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ZEROS OF THE DAVENPORT-HEILBRONN COUNTEREXAMPLE

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Abstract. We compute zeros off the critical line of a Dirichlet series considered by H. Davenport and H. Heilbronn. This computation is accomplished by deforming a Dirichlet series with a set of known zeros into the Davenport-Heilbronn series.

1. Introduction

Let $\xi = \big(\sqrt{10-2\sqrt{5}}-2\big)/(\sqrt{5}-1)$. For $s=\sigma+it$ with $\sigma>1$, let

$$f_1(s) = 1 + \frac{\xi}{2^s} - \frac{\xi}{3^s} - \frac{1}{4^s} + \frac{0}{5^s} + \cdots \tag{1}$$

be a Dirichlet series with periodic coefficients of period 5. Then $f_1(s)$ defines an entire function satisfying the following functional equation

$$f(s) = T^{-s+\frac{1}{2}}\,\chi_2(s)\,f(1-s) \tag{2}$$

with $T=5$ and

$$\chi_2(s) = 2(2\pi)^{s-1}\,\Gamma(1-s)\cos\big(\frac{\pi s}{2}\big). \tag{3}$$

In 1936, H. Davenport and H. Heilbronn (see [2]) showed that $f_1(s)$, as defined in (1), has zeros off the critical line $\sigma=1/2$. In 1994, R. Spira (see [3]) computed the following zeros of the Davenport-Heilbronn example:

$$.808517+85.699348i, \qquad .650830+114.163343i,$$
$$.574356+166.479306i, \qquad .724258+176.702461i.$$

In this note we present a scheme for computing additional zeros of the Davenport-Heilbronn Dirichlet series.

2. Continuity of zeros

In order to compute zeros of f_1, let us consider

$$f_0(s) = \Big(1+\frac{\sqrt{5}}{5^s}\Big)\zeta(s), \tag{4}$$

where $\zeta(s)$ is the Riemann zeta function. A good number of zeros of $\zeta(s)$ have been computed and are readily available. On the other hand

$$1+\frac{\sqrt{5}}{5^s}=0 \quad \text{if and only if} \quad s=\frac{1}{2}+\frac{2k+1}{\log 5}\pi i \quad \text{with} \quad k\in\mathbf{Z}.$$

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For each $\tau \in [0,1]$, let

$$f_\tau = f_0 \cdot (1-\tau) + f_1 \cdot \tau. \tag{5}$$

The next theorem shows that if ρ_0 is a zero of f_0, and $\tau > 0$ is small, then f_τ has a zero ρ_τ in a small neighborhood of ρ_0. Beginning with ρ_0 as initial data, it is easy (provided $\tau > 0$ is small) to numerically compute a zero ρ_τ of f_τ in a small neighborhood of ρ_0. Repeating this process a number of times, we end up with a zero ρ_1 of the Davenport-Heilbronn series f_1.

Theorem 1. *Let q be a fixed positive integer. Let $a_\tau(j): \mathbf{R} \times \mathbf{N} \to \mathbf{C}$ be a sequence of continuous functions such that $a_\tau(j+q) = a_\tau(j)$ for all $\tau \in \mathbf{R}$ and all $j \in \mathbf{N}$. Let $f_\tau(s)$ be the meromorphic function, defined initially for $\sigma > 1$ by*

$$f_\tau(s) = \sum_{j=1}^{\infty} \frac{a_\tau(j)}{j^s},$$

and extended to the whole complex plane by analytic continuation. Let ρ be such that $0 < \Re\mathrm{e}(\rho) < 1$ and $f_0(\rho) = 0$. If $\delta > 0$ and $\tau \in \mathbf{R}$ are sufficiently small, then there exists s such that $f_\tau(s) = 0$ and $|s - \rho| < \delta$.

Thus, our scheme of computation of zeros of f_1 is to keep track of zeros of f_0 while performing a 'deformation' of f_0 into f_1. By keeping track of the first known zeros of f_0 as defined in (4), we found the following additional zeros of the Davenport-Heilbronn Dirichlet series f_1 defined in (1):

$.86953 + 240.4046i$,	$.81955 + 320.8764i$,	$.76822 + 331.0502i$,
$.62850 + 366.6409i$,	$.81587 + 411.7967i$,	$.70882 + 440.4845i$,
$.51591 + 520.9438i$,	$.84695 + 531.2797i$,	$.72953 + 548.9067i$,
$.78655 + 566.5097i$,	$.58285 + 595.0233i$,	$.62825 + 611.7750i$,
$.61076 + 646.9868i$,	$.76059 + 657.1083i$,	$.78870 + 692.8924i$,
$.77736 + 737.7669i$,	$.85300 + 783.6530i$,	$.66855 + 811.7657i$,
$.56194 + 847.4657i$,	$.85610 + 857.2958i$,	$.68089 + 864.1180i$,
$.68843 + 892.1490i$,	$.75935 + 921.1726i$,	$.76249 + 983.7521i$,
$.69140 + 1012.019i$,	$.69809 + 1018.795i$,	$.58613 + 1029.004i$,
$.61106 + 1078.490i$,	$.85462 + 1092.454i$,	$.60577 + 1109.548i$.

3. Two hypotheses

In this section, we consider the case in which f_0 and f_1 are two linearly independent Dirichlet series satisfying a given functional equation. For example we might take f_0 to be as in equation (4) above, or we might take f_0 to be

$$L\big(s, \chi_2^{(5)}\big) = 1 - \frac{1}{2^s} - \frac{1}{3^s} + \frac{1}{4^s} + \frac{0}{5^s} + \cdots.$$

Notice that f_0 as given in (4) and $L\big(s, \chi_2^{(5)}\big)$ both satisfy the functional equation

$$f(s) = T^{-s+\frac{1}{2}}\, \chi_1(s)\, f(1-s) \tag{6}$$

with $T = 5$ and

$$\chi_1(s) = 2(2\pi)^{s-1}\, \Gamma(1-s) \sin\Big(\frac{\pi s}{2}\Big). \tag{7}$$

Thus we have two linearly independent Dirichlet series satisfying a given functional equation. Hence, by taking appropriate linear combinations of these, we can produce a Dirichlet series f_1 satisfying the above functional equation and having a zero off the critical line, in fact, at any preassigned place in the complex plane. With these f_0 and f_1, we let f_τ be as in (5). Then f_τ satisfies the functional equation (6) for all $\tau \in [0,1]$.

Since f_τ satisfies (6), then its nonreal zeros lie symmetrically about the critical line $\sigma = 1/2$. Hence, by Theorem 1 in section §2, a simple zero must move along the critical line. If we assume that all zeros of f_0 are simple, how then might we obtain any zero of f_1 lying off the critical line? It is easy to see that there must exist $0 \le \tau^* < 1$ such that f_{τ^*} has a zero in the critical line with an even multiplicity.

Loosely speaking, we might say that zeros of multiplicity greater than one must exist before the Riemann hypothesis fails.

4. Other periodic Dirichlet series

The Dirichlet series considered by Davenport and Heilbronn satisfies a functional equation akin to the functional equation satisfied by the Riemann zeta function. Moreover, this Dirichlet series of Davenport and Heilbronn is the unique solution of its functional equation. It is natural to consider all those periodic Dirichlet series arising as the unique solution to a fixed functional equation of the type satisfied by the Riemann zeta function. The following result will help us determine all such Dirichlet series; see [1].

Theorem 2. *Let $f(s)$ be a T-periodic Dirichlet series. Let $\chi_1(s)$ and $\chi_2(s)$ be as in* (7) *and* (3) *respectively. Let*

$$\mathcal{V}_{\alpha,\beta} = \Big\{ f(s) \,:\, f(s) = (-1)^\alpha \, T^{-s+\frac{1}{2}} \, \chi_\beta(s) f(1-s) \Big\}.$$

Let $d_1 = \dim \mathcal{V}_{0,1}$, $d_2 = \dim \mathcal{V}_{1,1}$, $d_3 = \dim \mathcal{V}_{0,2}$, and $d_4 = \dim \mathcal{V}_{1,2}$, where $\dim \mathcal{V}_{\alpha,\beta}$ is the dimension of $\mathcal{V}_{\alpha,\beta}$ as a vector space. Then d_j is given by the following table:

T	d_1	d_2	d_3	d_4
$4m$	$m+1$	m	m	$m-1$
$4m+1$	$m+1$	m	m	m
$4m+2$	$m+1$	$m+1$	m	m
$4m+3$	$m+1$	$m+1$	$m+1$	m

Thus, for $T = 2$,

$$\big(1 + \frac{\sqrt{2}}{2^s}\big)\zeta(s), \qquad \big(1 - \frac{\sqrt{2}}{2^s}\big)\zeta(s)$$

is the list of all 2-periodic Dirichlet series which are the unique solution of a functional equation.

For $T = 3$,

$$L\big(s, \chi_1^{(3)}\big), \qquad \big(1 + \frac{\sqrt{3}}{3^s}\big)\zeta(s), \qquad \big(1 - \frac{\sqrt{3}}{3^s}\big)\zeta(s)$$

is the list of all 3-periodic Dirichlet series which are the unique solution of a functional equation. Here $\chi_1^{(3)}$ is the nonprincipal character modulo 3.

For $T = 4$,

$$L\big(s, \chi_1^{(4)}\big), \qquad \big(1 - \frac{2}{4^s}\big)\zeta(s)$$

is the list of all 4-periodic Dirichlet series which are the unique solution of a functional equation. Here $\chi_1^{(4)}$ is the nonprincipal character modulo 4.

For $T = 5$,

$$\big(1 - \frac{\sqrt{5}}{5^s}\big)\zeta(s), \quad f_1(s), \quad f_2(s) = 1 - \frac{1/\xi}{2^s} + \frac{1/\xi}{3^s} - \frac{1}{4^s} + \frac{0}{5^s} + \cdots \tag{8}$$

is the list of all 5-periodic Dirichlet series which are the unique solution of a functional equation. Here f_1 is the Dirichlet series of Davenport and Heilbronn, and the constant ξ in the definition of f_2 is defined in the Introduction, §1.

For $T = 6$,

$$\Big(1 - \frac{1-\sqrt{2}}{1+2^s}\Big)L(s, \chi_1^{(6)}), \qquad \Big(1 - \frac{1+\sqrt{2}}{1+2^s}\Big)L(s, \chi_1^{(6)})$$

is the list of all 6-periodic Dirichlet series which are the unique solution of a functional equation. Here $\chi_1^{(6)}$ is the nonprincipal character modulo 6.

For $T = 7$,

$$f_3(s) = 1 - \frac{1+\alpha}{2^s} - \frac{\alpha}{3^s} + \frac{\alpha}{4^s} + \frac{1+\alpha}{5^s} - \frac{1}{6^s} + \frac{0}{7^s} + \cdots \tag{9}$$

is the only 7-periodic Dirichlet series which is the unique solution of a functional equation. Here $\alpha = 0.80194\cdots$.

For $T = 8$,

$$\big(1 - \frac{\sqrt{2}}{2^s}\big)L(s, \chi_1^{(8)})$$

is the only 8-periodic Dirichlet series which is the unique solution of a functional equation. Here $\chi_1^{(8)}(3) = -1$, $\chi_1^{(8)}(5) = 1$, $\chi_1^{(8)}(7) = -1$ is a Dirichlet character.

Of all these periodic Dirichlet series arising as the unique solution of a functional equation, only three are not Euler products. These three series are f_1 as given in (1), f_2 as given in (8) and f_3 as given in (9).

Now we list a few zeros off the critical line of f_2 as given in (8). Notice that most of these zeros have real part greater than 1:

$$\begin{array}{lll}
2.30862 + 8.91836i, & 1.94374 + 18.8994i, & 2.09106 + 26.5450i, \\
2.15626 + 36.5556i, & 1.50497 + 44.8057i, & 2.33262 + 54.4201i, \\
1.78509 + 64.3711i, & 2.17279 + 72.0637i, & 0.69340 + 77.3469i, \\
2.05503 + 82.0598i, & 1.83279 + 89.9631i, & 2.34551 + 99.8614i, \\
1.18952 + 107.106i, & 1.33795 + 109.439i, & 2.22293 + 117.572i.
\end{array}$$

Finally, we list a few zeros off the critical line of f_3 as given in (9). Notice that most of these zeros have real part greater than 1:

$$\begin{array}{lll}
1.34746 + 17.5286i, & 1.06162 + 28.4426i, & 1.30492 + 45.5320i, \\
1.01460 + 56.2793i, & 0.91718 + 63.7111i, & 1.33196 + 80.3522i, \\
1.22180 + 91.1756i, & 1.22009 + 108.402i, & 0.92165 + 119.323i, \\
1.28500 + 126.482i, & 1.08964 + 137.285i, & 0.91608 + 143.175i, \\
0.78002 + 146.163i, & 1.28909 + 154.268i, & 0.65384 + 161.521i.
\end{array}$$

5. Proof of Theorem 1

In order to prove the theorem in §2, we let

$$A_\tau = \frac{1}{q}\sum_{j=1}^{q} a_\tau(j) \qquad \text{and} \qquad A_\tau(x) = \sum_{j\le x} a_\tau(j).$$

For $\sigma > 0$ we have

$$f_\tau(s) = \frac{A_\tau s}{s-1} + s\int_1^\infty \frac{A_\tau(x) - A_\tau x}{x^{s+1}}dx.$$

Assume $\rho \in \mathbf{C}$ is such that $0 < \Re\mathrm{e}(\rho) < 1$ and $f_0(\rho) = 0$. There exist $\delta > 0$ such that $f_0(s)$ does not vanish for $0 < |s-\rho| \le \delta$. Let

$$\epsilon = \min\big\{|f_0(s)| \,:\, |s-\rho| = \delta\big\}.$$

Since

$$|A_\tau(x) - A_\tau x - A_0(x) + A_0 x| \le 2\sum_{j=1}^{q} |a_\tau(j) - a_0(j)|$$

and $a_\tau(j) \colon \mathbf{R}\times\mathbf{N} \to \mathbf{C}$ are continuous functions of τ, then it follows that

$$|f_\tau(s) - f_0(s)| \le \left\{\frac{1}{q}\left|\frac{s}{s-1}\right| + 2\left|\frac{s}{\sigma}\right|\right\}\sum_{j=1}^{q}|a_\tau(j) - a_0(j)| < \epsilon$$

provided $|\tau|$ and δ are sufficiently small and $|s-\rho| = \delta$. By Rouche's theorem

$$f_\tau(s) = f_0(s) + \Big\{f_\tau(s) - f_0(s)\Big\}$$

vanishes for some s such that $|s-\rho| < \delta$. □

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