moreover, it is easy to see that

$$\|A^\beta \eta_N(\mathcal{F}, h)\| = \|A^\beta(u(t) - u_N(t))\|$$

$$(2.24) \qquad \leq \frac{c\|A^{\alpha+\beta} u_0\|}{\alpha} \left\{ \frac{e^{-\pi d/h}}{\sinh(\pi d/h)} + exp[-a_I t \cosh((N+1)h) - \alpha(N+1)h] \right\}$$

$$\forall\, \beta \geq 0, \quad \forall\, \alpha > 0.$$

Since we also apply this quadrature later for $t \to 0$, it is important that it converges exponentially for all $t \geq 0$. Equalizing both exponentials for $t = 0$ by

$$(2.25) \qquad \frac{2\pi d}{h} = \alpha(N+1)h$$

we get for the step size

$$(2.26) \qquad h = \sqrt{\frac{2\pi d}{\alpha(N+1)}}$$

the following error estimates:

$$(2.27) \qquad \|\eta_N(\mathcal{F}, h)\| \leq \frac{c}{\alpha} \exp\left(-\sqrt{\frac{\pi d\alpha}{2}(N+1)}\right)\|A^\alpha u_0\|$$

and

$$(2.28) \quad \|A^\beta \eta_N(\mathcal{F}, h)\| \leq \frac{c}{\alpha} \exp\left(-\sqrt{\frac{\pi d\alpha}{2}(N+1)}\right)\|A^{\alpha+\beta} u_0\| \quad \forall\, \beta \geq 0, \quad \forall\alpha > 0$$

with a constant c independent of t, N.

The first summand in the exponent of $e^{[-a_I t \cosh((N+1)h) - \alpha(N+1)h]}$ contributes mainly to the error order. Setting $h = c_1 \ln N/N$ with some positive constant c_1 in (2.23) we remain asymptotically for a fixed $t > 0$ with an error

$$(2.29) \qquad \|\eta_N(\mathcal{F}, h)\| \leq c \left[e^{-\pi dN/(c_1 \ln N)} + e^{-c_1 a_I tN/2 - c_1 \alpha \ln N} \right] \|A^\alpha u_0\|,$$

where c is a positive constant. Thus, we have proven the following result.

Theorem 2.1. *Let A be a densely defined, closed, strongly positive linear operator and $u_0 \in D(A^\alpha)$, $\alpha \in (0, 1)$. Then Sinc-quadrature (2.21) represents an approximate solution of the homogeneous initial value problem (1.1) (i.e. $u(t) = e^{-At}u_0$) and possesses a uniform with respect to $t \geq 0$ exponential convergence rate with estimate (2.23) which is of the order $\mathcal{O}(e^{-c\sqrt{N}})$ uniformly in $t \geq 0$ provided that $h = \mathcal{O}(1/\sqrt{N})$ and of the order*

$$\mathcal{O}\left(max\left\{ e^{-\pi dN/(c_1 \ln N)}, \; e^{-c_1 a_I tN/2 - c_1 \alpha \ln N} \right\} \right)$$

for each fixed $t \geq 0$ provided that $h = c_1 \ln N/N$.

3. A discretization scheme of Chebyshev type

Changing in (1.2) the variables by

$$t = \frac{x+1}{2} \tag{3.1}$$

we transform problem (1.2) to the following problem on the interval $[-1, 1]$:

$$u(\frac{x+1}{2}) = g_h(x) + g_{nl}(x, u) \tag{3.2}$$

with

$$g_h(x) = e^{-A\frac{x+1}{2}} u_0,$$

$$(3.3) \qquad g_{nl}(x, u) = \frac{1}{2} \int_{-1}^{x} e^{-A\frac{x-\xi}{2}} f(\frac{\xi+1}{2}, u(\frac{\xi+1}{2})) d\xi.$$

Using the representation of the operator exponential by the Dunford-Cauchy integral along the integration path Γ_I defined above in (2.4), (2.5) and enveloping the spectral curve Γ_0 we obtain

$$g_h(x) = e^{-A\frac{x+1}{2}} u_0 = \frac{1}{2\pi i} \int_{\Gamma_I} e^{-z\frac{x+1}{2}} [(zI - A)^{-1} - \frac{1}{z} I] u_0 dz,$$

$$(3.4) \qquad g_{nl}(x, u) = \frac{1}{2} \int_{-1}^{x} e^{-A\frac{x-\eta}{2}} f(\frac{\eta+1}{2}, u(\frac{\eta+1}{2})) d\eta$$

$$= \frac{1}{4\pi i} \int_{-1}^{x} \int_{\Gamma_I} e^{-z\frac{x-\eta}{2}} [(zI - A)^{-1} - \frac{1}{z} I] f(\frac{\eta+1}{2}, u(\frac{\eta+1}{2})) dz d\eta$$

(note, that $P.V. \int_{\Gamma_I} z^{-1} dz = 0$, but this term in the resolvent provides the numerical stability of the algorithm below when $t \to 0$; see [20] for details). After parametrizing the first integral in (3.4) by (2.4) we have

$$g_h(x) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \mathcal{F}_h(x, \xi) d\xi \tag{3.5}$$

with

$$\mathcal{F}_h(x, \xi) = F_A((x+1)/2, \xi) u_0 \tag{3.6}$$

(in the case $A = 0$ we define $F_A(t, \xi) = 0$).

We approximate integral (3.5) by the following Sinc-quadrature (see (2.21), (2.26), (2.27)):

$$g_{h,N_1}(x) = \frac{h}{2\pi i} \sum_{k=-N_1}^{N_1} \mathcal{F}_h(x, kh), \quad h = \sqrt{\frac{2\pi d}{\alpha(N_1 + 1)}}, \tag{3.7}$$

with the error

$$(3.8) \quad \|\eta_{N_1}(\mathcal{F}_h, h)\| = \|\mathcal{E}((x+1)/2) u_0\| \leq \frac{c}{\alpha} exp\left(-\sqrt{\frac{\pi d\alpha}{2}}(N_1 + 1)\right) \|A^\alpha u_0\|,$$

where

$$(3.9) \quad \mathcal{E}((x-\eta)/2) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} F_A((x-\eta)/2, \xi) d\xi - \frac{1}{2\pi i} \sum_{k=-N_1}^{N_1} F_A((x-\eta)/2, kh)$$

and the constant c independent of x, N_1. Analogously we transform the second integral in (3.4) to

$$
\begin{aligned}
(3.10) \quad g_{nl}(x, u) &= \frac{1}{4\pi i} \int_{-1}^{x} \int_{\Gamma_I} e^{-z\frac{x-\eta}{2}} [(zI - A)^{-1} - \frac{1}{z}I] f(\frac{\eta+1}{2}, u(\frac{\eta+1}{2})) dz d\eta \\
&= \frac{1}{4\pi i} \int_{-1}^{x} \int_{-\infty}^{\infty} \mathcal{F}_{nl}(x, \xi, \eta) d\xi d\eta,
\end{aligned}
$$

where

$$
(3.11) \qquad \mathcal{F}_{nl}(x, \xi, \eta) = F_A((x - \eta)/2, \xi) f(\frac{\eta+1}{2}, u(\frac{\eta+1}{2})).
$$

Replacing the infinite integral by quadrature rule (3.7) we arrive at the approximation

$$
(3.12) \qquad g_{nl,N_1}(x, u) = \frac{h}{4\pi i} \int_{-1}^{x} \sum_{k=-N_1}^{N_1} \mathcal{F}_{nl}(x, kh, \eta) d\eta.
$$

In order to approximate the nonlinear operator $g_{nl,N_1}(x, u)$ we choose the mesh $\omega_N = \{x_{k,N} = \cos \frac{(2k-1)\pi}{2N}, k = 1, ..., N\}$ on $[-1, 1]$, where $x_{k,N}$ are zeros of Chebyshev orthogonal polynomial of first kind $T_N(x) = \cos(N \arccos x)$. For the stepsizes $\tau_{k,N} = x_{k,N} - x_{k-1,N}$ it is well known that (see [38],Ch.6, Th.6.11.12, [39], p. 123)

$$
\begin{aligned}
& \tau_{k,N} = x_{k+1,N} - x_{k,N} < \frac{\pi}{N}, k = 1, ..., N, \\
(3.13) \quad & \tau_{max} = \max_{1 \le k \le N} \tau_{k,N} < \frac{\pi}{N}.
\end{aligned}
$$

Let

$$
(3.14) \qquad P_{N-1}(x; f(\cdot, u)) = \sum_{p=1}^{N} f((x_{p,N} + 1)/2, u((x_{p,N} + 1)/2)) L_{p,N-1}(x)
$$

be the interpolation polynomial for the function $f(x, u(x))$ on the mesh ω_N, i.e. $P_{N-1}(x_{k,N}; f(\cdot, u)) = f((x_{k,N}+1)/2, u((x_{k,N}+1)/2)), k = 1, 2, ..., N$, where $L_{p,N-1} = \frac{T_N(x)}{T'_N(x_{p,N})(x-x_{p,N})}, p = 1, ..., N$, are the Lagrange fundamental polynomials. Given a vector $y = (y_1, ..., y_N), y_i \in X$, let

$$
(3.15) \qquad P_{N-1}(x; f(\cdot, y)) = \sum_{p=1}^{N} f((x_{p,N} + 1)/2, y_p) L_{p,N-1}(x)
$$

be the polynomial which interpolates $f(x, y)$, i.e.

$$
P_{N-1}(x_{k,N}; f(\cdot, y)) = f((x_{k,N} + 1)/2, y_k), k = 1, 2, ..., N.
$$

Substituting $P_{N-1}(t; f(\cdot, y))$ instead of $f(t, u)$ into (3.11), (3.12) we get the approximation

$$
(3.16) \qquad g_{nl,N,N_1}(x, y) = \frac{h}{4\pi i} \int_{-1}^{x} \sum_{k=-N_1}^{N_1} F_A((x - \eta)/2, kh) P_{N-1}(\eta; f(\cdot, y)) d\eta.
$$

Substituting approximations (3.7) and (3.16) into (3.2) and collocating the resulting equation on the grid ω_N we arrive at the following **Algorithm A1** for solving problem (3.2): find $y = (y_1, ..., y_N), y_i \in X$, such that

$$
(3.17) \qquad y_j = g_{h,N_1}(x_{j,N}) + g_{nl,N,N_1}(x_{j,N}, y), j = 1, ..., N,
$$

or

$$
y_j = \frac{h}{2\pi i} \sum_{k=-N_1}^{N_1} \mathcal{F}_h(x_{j,N}, kh)
$$

(3.18)
$$
+ \frac{h}{4\pi i} \sum_{k=-N_1}^{N_1} \int_{-1}^{x_{j,N}} F_A((x_{j,N} - \eta)/2, kh) P_{N-1}(\eta; f(\cdot, y)) d\eta,
$$

$$
j = 1, ..., N.
$$

Equations (3.17) or (3.18) define a nonlinear operator $\mathcal{A}$ so that

(3.19)
$$
y = \mathcal{A}(y) + \phi,
$$

where

$$
y = (y_1, y_2, ..., y_N), \quad y_i \in X,
$$

$$
[\mathcal{A}(y)]_j = \frac{h}{4\pi i} \sum_{k=-N_1}^{N_1} \int_{-1}^{x_{j,N}} F_A((x_{j,N} - \eta)/2, kh) P_{N-1}(\eta; f(\cdot, y)) d\eta,
$$

(3.20)

$$
(\phi)_j = \frac{h}{2\pi i} \sum_{k=-N_1}^{N_1} \mathcal{F}_h(x_{j,N}, kh) = \frac{h}{2\pi i} \sum_{k=-N_1}^{N_1} F_A((x_{j,N} + 1)/2, kh) u_0,
$$

$$
j = 1, ..., N.
$$

This is a system of nonlinear equations which can be solved by an iteration method. Since the integrands in
(3.21)
$$
I_{j,k} = \int_{-1}^{x_{j,N}} F_A((x_{j,N} - \eta)/2, kh) P_{N-1}(\eta; f(\cdot, y)) d\eta, \ j = 1, ..., N, \ k = -N_1, ..., N_1,
$$

are products of the exponential function and polynomials, these integrals can be calculated analytically, for example, by computer algebra tools.

Given the vector $y = (y_1, ..., y_N)$ the interpolation polynomial $\tilde{u}(x) = P_{N-1}(x; y)$ represents an approximation for $u((x + 1)/2) = u(t)$, i.e. $u((x + 1)/2) = u(t) \approx P_{N-1}(x; y)$.

4. The error analysis for a small Lipschitz constant

In this section we investigate the error of algorithm (3.18). The projection of the exact equation (3.2) onto the grid ω_N provides

(4.1)
$$
u(t_j) = e^{-At_j} u_0 + \frac{1}{2} \int_{-1}^{x_{j,N}} e^{-A(x_{j,N} - \xi)/2} f(\frac{1+\xi}{2}, u(\frac{1+\xi}{2})) d\xi,
$$

$$
t_j = \frac{1 + x_{j,N}}{2}, \quad x_{j,N} = \cos \frac{(2j-1)\pi}{2N}, \quad j = 1, ..., N.
$$

Using equations (3.18) we represent the error of the algorithm in the form

(4.2)
$$
Z_j = u(t_j) - y_j
$$
$$
= \psi_j^{(0)} + \psi_j^{(1)} + \psi_j^{(2)} + \psi_j^{(3)}, \quad j = 1, ..., N,
$$

where
(4.3)

$$\psi_j^{(0)} = \eta_N(\mathcal{F}_h, h)$$

$$= \left\{ e^{-At_j} - \frac{h}{2\pi i} \sum_{k=-N_1}^{N_1} z'(kh) e^{-t_j z(kh)} \left[(z(kh)I - A)^{-1} - \frac{1}{z(kh)} I \right] \right\} u_0$$

$$\psi_j^{(1)} = \frac{1}{2} \int_{-1}^{x_{j,N}} f\left(\frac{1+\eta}{2}, u\left(\frac{1+\eta}{2} \right) \right) \left\{ e^{-A\frac{(x_{j,N}-\eta)}{2}} \right.$$

$$\left. - \frac{h}{2\pi i} \sum_{k=-N_1}^{N_1} z'(kh) e^{-z(kh)\frac{(x_{j,N}-\eta)}{2}} \left[(z(kh)I - A)^{-1} - \frac{1}{z(kh)} I \right] \right\} d\eta,$$

$$\psi_j^{(2)} = \frac{h}{2\pi i} \sum_{k=-N_1}^{N_1} z'(kh) \frac{1}{2} \int_{-1}^{x_{j,N}} e^{-z(kh)(x_{j,N}-\eta)/2} \left[(z(kh)I - A)^{-1} - \frac{1}{z(kh)} I \right]$$

$$\times \left[f\left(\frac{1+\eta}{2}, u\left(\frac{1+\eta}{2} \right) \right) - \sum_{l=1}^{N} f(t_l, u(t_l)) L_{l,N-1}(\eta) \right] d\eta,$$

$$\psi_j^{(3)} = \frac{h}{2\pi i} \sum_{k=-N_1}^{N_1} z'(kh) \frac{1}{2} \int_{-1}^{x_{j,N}} e^{-z(kh)(x_{j,N}-\eta)/2} \left[(z(kh)I - A)^{-1} - \frac{1}{z(kh)} I \right]$$

$$\times \left[\sum_{l=1}^{N} [f(t_l, u(t_l)) - f(t_l, y_l)] L_{l,N-1}(\eta) \right] d\eta,$$

$$z(\xi) = a_I \cosh \xi - i b_I \sinh \xi.$$

Using notations (3.6), (3.9) we can write down

$$\psi_j^{(0)} = \mathcal{E}((x_{j,N} + 1)/2) u_0$$

$$\psi_j^{(1)} = \frac{1}{2} \int_{-1}^{x_{j,N}} \mathcal{E}((x_{j,N} - \eta)/2) f(\frac{1+\eta}{2}, u(\frac{1+\eta}{2})) d\eta,$$

$$\psi_j^{(2)} = \frac{1}{2} \int_{-1}^{x_{j,N}} \frac{h}{2\pi i} \sum_{k=-N_1}^{N_1} F_A((x_{j,N} - \eta)/2, kh)$$

(4.4)

$$\times \left[f(\frac{1+\eta}{2}, u(\frac{1+\eta}{2})) - P_{N-1}(\eta; f(\cdot, u(\cdot))) \right] d\eta,$$

$$\psi_j^{(3)} = \frac{1}{2} \int_{-1}^{x_{j,N}} \frac{h}{2\pi i} \sum_{k=-N_1}^{N_1} F_A((x_{j,N} - \eta)/2, kh)$$

$$\times \left[P_{N-1}(\eta; f(\cdot, u(\cdot)) - f(\cdot, y(\cdot))) \right] d\eta$$

$$= [\mathcal{A}(u)]_j - [\mathcal{A}(y)]_j$$

where $u = (u(t_1), ..., u(t_N)), y = (y_1, ..., y_N)$ and $\psi_j = \psi_j^{(0)} + \psi_j^{(1)} + \psi_j^{(2)}$ is the truncation error.

For the first summand we have estimate (3.8):

(4.5)
$$\|\psi_j^{(0)}\| \le \frac{c}{\alpha} \exp\left(-\sqrt{\frac{\pi d\alpha}{2}} (N_1 + 1) \right) \|A^\alpha u_0\|$$

and obviously the estimate

$$(4.6) \qquad \|A^\alpha \psi_j^{(0)}\| \le \frac{c}{\beta - \alpha} \exp\left(-\sqrt{\frac{\pi d\alpha}{2}}(N_1 + 1)\right) \|A^\beta u_0\| \quad \forall \, \beta > \alpha > 0.$$

In order to estimate $\psi_j^{(1)}$ we assume that
(i)

$$(4.7) \qquad f(t, u(t)) \in D(A^\alpha) \, \forall \, t \in [0, 1] \quad \text{and} \quad \int_0^1 \|A^\alpha f(t, u(t))\| dt < \infty.$$

Using this assumption we obtain analogously to (4.5), (4.6)

$$(4.8) \qquad \|\psi_j^{(1)}\| \le \frac{c}{\alpha} \exp\left(-\sqrt{\frac{\pi d\alpha}{2}}(N_1 + 1)\right) \int_0^1 \|A^\alpha f(t, u(t))\| dt$$

and
(4.9)

$$\|A^\alpha \psi_j^{(1)}\| \le \frac{c}{\beta - \alpha} \exp\left(-\sqrt{\frac{\pi d\alpha}{2}}(N_1 + 1)\right) \int_0^1 \|A^\beta f(t, u(t))\| dt \quad \forall \, \beta > \alpha > 0.$$

In order to estimate $\psi_j^{(2)}$ we assume in addition to (4.7) that

(ii) the vector valued function $A^\alpha f(\frac{1+\xi}{2}, u(\frac{1+\xi}{2}))$ of ξ can be analytically extended from the interval $B = [-1, 1]$ into the domain $\mathcal{D}_\rho$ enveloped by the so called Bernstein's regularity ellipse $\mathcal{E}_\rho = \mathcal{E}_\rho(B)$ (with the foci at $z = \pm 1$ and the sum of semi-axes equal to $\rho > 1$):

$$\mathcal{E}_\rho = \{z \in \mathbb{C} : z = \frac{1}{2}\left(\rho e^{i\varphi} + \frac{1}{\rho} e^{-i\phi}\right)\}$$

$$= \{(x, y) : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \, a = \frac{1}{2}\left(\rho + \frac{1}{\rho}\right), \, b = \frac{1}{2}\left(\rho - \frac{1}{\rho}\right)\}.$$

Using (2.10) with $m = 0$, the first inequality (2.18) with $\nu = 0$ and the fact that the Lebesque constant for the Chebyshev interpolation process is bounded by $c \ln N$ we obtain

$$(4.10) \qquad \|\psi_j^{(2)}\| \le c \cdot S_{N_1} \cdot \ln N \cdot E_N \left(A^\alpha f\left(\cdot, u\left(\cdot\right)\right)\right)$$

where $S_{N_1} = \sum_{k=-N_1}^{N_1} h|z'(kh)|/|z(kh)|^{1+\alpha}$, c is a constant independent of N, N_1, η and $E_N\left(A^\alpha f\left(\cdot, u\left(\cdot\right)\right)\right)$ is the value of the best approximation of $A^\alpha f(t, u(t))$ by polynomials of degree not greater than $N - 1$ in the maximum norm with respect to t. Using the estimate

$$(4.11) \qquad \begin{aligned} |z(kh)| &= \sqrt{a_I^2 \cosh^2(kh) + b_I^2 \sinh^2(kh)} \\ &\ge a_I \cosh(kh) \ge a_I e^{|kh|}/2 \end{aligned}$$

the last sum can be estimated by

$$(4.12) \qquad |S_{N_1}| \le \frac{c}{\sqrt{N_1}} \sum_{k=-N_1}^{N_1} e^{-\alpha|k|/\sqrt{N_1}} \le c \int_{-\sqrt{N_1}}^{\sqrt{N_1}} e^{-\alpha t} dt \le c/\alpha.$$

Due to assumption **(ii)** we have for the value of the best polynomial approximation [5, 20]

$$(4.13) \qquad E_N\left(A^\alpha f\left(\cdot, u\left(\cdot\right)\right)\right) \le \frac{\rho^{-N}}{1-\rho} \sup_{z \in D_\rho} \|A^\alpha f(z, u(z))\|$$

which together with (4.10)and (4.12) yields

$$(4.14) \qquad \|\psi_j^{(2)}\| \le \frac{c}{\alpha} \ln N \rho^{-N} \sup_{z \in D_\rho} \|A^\alpha f(z, u(z))\|$$

and

$$(4.15) \qquad \|A^\alpha \psi_j^{(2)}\| \le \frac{c}{\beta-\alpha} \ln N \rho^{-N} \sup_{z \in D_\rho} \|A^\beta f(z, u(z))\| \quad \forall\, \beta > \alpha > 0.$$

Before we go over to the estimating of $\psi_j^{(3)}$ let us introduce the functions

$$(4.16) \qquad \Lambda_j^{(1)}(\xi) = \sum_{k=1}^{N} \left| \int_{-1}^{\xi} \chi_j(\eta) L_{k,N-1}(\eta) d\eta \right|, \ j = 1, ..., N,$$

and

$$(4.17) \qquad \Lambda_j^{(2)}(\xi) = \sum_{k=1}^{N} \left| \int_{\xi}^{x_{j,N}} \chi_j(\eta) L_{k,N-1}(\eta) d\eta \right|, \ j = 1, ..., N,$$

with some bounded functions $\chi_j(\eta)$:

$$(4.18) \qquad |\chi_j(\eta)| \le \kappa_j \ \forall \eta \in [-1,1], \ j = 1, ..., N,$$

and prove the following auxiliary assertion.

Lemma 4.1. *There holds*

$$(4.19) \qquad \begin{aligned} &\Lambda_j^{(1)}(\xi) \le \kappa_j \sqrt{\pi(\xi+1)}, \\ &\Lambda_j^{(2)}(\xi) \le \kappa_j \sqrt{\pi(x_{j,N}-\xi)}, \ \xi \in (-1, x_{j,N}), \ j = 1, ..., N. \end{aligned}$$

Proof. Let

$$(4.20) \qquad \begin{aligned} \epsilon_{k,j}^{(1)} &= \mathrm{sign}\left\{ \int_{-1}^{\xi} \chi_j(\eta) L_{k,N-1}(\eta) d\eta \right\}, \\ \epsilon_{k,j}^{(2)} &= \mathrm{sign}\left\{ \int_{\xi}^{x_{j,N}} \chi_j(\eta) L_{k,N-1}(\eta) d\eta \right\}. \end{aligned}$$

Then taking into account that all coefficients of the Gauß quadrature relating to the Chebyshev orthogonal polynomials of the first kind are equal to π/N and the Lagrange fundamental polynomials $L_{k,N}(\eta)$ are orthogonal [38] with the weight

$1/\sqrt{1 - \eta^2}$ we obtain

$$\Lambda_j^{(1)}(\xi) = \int_{-1}^{\xi} \chi_j(\eta) \sum_{k=1}^{N} \epsilon_{k,j}^{(1)} L_{k,N-1}(\eta) d\eta$$

$$\leq \sqrt{\xi + 1} \left\{ \int_{-1}^{\xi} \chi_j^2(\eta) \left[\sum_{k=1}^{N} \epsilon_{k,j}^{(1)} L_{k,N-1}(\eta) \right]^2 d\eta \right\}^{1/2}$$

$$\leq \kappa_j \sqrt{\xi + 1} \left\{ \int_{-1}^{\xi} \left[\sum_{k=1}^{N} \epsilon_{k,j}^{(1)} L_{k,N-1}(\eta) \right]^2 / \sqrt{1 - \eta^2} d\eta \right\}^{1/2}$$

(4.21)

$$\leq \kappa_j \sqrt{\xi + 1} \left\{ \sum_{k,p=1}^{N} \epsilon_{k,j}^{(1)} \epsilon_{p,j}^{(1)} \int_{-1}^{1} L_{k,N-1}(\eta) L_{p,N-1}(\eta) / \sqrt{1 - \eta^2} d\eta \right\}^{1/2}$$

$$= \kappa_j \sqrt{\xi + 1} \left\{ \sum_{k=1}^{N} \left(\epsilon_{k,j}^{(1)} \right)^2 \int_{-1}^{1} L_{k,N-1}^2(\eta) / \sqrt{1 - \eta^2} d\eta \right\}^{1/2}$$

$$= \kappa_j \sqrt{\pi / N} \sqrt{\xi + 1} \left\{ \sum_{k=1}^{N} 1 \right\}^{1/2}$$

$$= \kappa_j \sqrt{\pi (1 + \xi)}.$$

Analogously we obtain

$$\Lambda_j^{(2)}(\xi) = \int_{\xi}^{x_{j,N}} \chi_j(\eta) \sum_{k=1}^{N} \epsilon_{k,j}^{(2)} L_{k,N-1}(\eta) d\eta$$

$$\leq \sqrt{x_{j,N} - \xi} \left\{ \int_{\xi}^{x_{j,N}} \chi_j^2(\eta) \left[\sum_{k=1}^{N} \epsilon_{k,j}^{(2)} L_{k,N-1}(\eta) \right]^2 d\eta \right\}^{1/2}$$

(4.22)

$$\leq \kappa_j \sqrt{x_{j,N} - \xi} \left\{ \int_{\xi}^{x_{j,N}} \left[\sum_{k=1}^{N} \epsilon_{k,j}^{(2)} L_{k,N-1}(\eta) \right]^2 / \sqrt{1 - \eta^2} d\eta \right\}^{1/2}$$

$$\leq \kappa_j \sqrt{x_{j,N} - \xi} \left\{ \sum_{k,p=1}^{N} \epsilon_{k,j}^{(2)} \epsilon_{p,j}^{(2)} \int_{-1}^{1} L_{k,N-1}(\eta) L_{p,N-1}(\eta) / \sqrt{1 - \eta^2} d\eta \right\}^{1/2}$$

$$= \kappa_j \sqrt{\pi (x_{j,N} - \xi)} = \sqrt{2\pi} \kappa_j \cos \frac{(2j - 1)\pi}{4N}.$$

The proof is complete. $\square$

Corollary 4.2. *We define the numbers*

(4.23) $$\Lambda_j^{(1)} = \Lambda_j^{(1)}(x_{j,N}) = \sum_{k=1}^{N} \left| \int_{-1}^{x_{j,N}} \chi_j(\eta) L_{k,N-1}(\eta) d\eta \right|, \; j = 1, ..., N.$$

Then using Lemma 4.1 we derive

$$(4.24) \qquad \Lambda_j^{(1)} \le \kappa_j \sqrt{\pi(1 + \cos \frac{(2j-1)\pi}{2N})} = \sqrt{2\pi}\kappa_j \cos \frac{(2j-1)\pi}{4N}, \; j = 1, ..., N.$$

Conjecture. *Setting $\chi(\eta) = 1$, we obtain from (4.24) that*

$$(4.25) \qquad \Lambda_j^{(1)} \le \Lambda_{u,j}^{(1)} \le \sqrt{2\pi} \cos \frac{(2j-1)\pi}{4N}$$

and the upper bound $\Lambda_{u,j}^{(1)}$ of $\Lambda_j^{(1)}$ is monotonically decreasing in j. At the same time, calculations indicate the behavior of $\Lambda_j^{(1)}$ given by Table 1. Our hypothesis is that each $\Lambda_j^{(1)}$ is also monotonically decreasing in j and $\Lambda_{u,j}^{(1)} = 2 \cos \frac{(2j-1)\pi}{4N}$.

TABLE 1. The behavior of $\Lambda_j^{(1)}$ and of $\Lambda_{u,j}^{(1)}$ for $N = 8$.

j	1	2	3	4	5	6	7	8
Λ_j	1.9807	1.8516	1.6037	1.2630	.8934	.5213	.2462	$.3097e - 1$
$\Lambda_{u,j}$	1.9903	1.9138	1.76384	1.5460	1.2687	.9427	.5805	.1960

Corollary 4.3. *The numbers $\Lambda_j^{(1)}$ remain bounded also if $\chi_j(\eta) = e^{-z(x_{j,N} - \eta)}$, where $z = \rho e^{i\theta}$ is a complex number with $\rho \ge 0$, $\theta \in (-\pi/2, \pi/2)$.*

Actually, we have in this case

$$\Lambda_j^{(1)} = \sum_{k=1}^{N} \left| \int_{-1}^{x_{j,N}} e^{-\rho exp\{i\theta\}(x_{j,N}-\eta)} L_{k,N-1}(\eta) d\eta \right|$$

$$(4.26) \qquad \le \sum_{k=1}^{N} \left| \int_{-1}^{x_{j,N}} \chi_{j,1}(\eta) L_{k,N-1}(\eta) d\eta \right|$$

$$+ \sum_{k=1}^{N} \left| \int_{-1}^{x_{j,N}} \chi_{j,2}(\eta) L_{k,N-1}(\eta) d\eta \right|$$

where

$$(4.27) \qquad \begin{aligned} \chi_{j,1}(\eta) &= e^{-\rho \cos\theta(x_{j,N}-\eta)} \cos[\rho \sin\theta(x_{j,N} - \eta)], \\ \chi_{j,2}(\eta) &= e^{-\rho \cos\theta(x_{j,N}-\eta)} \sin[\rho \sin\theta(x_{j,N} - \eta)]. \end{aligned}$$

Applying Lemma 4.1 for each summand and each of functions $\chi_{j,1}(\eta) \le 1$ and $\chi_{j,1}(\eta) \le 1$ we arrive at the estimate

$$(4.28) \qquad \Lambda_j^{(1)} \le 2\sqrt{2\pi} \cos \frac{(2j-1)\pi}{4N}, \; j = 1, ..., N.$$

Now, we are in the position to estimate $\psi_j^{(3)}$.

To this end we assume that

(iii) The function $f(t, y) = f(t, y; N)$ in the domain

$$G = \{(t, y, N): \; 0 \le t \le 1, \; \|\|y - u\|\| < \gamma, N \ge N_0\},$$

in addition to **(i)**, **(ii)**, satisfies

$$(4.29) \qquad \|\|A^\alpha[f(t, y_1) - f(t, y_2)]\|\| \le L \|\|y_1 - y_2\|\| \; \forall y_1, \; y_2 \in G,$$

for all $(t, y_i, N) \in G$, $i = 1, 2$, where $|||Z||| = |||y - u||| = \max_{j=1,\ldots,N} \|y_j - u(t_j)\|$, γ is a positive real constant and N_0 is a fixed natural number large enough.

Under this assumption and taking into account Lemma 4.1 and Corollary 4.3 as well as (4.11), (4.12), we have

(4.30)
$$\|\psi_j^{(3)}\| \leq cLh \sum_{k=-N_1}^{N_1} \frac{|z'(kh)|}{|z(kh)|^{1+\alpha}} \sum_{l=1}^{N} \left| \int_{-1}^{x_{j,N}} e^{-z(kh)(x_{j,N}-\eta)} L_{l,N-1}(\eta) d\eta \right| \|Z_l\|$$

$$\leq |||Z|||cLh \sum_{k=-N_1}^{N_1} \frac{|z'(kh)|}{|z(kh)|^{1+\alpha}} \Lambda_j^{(1)} \leq cLS_{N_1}\Lambda_j^{(1)}|||Z||| \leq \frac{c^*}{\alpha}L|||Z|||$$

with a new positive constant c^*. This estimate together with (4.2) implies

(4.31)
$$|||Z||| \leq \frac{\alpha}{\alpha - c^*L}|||\psi|||$$

with a constant c independent of α, N provided that $c^*L/\alpha < 1$. Analogously we obtain

(4.32)
$$\|A^\alpha \psi_j^{(3)}\| \leq \frac{c^*}{\beta - \alpha}L|||A^\beta Z||| \quad \forall \beta > \alpha > 0$$

and

(4.33)
$$|||A^\alpha Z||| \leq \frac{\beta - \alpha}{\beta - \alpha - c^*L}|||A^\beta \psi||| \quad \forall \beta > \alpha > 0$$

with a constant c^* independent of α, β, N provided that $c^*L/(\beta - \alpha) < 1$.

Taking into account (4.2) and estimates (4.5), (4.8), (4.14) as well as (4.30) we arrive at the estimate

(4.34)
$$|||Z||| \leq \frac{c}{\alpha - c^*L} \times \left(e^{-c_1\sqrt{N_1}} \left(\|A^\alpha u_0\| + \int_0^1 \|A^\alpha f(t, u(t))\| dt \right) \right.$$
$$\left. + \ln N \rho^{-N} \sup_{z \in D_\rho} \|A^\alpha f(z, u(z))\| \right)$$

provided that the Lipschitz constant L is such that $c^*L/\alpha < 1$. Equalizing the exponents by $N \asymp \sqrt{N_1}$ (i.e. the number of the interpolation points must be proportional to the square root of the number of nodes in the Sinc-quadrature) we obtain

(4.35)
$$|||Z||| = |||u - y||| \leq \frac{c}{\alpha - c^*L} \ln N_1 e^{-c_1\sqrt{N_1}}$$
$$\times \left(\|A^\alpha u_0\| + \int_0^1 \|A^\alpha f(t, u(t))\| dt + \sup_{z \in D_\rho} \|A^\alpha f(z, u(z))\| \right),$$

and due to (4.6), (4.9), (4.15), (4.33) in a more strong norm

(4.36)
$$|||A^\alpha Z||| = |||A^\alpha(u - y)||| \leq \frac{c}{\beta - \alpha - c^*L} \ln N_1 e^{-c_1\sqrt{N_1}}$$
$$\times \left(\|A^\beta u_0\| + \int_0^1 \|A^\beta f(t, u(t))\| dt + \sup_{z \in D_\rho} \|A^\beta f(z, u(z))\| \right),$$
$$\forall \beta > \alpha > 0, \ \beta - \alpha > c^*L.$$

These estimates show in particular that the operator $\mathcal{A}$ is contractive on G provided that $c^* L / \alpha < 1$ and $N \geq N_0$. Taking into account the Banach fixed point theorem we obtain by usual arguments that there exists the unique solution of (3.19) in G for which the estimate (4.35) holds.

Thus, we have proven the following main result of this paper.

Theorem 4.4. *Let A be a densely defined, closed, strongly positive linear operator with the domain $D(A)$ in a Banach space X and let the assumptions* **(i)**, **(ii)**, **(iii)** *hold. Then algorithm* **A1** *defined by* (3.18) *for the numerical solution of the nonlinear problem* (1.1) *possesses a uniform with respect to t exponential convergence rate with estimates* (4.35), (4.36) *provided that $N \asymp \sqrt{N_1}$ and the Lipschitz constant L is sufficiently small.*

Remark 4.5. The same result can be obtained if one uses the interpolation polynomial on the Chebyshev-Gauss-Lobatto grid

$$(4.37) \qquad \omega_N^{CGL} = \{x_{k,N} = x_{k,N}^{CGL} = \cos \frac{(N-j)\pi}{N}, k = 0, 1, ..., N\}$$

where the nodes are zeros of the polynomial $(1 - x^2) T_N'(x)$.

Example 4.6. In order to have a view of the possible size of the Lipschitz constant let us consider the following nonlinear Cauchy problem:

$$(4.38) \qquad \begin{aligned} \frac{d\vec{u}(t)}{dt} + A\vec{u}(t) &= \vec{f}(t, \vec{u}(t)), \ t > 0, \\ \vec{u}(0) &= \vec{u}_0 \end{aligned}$$

with a linear self-adjoint positive definite operator A such that

$$(4.39) \qquad\qquad\qquad A = A^* \geq \lambda_0 I, \ \lambda_0 > 0.$$

In this case algorithm (3.18) takes the form

$$(4.40) \qquad \begin{aligned} \vec{y}(t_j) = \vec{y}_j &= e^{-At_j} \vec{u}_0 + \sum_{p=1}^N \frac{1}{2} \int_{-1}^{x_{j,N}} e^{-A(x_{j,N}-\eta)/2} L_{p,N-1}(\eta) d\eta \vec{f}(t_p, \vec{y}_p), \\ & j = 1, ..., N. \end{aligned}$$

For the error $\vec{z}_j = \vec{y}_j - \vec{u}(t_j) = \vec{y}_j - \vec{u}_j$ we have the equation

$$(4.41) \qquad \begin{aligned} \vec{z}_j &= \sum_{p=1}^N \frac{1}{2} \int_{-1}^{x_{j,N}} e^{-A(x_{j,N}-\eta)/2} L_{p,N-1}(\eta) d\eta [\vec{f}(t_p, \vec{y}_p) - \vec{f}(t_p, \vec{u}_p)] + \vec{\psi}_j, \\ & j = 1, ..., N, \end{aligned}$$

where

$$(4.42) \qquad \vec{\psi}_j = \frac{1}{2} \int_{-1}^{x_{j,N}-\eta} e^{-A(x_{j,N}-\eta)/2} [P_{N-1}(\eta, \vec{u}) - \vec{f}(\frac{\eta+1}{2}, u(\frac{\eta+1}{2}))] d\eta$$

is the truncation error. Using the equation

$$\int_{-1}^{x_{j,N}} e^{-A(x_{j,N}-\eta)/2} L_{p,N-1}(\eta) d\eta$$

$$= -\int_{-1}^{x_{j,N}} e^{-A(x_{j,N}-\eta)/2} \frac{d}{d\eta} \int_{\eta}^{x_{j,N}} L_{p,N-1}(\xi) d\xi d\eta$$

(4.43)

$$= e^{-A(x_{j,N}+1)/2} \int_{-1}^{x_{j,N}} L_{p,N-1}(\eta) d\eta$$

$$+ \frac{1}{2} \int_{-1}^{x_{j,N}} A e^{-A(x_{j,N}-\eta)/2} \int_{\eta}^{x_{j,N}} L_{p,N-1}(\xi) d\xi d\eta$$

we obtain

$$\vec{z}_j = \sum_{p=1}^{N} \frac{1}{2} \left\{ e^{-A(x_{j,N}+1)/2} \int_{-1}^{x_{j,N}} L_{p,N-1}(\eta) d\eta \right.$$

(4.44)

$$\left. + \frac{1}{2} \int_{-1}^{x_{j,N}} A e^{-A(x_{j,N}-\eta)/2} \int_{\eta}^{x_{j,N}} L_{p,N-1}(\xi) d\xi d\eta \right\} [\vec{f}(t_p, \vec{y}_p) - \vec{f}(t_p, \vec{u}_p)]$$

$$+ \vec{\psi}_j, \quad j = 1, ..., N.$$

Since A is a self-adjoint, positive definite operator we have

(4.45)

$$\|A e^{-A(x_{j,N}-\eta)/2}\| = \max_{\lambda_0 \le \lambda < \infty} (\lambda e^{-\lambda(x_{j,N}-\eta)/2}) \le \frac{2}{e(x_{j,N}-\eta)},$$

$$\|e^{-A(x_{j,N}+1)/2}\| \le 1.$$

This estimate together with (4.44) and Lemma 4.1 implies

$$\|\vec{z}_j\| \le L \sum_{p=1}^{N} \frac{1}{2} \left| \int_{-1}^{x_{j,N}} L_{p,N-1}(\eta) d\eta \right| \|\|\vec{z}\|\|$$

$$+ L \frac{1}{2} \sum_{p=1}^{N} \frac{1}{2} \int_{-1}^{x_{j,N}} \|A e^{-A(x_{j,N}-\eta)/2}\| \cdot \left| \int_{-1}^{\eta} L_{p,N}(\xi) d\xi \right| d\eta \cdot \|\|\vec{z}\|\| + \|\vec{\psi}_j\|$$

(4.46)

$$\le \frac{1}{2} L \Lambda_j^{(1)}(x_{j,N}) \|\|\vec{z}\|\| + \frac{L}{2e} \int_{-1}^{x_{j,N}} \frac{1}{x_{j,N}-\eta} \Lambda_j^{(2)}(\eta) \|\|\vec{z}\|\| + \|\vec{\psi}_j\|$$

$$\le \frac{L}{2} \Lambda_j \|\|\vec{z}\|\| + \frac{L\sqrt{\pi}}{2e} \int_{-1}^{x_{j,N}} \frac{1}{\sqrt{x_{j,N}-\eta}} \|\|\vec{z}\|\| + \|\vec{\psi}_j\|$$

$$\le \left(\sqrt{\frac{\pi}{2}} + \frac{\sqrt{2\pi}}{e} \right) L \|\|\vec{z}\|\| + \|\vec{\psi}_j\|.$$

The last inequality yields the following condition:

(4.47)

$$L < \frac{\sqrt{2}e}{\sqrt{\pi}e + 2\sqrt{\pi}}$$

on the Lipschitz constant L which provides the convergence of the fixed point
iteration and the corresponding a priori estimate for $\|\|\vec{z}\|\|$.

Remark 4.7. Given N_1 choose the integer number $N_2 = [\sqrt{N_1}]$ and set $y(t) = P_{N_2-1}(2t - 1; y)$ with y defined by algorithm (3.18). In order to get an error

estimate for all $t \in [0,1]$ we represent

$$Z(t) = u(t) - P_{N_2-1}(2t-1;y)$$
$$= u(t) - P_{N_2-1}(2t-1;u) + [P_{N_2-1}(2t-1;u) - P_{N_2-1}(2t-1;y)].$$

Taking into account that the Lebesque constant relating to the Chebyshev interpolation nodes is bounded by $c \ln N_2$ and using the estimates (4.35), $\|P_{N_2-1}(2t-1;u) - P_{N_2-1}(2t-1;y)\| \leq c \ln N_2 \|\|Z\|\| \leq c \ln^2 N_2 e^{-c_1\sqrt{N_2}}$ and $\|u(t) - P_{N_2-1}(2t-1;u)\| \leq c \ln N_2 e^{-c_1\sqrt{N_2}}$, we derive

$$(4.48) \qquad \max_{0 \leq t \leq 1} \|u(t) - P_{N_2-1}(2t-1;y)\| \leq c \ln^2 N_2 e^{-c_1\sqrt{N_2}}.$$

5. Modified algorithm for arbitrary Lipschitz constant

In this section we show how the algorithm above can be modified for a nonlinearity with an arbitrary Lipschitz constant. To this end we suppose that $u(t) \in D(A^\sigma)$, $\sigma > c^* L/2$. We cover the interval $[0,1]$ by the grid $\omega_G = \{t_i = i \cdot \tau : i = 0,1,...,K, \tau = 1/K\}$ and consider problem (1.1) on each subinterval $[t_{k-1}, t_k], k = 1, ..., K$. The substitution $t = t_{k-1}(1-\xi)/2 + t_k(1+\xi)/2, v(\xi) = u(t_{k-1}(1-\xi)/2 + t_k(1+\xi)/2)$ translates the original equation into the differential equation

$$(5.1) \qquad v'(\xi) + \tilde{A}v = \tilde{f}(\xi,v)$$

on the reference interval $[-1,1]$ with $\tilde{A} = \frac{\tau}{2}A$ and with the function $\tilde{f}(\xi,v) = \frac{\tau}{2}f(t_{k-1}(1-\xi)/2+t_k(1+\xi)/2, u(t_{k-1}(1-\xi)/2+t_k(1+\xi)/2))$ satisfying the Lipschitz condition with the Lipschitz constant $\tilde{L} = \tau L/2$ which can be made arbitrarily small by the appropriate choice of τ. We cover each subinterval $[t_{k-1}, t_k]$ by the Chebyshev-Gauss-Lobatto grid

$$(5.2) \qquad \begin{aligned} \omega_{k,N}^{CGL} &= \{t_{k,j} : t_{k,j} = t_{k-1}(1-x_{j,N})/2 + t_k(1+x_{j,N})/2, j = 0,1,...,N\}, \\ x_{j,N} &= \cos\left(\pi(N-j)/N\right) \end{aligned}$$

and denote $v_k(x_{j,N}) = v_{k,j} = u(t_{k,j}), v_{k,0} = v_k, u(t_{k,0}) = u(t_k) = u_k, \vec{v}_k = [v_{k,j}]_{j=1,...,N}, \vec{u}_k = [u(t_{k,j})]_{j=1,...,N}$. Then, algorithm (3.18) with the corresponding Chebyshev-Gauss-Lobatto interpolation polynomial can be applied which provides an exponential accuracy on the subinterval $[t_{k-1}, t_k]$ under the assumption that the initial vector u_{k-1} is known. This is exactly the case for $k = 1$, and by algorithm (3.18) we obtain a value $v_{1,N} = v_1$ as an approximation for $u(t_1)$. Starting on the subinterval $[t_1, t_2]$ with the approximate initial value v_1 we obtain an approximate solution for this subinterval and so on.

In order to write down this idea as an algorithm we derive from (5.1) the relation

$$(5.3) \qquad v_{k,j} = e^{-\tilde{A}(1+x_{j,N})}u_{k-1} + \int_{-1}^{x_{j,N}} e^{-\tilde{A}(x_{j,N}-\eta)}\tilde{f}(\eta, v_k(\eta))d\eta.$$

Denoting by $y_{k,j}$ approximations to $v_{k,j}$, approximating the operator exponential by (2.22) with N_1 nodes and the nonlinearity by the Chebyshev-Gauss-Lobatto

interpolation polynomial

$$P_N(\eta, \vec{\tilde{f}}) = \sum_{l=0}^{N} \tilde{f}(x_{l,N}, y_{k,l}) L_{l,N}(\eta),$$

(5.4)
$$L_{l,N}(\eta) = \frac{(1 - \eta^2) T'_N(\eta)}{(\eta - x_{l,N}) \frac{d}{d\eta}[(1 - \eta^2) T'_N(\eta)]_{\eta = x_{l,N}}},$$

$$\vec{\tilde{f}} = [\tilde{f}(x_{j,N}, y_{k,j})]_{j=0}^{N}$$

we arrive at the following system of nonlinear equations (analogous to (3.18)):

(5.5)
$$y_{k,j} = e_{N_1}^{-\tilde{A}(1+x_{j,N})} y_{k-1} + \int_{-1}^{x_{j,N}} e_{N_1}^{-\tilde{A}(x_{j,N}-\eta)} P_N(\eta, \vec{\tilde{f}}) d\eta,$$

which expresses $y_{k,j}, j = 1, 2, ..., N$ (in particular $y_{k,N} = y_{k+1}$) through y_{k-1}.

Now, we can formulate the following algorithm.

Algorithm A2.

Given K satisfying (5.9), N_1 computes the approximate solution of nonlinear problem (1.1) with an arbitrary Lipschitz constant by solving of the nonlinear discrete system (5.5) on each subinterval

1. Choose K satisfying (5.9) and N_1, and set $\tau = 1/K$, $t_0 = 0$, $y_0 = u_0$.

2. For $i := 1$ step 1 to K do

2.1. Set $t_i = t_{i-1} + \tau$ and find the approximate solution $y_{i,j}, j = 1, 2, ..., N$, of problem (1.1) on the Chebyshev-Gauss-Lobatto grid (5.2) covering the interval $[t_{i-1}, t_i]$ by algorithm (5.5) using y_{i-1} as the initial value.

2.2. Set $y_i = y_{i,N}$.

Now, let us analyze the error $z_{k,j} = u(t_{k,j} - y_{k,j}) = v_{k,j} - y_{k,j}$ of this algorithm. We have the representation

(5.6)
$$z_{k,j} = \psi_{k,j} + \sum_{p=0}^{3} \psi_{k,j}^{(p)},$$

where

(5.7)
$$\psi_{k,j} = e_{N_1}^{-\tilde{A}(x_{j,N}+1)} z_{k-1},$$

$$\psi_{k,j}^{(0)} = [e^{-\tilde{A}(x_{j,N}+1)} - e_{N_1}^{-\tilde{A}(x_{j,N}+1)}] u_{k-1},$$

$$\psi_{k,j}^{(1)} = \int_{-1}^{x_{j,N}} [e^{-\tilde{A}(x_{j,N}-\eta)} - e_{N_1}^{-\tilde{A}(x_{j,N}-\eta)}] \tilde{f}(\eta, v_k(\eta)) d\eta,$$

$$\psi_{k,j}^{(2)} = \int_{-1}^{x_{j,N}} e_{N_1}^{-\tilde{A}(x_{j,N}-\eta)} [\tilde{f}(\eta, v_k(\eta)) - \sum_{l=0}^{N} \tilde{f}(x_{l,N}, v_{k,l}) L_{l,N}(\eta)] d\eta,$$

$$\psi_{k,j}^{(3)} = \int_{-1}^{x_{j,N}} e_{N_1}^{-\tilde{A}(x_{j,N}-\eta)} \sum_{l=0}^{N} [\tilde{f}(x_{l,N}, v_{k,l}) - \tilde{f}(x_{l,N}, y_{k,l})] L_{l,N}(\eta) d\eta.$$

Under the same assumptions and analogously to (4.5), (4.8), (4.14), (4.30) we obtain the following estimates:

$$\|\psi_{k,j}\| \le \frac{c}{\alpha_k}\|\tilde{A}^{\alpha_k}z_{k-1}\|,$$

$$\|\psi_{k,j}^{(0)}\| \le \frac{c}{\alpha_k}\exp\left(-\sqrt{\frac{\pi d\alpha_k}{2}}(N_1+1)\right)\|\tilde{A}^{\alpha_k}u_{k-1}\|,$$

(5.8)
$$\|\psi_{k,j}^{(1)}\| \le \frac{c}{\alpha_k}\exp\left(-\sqrt{\frac{\pi d\alpha_k}{2}}(N_1+1)\right)\int_{-1}^{1}\|\tilde{A}^{\alpha_k}\tilde{f}(t,v_k(t))\|dt,$$

$$\|\psi_{k,j}^{(2)}\| \le \frac{c}{\alpha_k}\ln N\rho^{-N}\sup_{z\in D_{\rho_k}}\|\tilde{A}^{\alpha_k}\tilde{f}(z,v_k(z))\|,$$

$$\|\psi_{k,j}^{(3)}\| \le \frac{c^*}{\alpha_k}\frac{L\tau}{2}|||\vec{z}_k|||,$$

where α_k are some positive numbers, D_{ρ_k} are the analyticity ellipses for $\tilde{A}^{\alpha_k}\tilde{f}(z,v_k(z))$ and $|||\vec{z}_k||| = \max_{1\le j\le N}\|z_{k,j}\|$. Choosing $\tau = 1/K$ such that

(5.9)
$$\frac{c^*}{\alpha_k}\frac{L\tau}{2} < 1$$

we obtain from (5.6), (5.8)

(5.10)
$$|||\vec{z}_k||| = \max_{1\le j\le N}\|z_{k,j}\| \le \frac{c(\tau/2)^{\alpha_k}}{\alpha_k - c^*L\tau/2}\left\{\|A^{\alpha_k}z_{k-1}\|\right.$$
$$+ \left[\int_{t_{k-1}}^{t_k}\|A^{\alpha_k}f(t,u(t))\|dt + \|A^{\alpha_k}u(t_{k-1})\|\right]\exp\left(-\sqrt{\frac{\pi d\alpha_k}{2}}(N_1+1)\right)$$
$$+ \frac{\tau}{2}\ln N\rho^{-N}\sup_{z\in D_{\rho_k}}\|A^{\alpha_k}f(t_k(z),u(t_k(z)))\|\left.\right\},$$
$$k = 2,3,...,K,$$
$$t_k(z) = t_{k-1}\frac{1-z}{2} + t_k\frac{1+z}{2}.$$

Equalizing the exponents by setting $N \asymp \sqrt{N_1}$ (i.e. the number of the interpolation points on each subinterval must be proportional to the square root of the number of nodes in the Sinc approximation of the operator exponential) we obtain from (5.10)

(5.11)
$$|||A^{\alpha_{k+1}}\vec{z}_k||| \le \frac{c(\tau/2)^{\alpha_k}}{\alpha_k - \alpha_{k+1} - c^*L\tau/2}\left\{|||A^{\alpha_k}z_{k-1}|||\right.$$
$$+ \ln N_1 e^{-c_1\sqrt{N_1}}\left[\int_{t_{k-1}}^{t_k}\|A^{\alpha_k}f(t,u(t))\|dt + \|A^{\alpha_k}u(t_{k-1})\|\right.$$
$$+ \frac{\tau}{2}\sup_{z\in D_{\rho_k}}\|A^{\alpha_k}f(t_k(z),u(t_k(z)))\|\left.\right]\left.\right\},$$
$$k = 2,3,...,K,$$

where α_k satisfy

$$(5.12) \qquad \begin{aligned} &\alpha_k - \alpha_{k+1} - c^* L\tau/2 > 0, \\ &0 < \alpha_k \leq \sigma, \quad k = 1, 2, ..., K. \end{aligned}$$

Taking into account that $z_0 = 0$ we have for $k = 1$ the estimate

(5.13)

$$|||A^{\alpha_2}\vec{z}_k||| \leq \frac{c(\tau/2)^{\alpha_1}}{\alpha_1 - \alpha_2 - c^* L\tau/2} \ln N_1 e^{-c_1\sqrt{N_1}} \left[\|A^{\alpha_1} u_0\| + \int_{t_0}^{t_1} \|A^{\alpha_1} f(t, u(t))\| dt \right.$$

$$\left. + \frac{\tau}{2} \sup_{z \in D_{\rho_1}} \|A^{\alpha_k} f(t_k(z), u(t_k(z)))\| \right].$$

Estimate (5.11) can be rewritten in the form

$$(5.14) \qquad w_k \leq \mu_k(g_k + w_{k-1}), \quad k = 1, 2, ..., K,$$

with

$$w_k = |||A^{\alpha_2}\vec{z}_k|||, \quad \mu_k = \frac{c(\tau/2)^{\alpha_1}}{\alpha_k - \alpha_{k+1} - c^* L\tau/2},$$

$$(5.15) \qquad g_k = \ln N_1 e^{-c_1\sqrt{N_1}} \left[\int_{t_{k-1}}^{t_k} \|A^{\alpha_k} f(t, u(t))\| dt + \|A^{\alpha_k} u(t_{k-1})\| \right.$$

$$\left. + \frac{\tau}{2} \sup_{z \in D_{\rho_k}} \|A^{\alpha_k} f(t_k(z), u(t_k(z)))\| \right]$$

which yields

$$(5.16) \qquad w_k \leq \mu_k(g_k + w_{k-1}), \quad k = 1, 2, ..., K,$$

and further recursively

$$(5.17) \qquad w_k \leq \mu_k g_k + \mu_{k-1}\mu_k g_{k-1} + \cdots + \mu_1\mu_2 \cdots \mu_k g_1.$$

Conditions (5.12) imply

$$(5.18) \qquad 0 < \alpha_{k+1} < \alpha_1 - c^* Lk\tau/2 > 0, \quad k = 1, 2, ..., K.$$

Let us choose $\alpha_1 = \sigma$, $\varepsilon \in (0, \frac{\sigma - c^* L/2}{c^* L})$ and

$$(5.19) \qquad \alpha_{k+1} = \sigma - \left(\frac{1}{2} + \varepsilon \right) c^* Lk\tau, \quad k = 1, 2, ..., K.$$

Then we have

$$(5.20) \qquad \rho_k = \frac{c(\tau/2)^{\alpha_k}}{\varepsilon c^* L\tau} = \frac{c(\tau/2)^{\alpha_k - 1}}{2\varepsilon c^* L} < \frac{c}{2\varepsilon c^* L}(\tau/2)^{\sigma - (0.5+\varepsilon)c^* L} = q$$

and (5.17) implies

$$(5.21) \qquad \max_{1 \leq k \leq K} w_k \leq \max\{q^K, q\} \sum_{p=1}^{K} g_p$$

or
(5.22)
$$\max_{1\leq k\leq K}|||A^{\alpha_2}\vec{z}_k|||$$

$$\leq \max\{q^K, q\} \ln N_1 e^{-c_1\sqrt{N_1}} \sum_{k=1}^{K} \left[\int_{t_{k-1}}^{t_k} \|A^{\alpha_k} f(t, u(t))\| dt + \|A^{\alpha_k} u(t_{k-1})\| \right.$$

$$\left. + \frac{\tau}{2} \sup_{z\in D_{\rho_k}} \|A^{\alpha_k} f(t_k(z), u(t_k(z)))\| \right].$$

Thus, we have proven the following second main result of this paper concerning the rate of convergence of Algorithm **A2**.

Theorem 5.1. *Let A be a densely defined closed strongly positive linear operator with the domain $D(A)$ in a Banach space X and let the assumptions* **(i)**, **(ii)**, **(iii)** *hold. If the solution of the nonlinear problem* (1.1) *belongs to the domain $D(A^\sigma)$ with $\sigma > c^* L/2$, then Algorithm* **A2** *possesses a uniform with respect to t exponential convergence rate with estimate* (5.22), *provided that $N \asymp \sqrt{N_1}$ and the chosen number of subintervals K satisfies* (5.9).

6. IMPLEMENTATION OF THE ALGORITHM

Algorithm (3.18) represents a nonlinear system of algebraic equations which can be solved by the fixed point iteration

$$y_j^{(m+1)} = \frac{h}{2\pi i} \sum_{k=-N_1}^{N_1} \mathcal{F}_h(x_{j,N}, kh)$$

(6.1)

$$+ \frac{h}{4\pi i} \int_{-1}^{x_j} \sum_{k=-N_1}^{N_1} F_A(\xi, x_j - \eta) P_{N-1}(\eta; f(\cdot, y^{(m)})) d\eta,$$

$$j = 1, ..., N, m = 0, 1, ...$$

Since the operator $\mathcal{A}$ is contractive we obtain the following inequality:

(6.2)
$$|||y_j^{(m+1)} - y_j^{(m)}||| \leq Lc^* |||y_j^{(m)} - y_j^{(m-1)}|||$$

which justifies the convergence of the fixed point iteration (6.1) with the speed of a geometric progression with the denominator $Lc^* < 1$, provided the assumptions of Theorem 4.4 hold.

Let us estimate the asymptotical computational costs of our method and a possible alternative polynomially convergent method (e.g. step-by-step implicit Euler method) in order to arrive at a given tolerance ε. Assuming the time step τ and the spatial step h in the Euler scheme to be equal we have asymptotically to make $\frac{t^*}{\varepsilon}$ steps in order to arrive at a tolerance ε at a given fixed point $t = t^*$. At each step the nonlinear equation $\tau f(t_{k+1}, y_{k+1}) - (\tau A y_{k+1} + I) = y_k$ should be solved, where y_k is an approximation for $u(t_k)$. Assuming the computational costs for the solution of this nonlinear equation to be M we arrive at the total computational costs for the Euler method $T_E \asymp t^* M/\varepsilon$. From the asymptotical relation $\ln N_1 e^{-c\sqrt{N_1}} < e^{-c_1\sqrt{N_1}} \asymp \varepsilon$ we obtain that in our algorithm $N \asymp \sqrt{N_1} \asymp \ln(1/\varepsilon)$. It is natural to assume that the computational costs for the numerical solution of the nonlinear equation (6.1) (or (3.19)) are not greater than $NM \asymp \sqrt{N_1} M$. Then the total costs of our algorithm are $T_O \asymp M \ln(1/\varepsilon) \ll T_E$ for ε small enough.

Example 6.1. Let us consider the nonlinear initial value problem

(6.3)
$$u'(t) + u(t) = \mu e^{-2t} - \mu[u(t)]^2, \ t \in (-1, 1],$$
$$u(-1) = e$$

with the exact solution $u(t) = e^{-t}$ (independent of μ). The equivalent Volterra integral equation is

(6.4)
$$u(t) = \varphi(t) - \mu \int_{-1}^{t} e^{-(t-s)} u^2(s) ds,$$

where

(6.5)
$$\varphi(t) = e^{-t} + \mu[e^{1-t} - e^{-2t}].$$

Algorithm (3.18) combined with the fixed point iteration takes in this case the form

(6.6)
$$y_j^{(m+1)} = \varphi_j - \mu \sum_{p=1}^{N} \alpha_{p,j} [y_p^{(m)}]^2,$$
$$y_j^{(0)} = 1/2, \ j = 1, ..., N, \ m = 0, 1, ...,$$

where

(6.7)
$$\alpha_{p,j} = e^{-x_{j,N}} \int_{-1}^{x_{j,N}} L_{p,N-1}(s) e^s ds,$$
$$y_j = y(x_{j,N}), \ \varphi_j = \varphi(x_{j,N}).$$

The algorithm was implemented in Maple 8 (Digits=30) for $\mu = 1/4$ where integrals (6.7) were computed analytically. We denote by It the number of iterations necessary to satisfy the interruption criterium $|y_{J,N}^{(m+1)} - y_{J,N}^{(m)}| < e^{-N} \cdot 10^{-2}$ and accept $y_j^{(It)} = y_{J,N}^{(m+1)}$ as the approximate solution. The error is computed as $\varepsilon_N = \|u - y\|_{N,\infty} = \max_{1 \le j \le N} |u(x_{j,N}) - y_j^{(It)}|$. The numerical results are given by Table 2 and confirm our theory.

TABLE 2. The error of algorithm (3.18) for problem (6.3).

N	ε_N	It
2	0.129406	6
4	0.626486 e-2	8
8	0.181353 e-5	9
16	0.162597 e-14	16
32	0.110000 e-28	26

Example 6.2. Let us consider the problem

(6.8)
$$\frac{\partial u}{\partial t} + Au = f(t, u(t)),$$
$$u(-1) = u_0$$

with the linear operator A given by

(6.9)
$$D(A) = \{w(x) \in H^2(0, 1) : \ w'(0) = 0, w'(1) = 0\},$$
$$Av = -w'' \ \forall w \in D(A),$$

with the nonlinear operator f given by

(6.10) $$f(t, u) = -2tu^2$$

and with the initial condition given by

(6.11) $$u_0 = u(-1, x) = 1/2.$$

Since the numerical algorithm above supposes that the operator coefficient is strongly positive, we shift its spectrum by the variables transform $u(t, x) = e^{d^2 t} v(t, x)$ with a real number d. Then we obtain the problem

(6.12) $$\frac{\partial v}{\partial t} + A_d v = f_d(t, v(t)),$$
$$v(-1) = v_0$$

with the linear operator A_d given by

(6.13) $$D(A_d) = D(A),$$
$$A_d w = Aw + d^2 w \ \forall w \in D(A_d),$$

with the nonlinear operator f_d given by

(6.14) $$f_d(t, v) = -2te^{d^2 t} v^2$$

and with the initial condition

(6.15) $$v_0 = v(-1, x) = e^{d^2}/2.$$

It is easy to check that the exact solution of this problem is

(6.16) $$v(t, x) = e^{-d^2 t}/(1 + t^2).$$

The equivalent Volterra integral equation for v has the form

(6.17) $$v(t, x) = \frac{1}{2} e^{-A_d(t+1)} e^{d^2} - 2 \int_{-1}^{t} e^{-A_d(t-s)} s e^{d^2 s} [v(s, \cdot)]^2 ds.$$

Returning to the unknown function u the integral equation takes the form

(6.18) $$u(t, x) = \frac{1}{2} e^{-A_d(t+1)} e^{d^2(t+1)} - 2 \int_{-1}^{t} e^{-A_d(t-s)} s e^{-d^2 s} [u(s, \cdot)]^2 ds.$$

Our algorithm was implemented in Maple with numerical results given by Table 3 where $\varepsilon_N = \max_{1 \leq j \leq N} \varepsilon_{j,N}$, $\varepsilon_{j,k,N} = |u(x_{j,N}, kh) - y_{j,k}|$, $j = 1, ..., N$, $k = -N_1, ..., N_1$. The numerical results are in good agreement with Theorem 4.4.

TABLE 3. The error of algorithm (3.18) for problem (6.3).

N	ε_N	It
4	0.8 e-1	12
8	0.7 e-3	10
16	0.5 e-6	11
32	0.3 e-12	12

Example 6.3. This example deals with the two-dimensional nonlinear problem

$$\frac{\partial u}{\partial t} + Au = f(t, u(t)),$$

(6.19)

$$u(0) = u_0$$

where

$$D(A) = \{w(x, y) \in H^2(\Omega) : \ w|_{\partial\Omega} = 0\},$$

(6.20)

$$Av = -\Delta v \ \forall v \in D(A),$$

$$\Omega = [0, 1] \times [0, 1]$$

with the nonlinear operator f given by

(6.21)

$$f(t, u) = -u^3 + e^{-6\pi^2 t} \sin^3 \pi x \sin^3 \pi y$$

and with the initial condition given by

(6.22)

$$u_0 = u(0, x, y) = \sin \pi x \sin \pi y.$$

The exact solution is given by $u = e^{-2\pi^2 t} \sin \pi x \sin \pi y$. Algorithm (3.18) with $N = \sqrt{N_1}$ Chebyshev-Gauss-Lobatto nodes combined with the fixed point iteration provides the error which is presented in Table 4.

TABLE 4. The error ε_N of algorithm (3.18) for problem (6.19)–(6.22).

N	ε_N	It
4	.3413e-6	12
8	.1761e-6	10
16	.8846e-7	14
32	.5441e-8	14

Example 6.4. Let us consider again the nonlinear initial value problem (6.3) and apply the Algorithm **A2** for various values of the Lipschitz constant 2μ. Inequality (4.47) guarantees the convergence of algorithm (3.18) combined with the fixed point iteration for $\mu < 0.4596747673$. Numerical experiments indicate the convergence also for $\mu > 0.4596747673$, but beginning with $\mu \approx 1$ the process becomes divergent and Algorithm **A2** should be applied. The corresponding results for various μ are presented in Table 5.

TABLE 5. The results of Algorithm **A2** for problem (6.3) with various values of the Lipschitz constant μ.

μ	K	It
0.9	1	22
1	2	20
10	32	20
20	50	25
50	128	25
100	256	24

Here the degree of the interpolation polynomial is $N = 16$, K is the number of subintervals of the whole interval $[-1; 1]$, It denotes the number of the iterations in order to arrive at the accuracy $\exp(-N) * 0.01$.

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