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THE LUCAS-PRATT PRIMALITY TREE

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Abstract. In 1876, E. Lucas showed that a quick proof of primality for a prime p could be attained through the prime factorization of $p-1$ and a primitive root for p. V. Pratt's proof that PRIMES is in NP, done via Lucas's theorem, showed that a certificate of primality for a prime p could be obtained in $O(\log^2 p)$ modular multiplications with integers at most p. We show that for all constants $C \in \mathbb{R}$, the number of modular multiplications necessary to obtain this certificate is greater than $C \log p$ for a set of primes p with relative asymptotic density 1.

1. Introduction

Over a quarter of a century before Agrawal, Kayal, and Saxena showed PRIMES is in P in [1], V. Pratt showed in [11] that PRIMES is in NP by utilizing the following theorem of Lucas:

Theorem 1. *Suppose $p > 1$ is an odd integer and*

$$\begin{cases} a^{\frac{p-1}{2}} \equiv -1 \mod p, \\ a^{\frac{p-1}{2q}} \not\equiv -1 \mod p \text{ for every odd prime } q \mid p-1. \end{cases} \tag{1.1}$$

Then p is prime and a is a primitive root of p. Conversely, if p is an odd prime, then every primitive root a of p satisfies conditions (1.1).

In 1877, Lucas [8] stated a result essentially equivalent to Theorem 1; it is based on his work from a year earlier. The actual statement presented here can be found in [3, Theorem 4.1.8].

To achieve this Lucas-Pratt certificate of primality, we would need to find a primitive root a modulo p, and we would need to certify each of the primes q appearing in the second condition. By charting this process, we find what is called a Lucas-Pratt tree. For example, to certify 9461 in this manner, we have the following tree:

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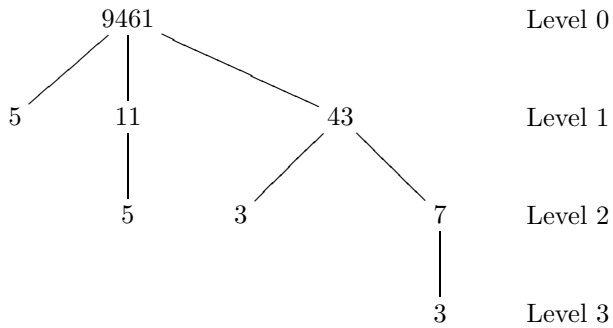

Notice that the branches beneath a prime r correspond to the odd primes dividing $\varphi(r) = r-1$. Pratt showed that the number of primes appearing in the Lucas-Pratt tree for a prime p, denoted $N(p)$, is $O(\log p)$. In fact, he showed:

Theorem 2. $N(p) < \log_2 p$.

Here $\log_2 p$ denotes the base-2 logarithm of p.

Let $M(p)$ denote the number of modular multiplications of integers with size at most p (using a standard modular exponentiation algorithm) needed to verify the primality of an integer p by this method. As a corollary to the above theorem, Pratt showed:

Corollary 1. $M(p) = O(\log^2 p)$.

See [11] for details of either proof, or [3, Theorem 4.1.9] for a simple exposition of both results. It is clear that $M(p)$ is at least $\log_2 p - 2$, as this is a lower bound for the number of multiplications necessary to evaluate the first condition of Theorem 1.

A relevant result of Pomerance is the following theorem (found in [10]):

Theorem 3. *For every prime p there is a proof that it is prime which requires for its verification $(\frac{5}{2} + o(1)) \log_2 p$ multiplications modulo p.*

Thus, in principle, for each prime p there exists a primality proof of order $O(\log p)$, and so one might ask if Pratt's result can be improved to $M(p) = O(\log p)$.

For any positive integer n, let $\varphi_k(n)$ denote the k-th iterate of the Euler φ-function at n, and let $\varphi_0(n) = n$. Define $F(p)$ as the product of the distinct odd primes dividing $\prod_{k\geq 0} \varphi_k(p)$. An odd prime $q \neq p$ appears in level $k > 1$ of the Lucas-Pratt tree for p if it divides $\varphi(r)$ for some prime r in level $k-1$ of the tree. Since this means that $q \mid \varphi_k(p)$, we have that $F(p)$ is equal to the squarefree product of the primes appearing in the Lucas-Pratt tree for p. As an example, $F(9461) = 9461 \cdot 5 \cdot 11 \cdot 43 \cdot 3 \cdot 7 = 469880565$.

We first prove a theorem analogous to a result in [7] and suggested in the same paper. The proof of this theorem follows the same plan as in [7], but certain extra tools are brought into play.

Theorem 4. *For each $\epsilon > 0$, the set of primes p for which*

$$\varphi(F(p)) > p^{(1-\epsilon)\frac{\log\log p}{\log\log\log p}}$$

has relative asymptotic density 1 within the set of all primes.

We use this theorem to obtain the following result:

Theorem 5. *Let $\epsilon > 0$ be arbitrary and let $C(p) = (1-\epsilon)\log\log p/\log\log\log p$. Then $M(p) > C(p)\log_2 p$ for a set of primes p with relative asymptotic density 1. In particular, for all $C \in \mathbb{R}$, $M(p) > C\log p$ for almost all primes p.*

As noted in Pratt's paper, naive multiplication modulo p has bit complexity $O(\log^2 p)$, and so, using naive multiplication, the bit complexity of the certificate of the prime p is $O(\log^4 p)$. One might ask if the bit complexity can be bounded below in a manner analogous to what we do here with $M(p)$. Limited attempts to use the result of this paper to answer this question have met with little success, but it would be interesting to see if a similar lower bound on the bit complexity is possible.

In [2], the related question of the height of the Lucas-Pratt tree is considered by Banks and Shparlinski, who prove a lower bound on the height for almost all primes p.

Throughout, p, q, and r will always be primes, and n will always be a positive integer.

2. Technical preparation

We now begin developing the tools we will use to prove Theorem 4.

Lemma 1. *For any value of $x \geq e^e$ and any prime $p < x$, the following holds:*

$$\sum_{\substack{r \leq \frac{x}{2} \\ r \equiv 1 \bmod p}} \frac{1}{r\log\frac{x}{r}} \leq \frac{30\log\log x}{p\log\frac{x}{p}}.$$

Proof. We use the Brun-Titchmarsh inequality of [9]; namely, for coprime integers k and ℓ, the number of primes $q \leq x$ with $q \equiv \ell \bmod k$, denoted $\pi(x;k,\ell)$, satisfies

$$\pi(x;k,\ell) \leq \frac{2x}{\varphi(k)\log\frac{x}{k}} \quad \text{for} \quad x > k.$$

For prime $p \geq 2$ and $(\ell, p) = 1$, we can use the fact that $\varphi(p) = p-1$ to see that

$$\pi(x;p,\ell) \leq \frac{2x}{\varphi(p)\log\frac{x}{p}} \leq \frac{4x}{p\log\frac{x}{p}} \quad \text{for} \quad x > p. \tag{2.1}$$

Note that the first prime $r \equiv 1 \bmod p$ is at least $\frac{3p}{2}$, so we need only consider $p \leq \frac{x}{3}$. We use partial summation to obtain

$$\begin{aligned}
\sum_{\substack{r \leq \frac{x}{2} \\ r \equiv 1 \bmod p}} \frac{1}{r\log\frac{x}{r}} &= \frac{\pi(\frac{x}{2};p,1)}{\frac{x}{2}\log 2} + \int_{\frac{3p}{2}}^{\frac{x}{2}} \pi(t;p,1)\Big(\frac{1}{t^2\log\frac{x}{t}} - \frac{1}{t^2(\log\frac{x}{t})^2}\Big)\,dt \\
&\leq \frac{4}{p\log\frac{x}{2p}\log 2} + \int_{\frac{3p}{2}}^{\frac{x}{2}} \frac{4}{p\,t\log\frac{t}{p}\log\frac{x}{t}}\,dt \\
&= \frac{4}{p\log\frac{x}{2p}\log 2} + \frac{4}{p\log\frac{x}{p}}\int_{\frac{3p}{2}}^{\frac{x}{2}} \Big(\frac{1}{\log\frac{t}{p}} + \frac{1}{\log\frac{x}{t}}\Big)\frac{dt}{t} \\
&= \frac{4}{p\log\frac{x}{2p}\log 2} + \frac{4}{p\log\frac{x}{p}} \cdot \Big(\log\log\frac{t}{p} - \log\log\frac{x}{t}\Big)\Big|_{\frac{3p}{2}}^{\frac{x}{2}}.
\end{aligned}$$

Since $\frac{4}{\log 2 \log \frac{x}{2p}} < \frac{16}{\log \frac{x}{p}}$ and $16 - 4\log\log 2 - 4\log\log 3/2 < 22$, we see that

$$\sum_{\substack{r \le \frac{x}{2} \\ r \equiv 1 \bmod p}} \frac{1}{r \log \frac{x}{r}} \le \frac{1}{p \log \frac{x}{p}} \Big(22 + 8 \log\log \frac{2x}{3p}\Big)$$
$$\le \frac{30}{p \log \frac{x}{p}} \log\log x,$$

where for the last step we use $x \ge e^e$. □

This allows us to prove the following proposition:

Proposition 1. *For all $x \ge e^e$, each prime $p < x$, and all positive integers k, the number of primes $q \le x$ with $p \mid \varphi_k(q)$ is bounded above by*

$$\frac{4x}{p \log \frac{x}{p}} (31 \log\log x)^{k-1}.$$

Proof. We prove the proposition by induction. For $k = 1$, we have

$$\sum_{\substack{q \le x \\ p \mid \varphi(q)}} 1 = \pi(x; p, 1),$$

and the result follows from (2.1).

For $k \ge 1$, assume the proposition holds up to k. Then if $p \mid \varphi_{k+1}(q)$, either $p^2 \mid \varphi_k(q)$ or there exists a prime $r \equiv 1 \bmod p$ such that $r \mid \varphi_k(q)$. Thus,

$$\sum_{\substack{q \le x \\ p \mid \varphi_{k+1}(q)}} 1 \le \sum_{\substack{q \le x \\ p^2 \mid \varphi_k(q)}} 1 + \sum_{r \equiv 1 \bmod p} \sum_{\substack{q \le x \\ r \mid \varphi_k(q)}} 1.$$

In the second term, r is an odd prime and it divides the even number $\varphi_k(q)$, so $r \le \frac{1}{2}\varphi_k(q) \le \frac{1}{2}q \le \frac{1}{2}x$. Hence, by the induction hypothesis and Lemma 1, we have

$$\sum_{\substack{q \le x \\ p \mid \varphi_{k+1}(q)}} 1 \le \frac{4x}{p \log \frac{x}{p}} (31 \log\log x)^{k-1} + \sum_{\substack{r \le \frac{x}{2} \\ r \equiv 1 \bmod p}} \frac{4x}{r \log \frac{x}{r}} (31 \log\log x)^{k-1}$$
$$\le \frac{4x}{p \log \frac{x}{p}} (31 \log\log x)^{k-1} \Big(1 + 30 \log\log x\Big)$$
$$\le \frac{4x}{p \log \frac{x}{p}} (31 \log\log x)^{k},$$

where for the last inequality we use $x \ge e^e$. This proves the proposition. □

We now state a lemma and a proposition from [7]. Define $\Omega(n)$ to be the number of prime divisors of n, including multiplicity. We have the following lemma, the proof of which is suggested in Exercise 05 of [5]:

Lemma 2. *There exists an absolute constant $C_1 > 0$ such that for each positive integer k,*

$$\sum_{\substack{n \le x \\ \Omega(n) \ge k}} 1 \le C_1 \frac{k}{2^k} x \log x.$$

Define

$$F_K(n) = \prod_{0 \le k \le K} \varphi_k(n).$$

Proposition 2. *There exists an absolute constant $C_2 > 0$ such that for all sufficiently large numbers x, all numbers $y \ge 1$, and all integers $K \ge 1$, the number of integers $n \le x$ with $p^2 \mid F_K(n)$ for some prime $p > y$ is at most*

$$\frac{x}{y} K (C_2 \log\log x)^{2K}.$$

Clearly, the bound in the proposition still holds if we restrict the count to prime n. Using the arguments from [4] and [7] together with a short computation, we may take $C_2 = 27$.

We need one last proposition before moving on to the proof of the theorems.

Proposition 3. *There exists an absolute constant $C_3 > 0$ such that for each $x \ge e^e$ and every positive integer K, the number of primes $p \le x$ for which $\Omega(F_K(p)) > 2(6 \log\log x)^K$ is bounded above by*

$$C_3 \frac{x}{\log^2 x} (31 \log\log x)^{K-1}.$$

Proof. By Lemma 2, for any $t \in (1, x]$,

$$\begin{aligned} \sum_{\substack{q \le t \\ \Omega(q-1) > 6 \log\log x}} 1 &\le \sum_{\substack{n \le t-1 \\ \Omega(n) > 6 \log\log x}} 1 \le \frac{6\, C_1 \log\log x}{2^{6 \log\log x}}\, t \log t \\ &\le 6\, C_1 \frac{t \log\log x}{(\log x)^{6 \log 2 - 1}} \le 6\, C_1 \frac{t \,\log\log x}{(\log x)^3}. \end{aligned} \tag{2.2}$$

Since

$$6\, C_1 \frac{t \log\log x}{\log^3 x} \le 6\, C_1 \frac{x}{\log^2 x},$$

the proposition holds for $K = 1$, provided $C_3 \ge 6\, C_1$; we now assume $K \ge 2$.

If $\Omega(q-1) \le 6 \log\log x$ for each prime q dividing $F_{K-1}(p)$, then for all integers $1 \le k \le K$, $\Omega(\varphi_k(p)) \le (6 \log\log x)^k$ and $\Omega(F_K(p)) \le 2(6 \log\log x)^K$. Therefore, if $\Omega(F_K(p)) > 2(6 \log\log x)^K$, there exists a prime q dividing $F_{K-1}(p)$ with $\Omega(q-1) > 6 \log\log x$. Clearly, such a q must be odd, and since $F_{K-1}(p) = \prod_{k \le K-1} \varphi_k(p)$, $q \mid \varphi_k(p)$ for some $k \le K-1$. If this $k = 0$, then $q = p$; if $k > 0$, $\varphi_k(p)$ is even, so $q \le \frac{x}{2}$. Hence, we need only consider either $q = p$ or $q \le \frac{x}{2}$. Thus, by (2.2) and Proposition 1, we have that

$$\begin{aligned} \sum_{\substack{p \le x \\ \Omega(F_K(p)) > 2(6 \log\log x)^K}} 1 &\le \sum_{\substack{q \le x \\ \Omega(q-1) > 6 \log\log x}} 1 + \sum_{\substack{q \le \frac{x}{2} \\ \Omega(q-1) > 6 \log\log x}} \sum_{1 \le k \le K-1} \sum_{\substack{p \le x \\ q \mid \varphi_k(p)}} 1 \\ &\le \frac{6\, C_1\, x \,\log\log x}{\log^3 x} + \sum_{\substack{q \le \frac{x}{2} \\ \Omega(q-1) > 6 \log\log x}} \sum_{1 \le k \le K-1} \frac{4x (31 \log\log x)^{k-1}}{q \log \frac{x}{q}} \\ &\le \frac{6\, C_1\, x \,\log\log x}{\log^3 x} + \sum_{\substack{q \le \frac{x}{2} \\ \Omega(q-1) > 6 \log\log x}} \frac{5x (31 \log\log x)^{K-2}}{q \log \frac{x}{q}}. \end{aligned}$$

Approaching this last sum with partial summation, increasing the range of the sum to $q \le x$, and using the bound in (2.2), we have

$$\begin{aligned}\sum_{\substack{q \le \frac{x}{2} \\ \Omega(q-1) > 6 \log\log x}} \frac{1}{q \log \frac{x}{q}} &< \sum_{\substack{q \le x \\ \Omega(q-1) > 6 \log\log x}} \frac{2}{q} \\ &\le \frac{6\, C_1\, x\, \log\log x}{\log^3 x} \cdot \frac{2}{x} + \int_3^x \frac{6\, C_1\, t\, \log\log x}{\log^3 x} \cdot \frac{2}{t^2}\, dt \\ &\le \frac{12\, C_1\, \log\log x}{\log^2 x}.\end{aligned}$$

Putting this together, we have that

$$\begin{aligned}\sum_{\substack{p \le x \\ \Omega(F_K(p)) > 2(6 \log\log x)^K}} 1 &\le \frac{6\, C_1\, x \log\log x}{\log^3 x} + \frac{2\, C_1\, x\, (31 \log\log x)^{K-1}}{\log^2 x} \\ &\le \frac{6\, C_1\, x\, (31 \log\log x)^{K-1}}{\log^2 x},\end{aligned}$$

so we may take $C_3 = 6\, C_1$, which proves the proposition. □

3. The proof of Theorem 4

We are now ready to prove that $\varphi(F(p))$ grows faster than any fixed power of p for almost all primes p.

Proof. We follow the plan of Theorem 3 in [7]. Let $x > e^e$ and fix $0 < \epsilon < 1$ arbitrarily small. Set

$$K = \lceil (1-\epsilon) \log\log x / \log\log\log x \rceil.$$

For $p \le x$ prime, factor $F_K(p) = A \cdot B$, where A is the product of the primes dividing $F_K(p)$ which are less than or equal to $\log^3 x$, and B is the product of the primes larger than $\log^3 x$ (including multiplicity in both A and B). Proposition 3 gives that with at most $O(\frac{x}{\log^2 x}(31 \log\log x)^{K-1}) = o(\pi(x))$ exceptions, all choices of p have

$$A \le (\log^3 x)^{2(6 \log\log x)^K} = \exp((6 \log\log x)^{K+1}) = x^{o(1)}.$$

For x sufficiently large, the minimal order of $\varphi(m)/m$ for $m \le x$ (see [6, Theorem 328]) gives that $\varphi_{j+1}(p)/\varphi_j(p) > (2 \log\log x)^{-1}$ holds for each $j \in [0, K-1]$. Since $\pi(\frac{x}{2 \log\log x}) = o(\pi(x))$, we may assume $p > x/(2 \log\log x)$. Then,

$$\begin{aligned}F_K(p) = p^{K+1} \prod_{i=0}^{K} \frac{\varphi_i(p)}{p} &= p^{K+1} \prod_{i=1}^{K} \prod_{j=0}^{i-1} \frac{\varphi_{j+1}(p)}{\varphi_j(p)} \\ &> p^{K+1}/(2 \log\log x)^{1+2+\cdots+K} = p^{K+1}/(2 \log\log x)^{K(K+1)/2} \\ &> x^{K+1}/(2 \log\log x)^{(K+1)(K+2)/2} > x^{K+1/2}.\end{aligned}$$

So with $o(\pi(x))$ exceptions, all choices of $p \le x$ have

$$B = \frac{F_K(p)}{A} > x^{K+1/4}.$$

By Proposition 2, for sufficiently large x, the number of primes $p \le x$ such that $q^2 \mid F_K(p)$ for some $q \mid B$ is bounded above by

$$\begin{aligned}\frac{x}{\log^3 x}\frac{(\log\log x)(\log\log x)^{2K}}{\log\log\log x}\,(C_2)^{2K} &\le \frac{x}{\log^3 x}(\log\log x)^{2K+1}\,(C_2)^{2K}\\ &\le \frac{x}{\log^3 x}(\log x)^{2-2\epsilon}(\log\log x)^3\,(C_2)^{2K}\\ &= \frac{x}{(\log x)^{1+2\epsilon+o(1)}}\\ &= o(\pi(x)).\end{aligned}$$

So, but for $o(\pi(x))$ choices of the prime $p \le x$, B is squarefree. Since $B \mid F(p)$, we have that $\varphi(B) \mid \varphi(F(p))$. Again applying the minimal order of the Euler φ-function, we have

$$\varphi(B) > \frac{B}{2\log\log B} > \frac{x^{K+1/4}}{2(\log(K+1/4)+\log\log x)} > \frac{x^{K+1/4}}{3\log\log x} > x^K$$

for sufficiently large x. Thus, with at most $o(\pi(x))$ exceptions, $\varphi(F(p)) > x^K \ge p^K$ holds for primes $p \le x$. This completes the proof of Theorem 4. □

Corollary 2. *For almost all primes p, $F(p)$ grows faster than any fixed power of p.*

4. Lucas-Pratt trees

Now to attain our primary result, we will use Pratt's Theorem on $N(p)$, the number of primes appearing in the Lucas-Pratt tree for a prime p.

Proof. For each prime q appearing in the Lucas-Pratt tree for p, we have to at least show that $a^{\frac{q-1}{2}} \equiv -1 \bmod q$ for some primitive root a. This takes at least $\log_2(\frac{q-1}{2})$ modular multiplications $\bmod q$. Thus,

$$M(p) \ge \sum_{q\mid F(p)} \log_2\left(\frac{q-1}{2}\right) = \sum_{q\mid F(p)} \Big(\log_2(q-1)-1\Big) \ge \log_2\varphi(F(p)) - N(p).$$

Thus by Theorems 2 and 4 above, for each fixed $\epsilon > 0$, but for $o(\pi(x))$ primes $p \le x$,

$$\begin{aligned}M(p) &> \frac{1-\epsilon}{\log 2}\frac{\log p\log\log p}{\log\log\log p} - \frac{1}{\log 2}\log p\\ &= \frac{1}{\log 2}\Big((1-\epsilon)\frac{\log\log p}{\log\log\log p} - 1\Big)\log p.\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, this completes the proof of Theorem 5. □

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References

1. M. Agrawal, N. Kayal, and N. Saxena, PRIMES is in P, *Annals of Mathematics*, 160 (2004), 781-793. MR2123939 (2006a:11170)
2. W. Banks and I. Shparlinski, On values taken by the largest prime factor of shifted primes, *Journal of the Australian Mathematical Society*, 82 (2007), 133–147.
3. R. Crandall and C. Pomerance, *Prime Numbers: A Computational Perspective*, Springer-Verlag, New York, 2001. MR1821158 (2002a:11007)
4. P. Erdős, A. Granville, C. Pomerance, and C. Spiro, On the normal behavior of the iterates of some arithmetic functions, in: *Analytic Number Theory, Proceedings of a Conference in Honor of P. T. Bateman*, Birkhäuser; Boston, Inc. 1990, 165-204. MR1084181 (92a:11113)
5. R. Hall and G. Tenenbaum, *Divisors*, Cambridge University Press, Cambridge, 1988. MR964687 (90a:11107)
6. G. Hardy and E. Wright, *An Introduction to the Theory of Numbers*, 4th ed., Clarendon Press, Oxford, UK, 1971. MR568909 (81i:10002)
7. F. Luca and C. Pomerance, Irreducible radical extensions and Euler-function chains, *Integers*, to appear, (Available at http://math.dartmouth.edu/∼carlp/PDF/radicalpaper5.pdf).
8. E. Lucas, Considérations nouvelles sur la théorie des nombres premiers et sur la division géométrique de la circonférence en parties égales, Association Française pour l'Avancement des Sciences, *Comptes Rendus*, 6 (1877), 159-167.
9. H. Montgomery and R. Vaughan, The large sieve, *Mathematika* (20) **40** (1973), 119-134. MR0374060 (51:10260)
10. C. Pomerance, Very short primality proofs, *Mathematics of Computation* (48) **177** (1987), 315-322. MR866117 (88b:11088)
11. V. Pratt, Every prime has a succinct certificate, *SIAM Journal on Computing*, 4 (1975), 214-220. MR0391574 (52:12395)

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