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A COUNTEREXAMPLE CONCERNING THE L_2-PROJECTOR ONTO LINEAR SPLINE SPACES

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Abstract. For the L_2-orthogonal projection P_V onto spaces of linear splines over simplicial partitions in polyhedral domains in $\mathbb{R}^d$, $d > 1$, we show that in contrast to the one-dimensional case, where $\|P_V\|_{L_\infty \to L_\infty} \leq 3$ independently of the nature of the partition, in higher dimensions the L_∞-norm of P_V cannot be bounded uniformly with respect to the partition. This fact is folklore among specialists in finite element methods and approximation theory but seemingly has never been formally proved.

1. Introduction

Variational methods based on piecewise polynomial approximations are a workhorse in numerical methods for PDEs and data analysis. In particular, least-squares methods lead to the study of the L_2-orthogonal projection operator $P_V : L_2(\Omega) \to V$ onto a given spline space V defined on a domain $\Omega \in \mathbb{R}^d$. A question of considerable interest is the uniform boundedness of the L_∞-norm

$$\|P_V\|_{L_\infty \to L_\infty} := \max_{f \in L_\infty(\Omega):\ \|f\|_{L_\infty}=1} \|P_V f\|_{L_\infty}$$

of P_V with respect to families of spline spaces V. If Ω is a bounded interval in $\mathbb{R}^1$, then the question has been intensively studied for the family of spaces of smooth splines of fixed degree r over arbitrary partitions, where Shadrin [7] has recently established that

$$\|P_V\|_{L_\infty \to L_\infty} \leq C(r) < \infty, \qquad r \geq 1, \tag{1}$$

for any partition. This result was known for a long time for small values of r, e.g., Ciesielski [3] proved that for linear splines one can take $C(1) = 3$, while de Boor [2] solved the case $r \leq 4$. The estimate (1) plays an important role in numerical analysis and for the investigation of orthonormal spline systems such as the Franklin system in L_p-based scales of function spaces, $1 \leq p \leq \infty$.

In higher dimensions, the study of the L_∞-norm of P_V arose mostly in the context of obtaining L_p error estimates in the finite element Galerkin method [5, 6], where sufficient conditions on the underlying partition and nodal basis $\{\phi_i\}$ of a finite element space V are formulated under which the norms $\|P_V\|_{L_\infty \to L_\infty}$ are bounded by a certain finite constant. Interestingly enough, these results suggest that such conditions on partitions or finite element type are essential for obtaining uniform bounds but formal proof of their *necessity* was not given.

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Similarly, in the theory of multivariate splines final results on the uniform boundedness of $\|P_V\|_{L_\infty \to L_\infty}$ could not be localized. Recently, Ciesielski [4] asked about the extension of his result for linear splines [3] to the higher-dimensional case, and the unanimous opinion of the audience was that in higher dimensions a similar result cannot hold. However, other than a vague reference to unpublished work by A. A. Privalov, no concrete proof could be found.

It is the aim of this paper to provide an elementary example of triangulations $\mathcal{T}_J$ of a square $\Omega \subset \mathbb{R}^2$ into $\mathrm{O}(J)$ triangles for which

$$\|P_{V(\mathcal{T}_J)}\|_{L_\infty \to L_\infty} \geq J, \qquad J \geq 1, \tag{2}$$

where $V(\mathcal{T}_J)$ is the space of linear C^0 splines (or finite element functions) on $\mathcal{T}_J$; see Theorem 1 below. As $J \to \infty$, the triangulations $\mathcal{T}_J$ will not satisfy the minimum angle condition, which is natural since for these types of triangulations a uniform bound can easily be established. However, they satisfy the maximum angle condition, [1], and do not possess vertices of high valence. The example can easily be extended to $d \geq 3$, and implies that $C(1) = \infty$ for all $d > 1$.

2. Notation and result

We concentrate on $d = 2$, the case $d \geq 3$ is mentioned in Section 3. Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain equipped with a finite triangulation $\mathcal{T}$ into nondegenerate closed triangles Δ satisfying the usual regularity condition that two different triangles may intersect at a common vertex respectively, edge only. The set of vertices of $\mathcal{T}$ is denoted by $\mathcal{V}_\mathcal{T}$. Let $|E|$ denote the Lebesgue measure of a measurable set $E \subset \mathbb{R}^2$. By $0 < \underline{\alpha}_\mathcal{T} \leq \bar{\alpha}_\mathcal{T} < \pi$ we denote the minimal and maximal interior angle of all triangles in $\mathcal{T}$.

Let $V(\mathcal{T})$ denote the linear space of all continuous functions g whose restriction to any of the triangles $\Delta \in \mathcal{T}$ is a linear polynomial. Any $g \in V(\mathcal{T})$ has a unique representation of the form w.r.t. the interpolating nodal basis $\Phi_\mathcal{T} := \{\phi_P,\ P \in \mathcal{V}_\mathcal{T}\}$:

$$g = \sum_{P \in \mathcal{V}_\mathcal{T}} g(P)\phi_P, \tag{3}$$

where the Courant hat functions $\phi_P \in V(\mathcal{T})$ are characterized by the conditions $\phi_P(P) = 1$, and $\phi_P(Q) = 0$, where $Q \neq P$ is any of the remaining vertices of $\mathcal{T}$. Thus, $\dim V(\mathcal{T}) = \#\mathcal{V}_\mathcal{T}$, and

$$\operatorname{supp} \phi_P = \Omega_P := \bigcup_{\Delta \in \mathcal{T}:\ P \in \Delta} \Delta.$$

The set Ω_P corresponds to the 1-ring neighborhood of P in $\mathcal{T}$, and we denote by $\mathcal{V}_P = \{Q \in \Omega_P \cap \mathcal{V} :\ Q \neq P\}$ the set of all neighboring vertices to P.

The L_2-orthogonal projection of a function $f \in L_2(\Omega)$ onto $V(\mathcal{T})$ is given by the unique $g := P_{V(\mathcal{T})} f \in V(\mathcal{T})$ such that

$$(f - g, \phi_P) = 0 \qquad \forall\ P \in \mathcal{V}_\mathcal{T}.$$

Here and in the sequel, $(\cdot, \cdot)$ stands for the $L_2(\Omega)$ inner product. Using (3) with unknown nodal values $x_P = g(P)$ as ansatz, this is equivalent to the linear system

$$\sum_{Q \in \mathcal{V}} (\phi_Q, \phi_P) x_Q = (f, \phi_P) \qquad \forall\ P \in \mathcal{V}. \tag{4}$$

A simple calculation shows that

$$(\phi_P, \phi_P) = \frac{|\Omega_P|}{6}.$$

Similarly, if $Q \in \mathcal{V}_P$ is a neighbor of P, then

$$(\phi_Q, \phi_P) = \sum_{\Delta \in \mathcal{T}:\, P,Q \in \Delta} \frac{|\Delta|}{12},$$

in all other cases we have $(\phi_Q, \phi_P) = 0$. This shows, in particular, that

$$(\phi_P, \phi_P) = \sum_{Q \neq P} (\phi_Q, \phi_P) = (1, \phi_P)/2.$$

For example, if we normalize (4) by (ϕ_P, ϕ_P), then (4) turns into a linear system

$$Ax = b, \qquad x := (x_Q :\ Q \in \mathcal{V}_{\mathcal{T}})^T, \quad b := (b_P = \frac{(f, \phi_P)}{(\phi_P, \phi_P)} :\ P \in \mathcal{V}_{\mathcal{T}})^T, \tag{5}$$

where $\|b\|_\infty \leq 2\|f\|_{L_\infty}$, and $A := (a_{PQ})$ satisfies

$$a_{PQ} = \begin{cases} 1, & Q = P, \\ \frac{(\phi_Q, \phi_P)}{(\phi_P, \phi_P)} > 0, & Q \in \mathcal{V}_P, \\ 0, & \text{otherwise}, \end{cases} \tag{6}$$

and

$$1 = \sum_{Q \neq P} a_{PQ} = \sum_{Q \in \mathcal{V}_P} a_{PQ}, \qquad \forall\, P \in \mathcal{V}_{\mathcal{T}}. \tag{7}$$

Thus, the matrix A is only weakly diagonally dominant, and not strictly diagonally dominant as in the one-dimensional case. Otherwise, we could estimate $\|A^{-1}\|_\infty$ in a trivial way, and use the inequality

$$\|P_{V(\mathcal{T})}\|_{L_\infty \to L_\infty} \leq 2\|A^{-1}\|_\infty = 2 \max_{\|Ay\|_\infty \leq 1} \|y\|_\infty, \tag{8}$$

which follows from the above, in conjunction with the obvious equality $\|P_{V(\mathcal{T})} f\|_{L_\infty} = \|x\|_\infty$. Let us mention without proof that (8) implies the following partial result.

Proposition 1. *If for any two neighboring vertices $P \neq Q$ from $\mathcal{V}_{\mathcal{T}}$ we have $a_{PQ} \geq c_0 > 0$, then*

$$\|P_{V(\mathcal{T})}\|_{L_\infty \to L_\infty} \leq (1 + 2c_0)c_0^{-2}.$$

For triangulations satisfying the minimum angle condition uniformly, i.e., $\underline{\alpha}_{\mathcal{T}} \geq \underline{\alpha}_0 > 0$, this result is applicable with c_0 determined solely by α_0, and thus covers the bounds considered in the finite element literature [5, 6]. The main result of this paper is the following:

Theorem 1. *For any $J \geq 1$, there is a triangulation $\mathcal{T}_J$ of a square into $8J + 4$ triangles such that the norm of the L_2-projector $P_{V(\mathcal{T}_J)}$ satisfies*

$$\|P_{V(\mathcal{T}_J)}\|_{L_\infty \to L_\infty} \geq 2J.$$

Thus, for spatial dimension $d = 2$ we have

$$C(1) := \sup_{\mathcal{T}} \|P_{V(\mathcal{T})}\|_{L_\infty \to L_\infty} = \infty.$$

We conjecture that in terms of the number of triangles our result is asymptotically sharp for bounded polygonal domains in $\mathbb{R}^2$, i.e.,

$$\sup_{\mathcal{T}:\, \#\mathcal{T}\le N} \|P_{V(\mathcal{T})}\|_{L_\infty\to L_\infty} \asymp N, \qquad N\to\infty. \tag{9}$$

That Ω is a square is not crucial. The examples given below can easily be modified to show $C(1)=\infty$ for simplicial partitions in higher dimensions as well.

3. Proof of Theorem 1

We will use the following notation. Let $S_a = [-a,a]^2$ be the square of side-length $2a$, $a>0$, with center at the origin. Set $\Omega := S_1$, with vertices denoted (in clockwise direction) by $P_{0,i}$, $i=1,\dots,4$, and fix the parameter $t\in(0,1)$. The triangulation $\mathcal{T}_J$, $J\ge 1$, of Ω is obtained by inserting the squares S_t, $S_{t^2},\dots,S_{t^J}$, whose vertices will be denoted similarly by $P_{j,i}$, $i=1,\dots,4$, $j=1,\dots,J$, placing an additional vertex P_{J+1} at the origin, connecting P_{J+1} by straight lines with the 4 vertices $P_{0,i}$ of S_1, and finally subdividing each of the remaining trapezoidal regions $P_{j-1,i-1}P_{j,i-1}P_{j,i}P_{j-1,i}$ into two triangles by connecting $P_{j-1,i-1}$ with $P_{j,i}$ in a consistent way. The outer rings of the resulting triangulation are shown in Figure 1a.

Let the function $f\in L_\infty(\Omega)$ with $\|f\|_{L_\infty}=1$ be defined as follows:

$$f(x)=(-1)^j, \qquad x\in\Omega_j := \begin{cases} S_{t^{j-1}}\backslash S_{t^j}, & j=1,\dots,J,\\ S_{t^J}, & j=J+1.\end{cases}$$

We use the same notation as above, and consider the linear system (5) corresponding to the L_2-orthogonal projection $g=P_{V(\mathcal{T}_J)}f$ of this f onto $V(\mathcal{T}_J)$. Because of uniqueness of orthogonal projections and the rotational symmetry of $\mathcal{T}_J$ and f, the entries of the vector x that represent the nodal values of g corresponding to the vertices $P_{j,i}$, $i=1,\dots,4$, are equal, and will be denoted by x_j, $j=0,\dots,J$ (the value at the origin is denoted by x_{J+1}). Moreover, the 4 equations in (5) corresponding to the 4 vertices of a square S_{t^j}, $j=0,\dots,J$, can be replaced by one, thus turning the original system $Ax=b$ of dimension $4J+5$ into a reduced tridiagonal system $\tilde{A}\tilde{x}=\tilde{b}$ of dimension $J+2$ for the vector $\tilde{x}=(x_0,\dots,x_{J+1})^T$. Since the equations in system (5) are invariant under affine transformations, and

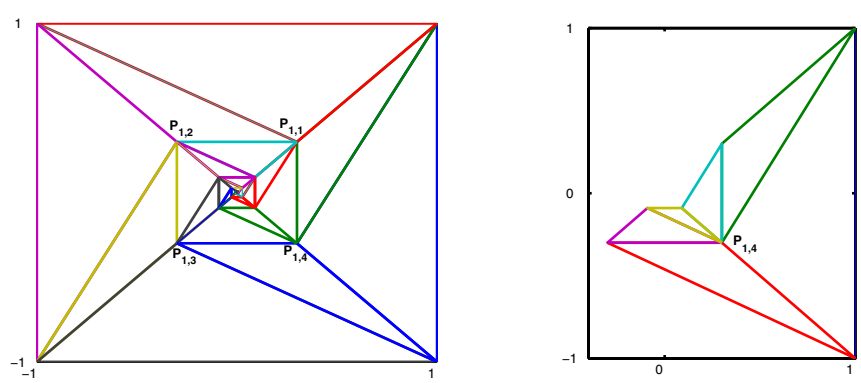

Figure 1. Triangulation $\mathcal{T}_J$ for $t=0.3$ (left), and typical $\Omega_{P_{j,i}}$ (right)

because of the definition of $\mathcal{T}_J$ via squares S_{tj} shrinking at a fixed geometric rate, it is easy to see that, with the exception of the first and last two, all equations in $\tilde{A}\tilde{x} = \tilde{b}$ have the same form:

$$\alpha x_{j-1} + \beta x_j + \gamma x_{j+1} = (-1)^j \delta, \qquad j = 1, \dots, J-1. \tag{10}$$

The coefficients can be found from the triangular neighborhood $\Omega_{P_{j,i}}$ of any of the $P_{j,i}$, $j = 1, \dots, J-1$ (see Figure 1 for an illustration), and the definitions leading to (5). We do not need their exact expressions, just their limit behavior as $t \to 0$, i.e., we will be looking for the entries of $\hat{A} := \lim_{t\to 0} \tilde{A}$ and $\hat{b} := \lim_{t\to 0} \tilde{b}$. Indeed, since for $t \to 0$ the whole $\Omega_{P_{j,i}}$ is essentially covered by the single triangle with vertices $P_{j,i}$, $P_{j-1,i-1}$, $P_{j-1,i}$, and since $f(x) = (-1)^j$ on the latter, we have

$$\alpha = 1 + \mathrm{O}(t), \quad \beta = 1 + \mathrm{O}(t), \quad \gamma = \mathrm{O}(t^2), \quad \delta = 2 + \mathrm{O}(t).$$

From (10), we obtain in the limit $t \to 0$ the equations

$$\hat{x}_{j-1} + \hat{x}_j = 2(-1)^j, \qquad j = 1, \dots, J-1,$$

where $\hat{x}_j = \lim_{t\to\infty} x_j$ (the existence of these limits follows from the invertibility of the limit matrix $\hat{A}$, see below). Similar considerations for the first and the last two equations of $\tilde{A}\tilde{x} = \tilde{b}$ yield the remaining three equations of $\hat{A}\hat{x} = \hat{b}$ as follows:

$$\frac{3}{2}\hat{x}_0 + \frac{1}{2}\hat{x}_1 = -2, \qquad \hat{x}_{j-1} + \hat{x}_j = 2(-1)^j, \quad j = J, J+1.$$

The resulting matrix $\hat{A}$ is obviously invertible. After finding $\hat{x}_0 = \hat{x}_1 = -1$ from the first two equations, forward substitution gives $\hat{x}_j = (2j-1)(-1)^j$, $j = 2, \dots, J+1$. This implies that for any $\epsilon > 0$ one can find a sufficiently small $t > 0$ such that

$$\|\tilde{x}\|_\infty \geq \|\hat{x}\|_\infty - \epsilon = 2J + 1 - \epsilon.$$

This proves Theorem 1. Note that the above reasoning does not work for type-I triangulations of a square obtained from a non-uniform rectangular tensor-product partition.

We conclude with the straightforward extension of the above example to arbitrary $d > 2$. Let e^m denote the m-th unit coordinate vector in $\mathbb{R}^d$, $m = 1, \dots, d$. As Ω we take the convex polyhedral domain with vertices

$$P_{0,1} = e^1 + e^2, \quad P_{0,2} = e^1 - e^2, \quad P_{0,3} = -e^1 - e^2, \quad P_{0,4} = -e^1 + e^2,$$

and $P'_m = e^m$, $m = 3, \dots, d$. For $d = 3$, this domain is a pyramid with square base in the xy-plane, and tip on the z-axis.

A suitable simplicial partition of Ω is obtained as follows. The base square with vertices $P_{0,i}$, $i = 1, \dots, 4$, is triangulated into $\mathcal{T}_J$ which depends on the parameters $0 < t < 1$ and J as described for $d = 2$. The resulting triangulation (now embedded into $\mathbb{R}^d$) and its vertices are again denoted by $\mathcal{T}_J$ resp. by $P_{j,i}$, $i = 1, \dots, 4$, $j = 0, \dots, J$, and P_{J+1}. Then each simplex in the associated simplicial partition $\mathcal{P}_J$ of Ω is generated by the $d-2$ vertices P'_m, $m = 3, \dots, d$, and the three vertices of a triangle in $\mathcal{T}_J$. The latter is called base triangle of the associated simplex. Obviously, the d-dimensional volume of each simplex is proportional to the 2-dimensional area of its base triangle, with proportionality constant $2/d!$. To obtain a suitable function $f \in L_\infty(\Omega)$ with $\|f\|_{L_\infty} = 1$ we prescribe values ± 1 on the simplices by inheritance from the values on the base triangles of the above 2-dimensional f.

From the symmetry properties of f and $\mathcal{P}_J$, it is obvious that the L_2-orthogonal projection $P_{V(\mathcal{P}_J)} f$ of f onto the linear spline space $V(\mathcal{P}_J)$ is characterized by its

value x_{J+1} at the origin P_{J+1}, by values x_j taken at the vertices $P_{j,i}$, $i = 1, \dots, 4$, where $j = 0, \dots, J$, and a common value x' taken at the remaining vertices P'_m, $m = 3, \dots, d$. To estimate these values which, in complete analogy to the 2-dimensional case, are represented by the solution vector $\tilde{x}$ of a certain linear system $\tilde{A}\tilde{x} = \tilde{b}$ (now of dimension $J + 3$), we need the limit version $\hat{A}\hat{x} = \hat{b}$ of this linear system for $t \to 0$. We spare the reader the elementary calculations, and state it without proof:

$$\begin{aligned}
\hat{x}_j + \hat{x}_{j-1} + \frac{d-2}{2}\hat{x}' &= (-1)^j \frac{d+2}{2}, \qquad j = 1, \dots, J+1, \\
\frac{3}{2}\hat{x}_0 + \frac{1}{2}\hat{x}_1 + \frac{d-2}{2}\hat{x}' &= -\frac{d+2}{2}, \\
\hat{x}_0 + \frac{1}{2}\hat{x}_1 + (1 + \frac{d-3}{2})\hat{x}' &= -\frac{d+2}{2}.
\end{aligned}$$

From this system one easily concludes that $\|\hat{x}\|_\infty \geq cJ$, and consequently $\|P_{V(\mathcal{P}_J)} f\|_{L_\infty} \geq cJ$ for a small enough $t > 0$ which implies the desired result. The lower bound cJ obtained does not seem to accurately reflect the possible growth of the projector norms in L_∞ as a function of the number of simplices, for $d \geq 3$ we would rather expect an exponential rate.

References

[1] I. Babuska, A. K. Aziz, On the angle condition in the finite element method, SIAM J. Numer. Anal. **13** (1976), 214–226. MR0455462 (56:13700)

[2] C. de Boor, On a max-norm bound for the least-squares spline approximant, in *Approximation and Function Spaces* (Gdansk, 1979), Z. Ciesielski (ed.), pp. 163–175, North-Holland, Amsterdam, 1981. MR649424 (84k:41012)

[3] Z. Ciesielski, Properties of the orthonormal Franklin system, Studia Math. **23** (1963), 141–157. MR0157182 (28:419)

[4] Z. Ciesielski, Private communication, Int. Conf. *Approximation Theory and Probability*, Bedlowo, 2004.

[5] J. Desloux, On finite element matrices, SIAM J. Numer. Anal. **9**, 2 (1972), 260–265. MR0309292 (46:8402)

[6] J. Douglas, Jr., T. Dupont, L. Wahlbin, The stability in L^q of the L^2-projection into finite element function spaces, Numer. Math. **23** (1975), 193–197. MR0383789 (52:4669)

[7] A. Yu. Shadrin, The L_∞-norm of the L_2-spline projector is bounded independently of the knot sequence: a proof of de Boor's conjecture, Acta Math. **187** (2001), 59–137. MR1864631 (2002j:41007)

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