MATHEMATICS OF COMPUTATION
Volume 77, Number 262, April 2008, Pages 773–788
S 0025-5718(07)02079-0
Article electronically published on October 29, 2007

NUMERICAL ANALYSIS OF AN EXPLICIT APPROXIMATION SCHEME FOR THE LANDAU-LIFSHITZ-GILBERT EQUATION

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ABSTRACT. The Landau-Lifshitz-Gilbert equation describes magnetic behavior in ferromagnetic materials. Construction of numerical strategies to approximate weak solutions for this equation is made difficult by its top order nonlinearity and nonconvex constraint. In this paper, we discuss necessary scaling of numerical parameters and provide a refined convergence result for the scheme first proposed by Alouges and Jaisson (2006). As an application, we numerically study discrete finite time blowup in two dimensions.

1. Introduction

The Landau-Lifshitz-Gilbert equation (LLG) records the exchange interaction between magnetic moments in a magnetic spin system on a square lattice. In this setting, the energy is given by the Hamiltonian

$$H = -K \sum_{i,j} \mathbf{S}_{i,j} \cdot \{\mathbf{S}_{i+1,j} + \mathbf{S}_{i,j+1}\},$$

where $\mathbf{S}_{i,j}$ is the spin vector of unit length at site (i,j) and K is a positive exchange constant. The dynamics of this system is given by the nearest neighbor interaction:

$$\dot{\mathbf{S}}_{i,j} = -K\mathbf{S}_{i,j} \times (\mathbf{S}_{i+1,j} + \mathbf{S}_{i-1,j} + \mathbf{S}_{i,j+1} + \mathbf{S}_{i,j-1}).$$

Assigning $\mathbf{S}_{i,j} = u(ih, jh, t)$ for $u : \mathbb{R}^2 \times \mathbb{R} \to S^2$, we have

$$\partial_t u = Kh^2 u \times \Delta u + O(h^3).$$

We adopt a standard usage of K to be inversely proportional to the square of h and arrive at the continuum limit (Heisenberg equation)

$$\partial_t u = u \times \Delta u \tag{1.1}$$

with an associated energy given by the Dirichlet energy functional. To incorporate the Gilbert damping law, whose origin lies in the observation that such systems reach equilibrium and must have decreasing energy over time, a dissipative term can be added on, resulting in the LLG equation:

$$\partial_t u = u \times \Delta u - \lambda u \times (u \times \Delta u)\,, \qquad \lambda > 0\,. \tag{1.2}$$

Received by the editor May 9, 2005 and, in revised form, November 23, 2006.
2000 *Mathematics Subject Classification.* Primary 65N12, 65N30, 35K55.
The first author was supported by Deutsche Forschungsgemeinschaft through the DFG Research Center MATHEON "Mathematics for key technologies" in Berlin.
The second author was partially supported by NSF grant DMS-0402788.

In this version of the LLG equation, Δu is a simple approximation of an effictive field which is in more general models replaced by $H_{\rm eff}(u) = -\nabla_{L^2} E^{LL}(u)$, where E^{LL} is the Landau-Lifshitz energy of micromagnetics; cf., e.g., [11].

The Cauchy problem for LLG with natural boundary conditions, then, is the problem of finding u, given initial data $u_0 : \Omega \subseteq \mathbb{R}^n \to S^2$ satisfying

$$\left\{ \begin{array}{lll} \partial_t u = u \times \Delta u - \lambda u \times (u \times \Delta u) & \text{on} & \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on} & \partial\Omega, \\ u(x,0) = u_0(x). & & \end{array} \right. \tag{1.3}$$

We will refer to the first term as the gyroscopic term and the second as the damping term. When only the damping term is present, this equation is the *harmonic map heat flow problem*. There are several standard forms of (1.2) which are equivalent for smooth solutions and which we will make use of in this paper. The first results from the vector identity $-\xi \times (v \times \xi) = -v + (v, \xi)\xi$, which holds for ξ a unit vector. From this, (1.2) can be rewritten as

$$\partial_t u = u \times \Delta u + \lambda(\Delta u + |\nabla u|^2 u). \tag{1.4}$$

From (1.2) and (1.4), we can derive the following two additional formulations:

$$\partial_t u + \lambda u \times \partial_t u = (1 + \lambda^2) u \times \Delta u, \tag{1.5}$$

$$\lambda \partial_t u - u \times \partial_t u = (1 + \lambda^2)(\Delta u + |\nabla u|^2 u). \tag{1.6}$$

Global weak solutions, even partially regular ones, have been shown to exist for LLG in two and three dimensions given initial data with finite Dirichlet energy. Amongst these is the work of Alouges and Soyeur [2] who have made use of the definition of a weak solution naturally arising from (1.5) to show that energy bounds are sufficient for the existence of such a weak solution in three dimensions. Guo and Hong [8] successfully carried through the argument that Struwe in [15] employed for the harmonic map heat flow to exhibit a *Struwe solution* in two dimensions, i.e., a partially regular solution that satisfies an energy inequality and is smooth away from a finite set of point singularities. Recently, Ko [10] in two dimensions and Melcher [12] in three dimensions independently constructed partially regular solutions to LLG smooth away from a locally finite n-dimensional parabolic Hausdorff measure set. While it is known that weak solutions are nonunique in general [2], uniqueness or nonuniqueness in the class of partially regular solutions is, however, still an open question. For the harmonic map heat flow, there exist nonunique solutions due to the appearance of singularities [5]. While this related question of singularity formation, i.e., whether singularities develop from smooth initial data in finite time, has been demonstrated for the harmonic map heat flow, no such initial data has been produced for LLG. An inquiry into this problem is a natural start to the problem of nonuniqueness as well as the broader issues regarding the validity of the model and selection criteria for "correct" solutions.

Very little is known about singularities and blow-up dynamics for LLG; cf. [9] for partial results. As long as $\|\nabla u\|_{L^\infty}$ is bounded, the solution remains regular for all time, so singularities in this case are indicated by loss of control on $\|\nabla u\|_{L^\infty}$. The presence of the gyroscopic term precludes the successful application of standard analytical arguments to show blowup of solutions such as convexity arguments, scaling arguments, and constructions of explicit solutions. The breakdown of these methods and the subsequent need to understand the contribution of the gyroscopic term have inspired recent efforts for singularity formation in the limiting case $\lambda = 0$.

Shatah and Zeng [14] have produced weak-L^2 initial data which are locally smooth that develop a singularity in finite time. However, these initial data fail to be of finite energy. The work [9] demonstrates orbital stability about the known explicit harmonic maps in the equivariant setting which are equilibrium solutions to LLG. However, there is no guarantee that blowup occurs, much less that it occurs in finite time.

Proper numerical treatment of LLG is made difficult by the fact that the nonlinearity occurs in the highest order derivative and the nonconvexity requirement $|u| = 1$. Explicit time integrators of high order coupled with occasional updates to ensure $|u| = 1$ are the most common strategies in the engineering literature but suffer from nonreliable dynamics. On the other hand, implicit strategies to discretize LLG in time often introduce artificial damping which prevents computed iterates from remaining on the sphere, and which also precludes a (discrete) energy law. Recent remedies have been made, partially addressing the dual requirements of efficiency and reliability: (i) projection methods have been constructed [6, 7, 16], independently dealing with the nonconvex algebraic constraint; however, no (discrete) energy principle is available, and convergence to LLG is only known in the case of existing strong solutions to LLG; (ii) explicit/implicit discretizations of Ginzburg-Landau penalizations that involve an additional parameter $\varepsilon > 0$ are used, which allow for a discrete energy principle, possibly for restricted choices of spatio-temporal discretization parameters. We refer to [11] for a more detailed discussion in this direction. Alouges and Jaisson [1] propose a finite element plus projection scheme, which is shown to converge if successively the time-step size and the mesh size tend to zero. The scheme is well suited for our study of weak solutions, and we are able to extend their results by supplying sufficient conditions for involved parameters yielding convergence. Moreover, we propose a modification to increase its efficiency. This yields a practical, stable and convergent numerical scheme which holds for arbitrarily small λ.

Such a reliable scheme is prerequisite to any study of qualitative properties of weak solutions. One problem (open for any choice of λ) is the question of whether blowup occurs for smooth initial data. Very little is known for this question. As long as $\|\nabla u\|_{L^\infty}$ is bounded, the solution remains regular for all time, so singularities in this case are indicated by loss of control on $\|\nabla u\|_{L^\infty}$. The presence of the gyroscopic term precludes a traditional application of analytical arguments to show blowup of solutions such as convexity arguments, scaling arguments, and constructions of explicit solutions. To our knowledge only the work of Pistella and Valente [13] has made an attempt to seek blowup solutions numerically, using a stable scheme with a fourth order regularizing term, whose convergence behavior is not known so far. Their study is heavily motivated by the work of Chang, Ding and Ye [4] on the blowup of the harmonic map heat flow. They specify equivariant data of degree greater than one, which is known to blow up for the harmonic map heat flow with fixed boundary data. Introducing a parameter β in front of the gyroscopic term in (1.2), they fix $\lambda = 1$ and steadily increase the value of β and notice that for $\beta \sim 10^{-4}$, blowup still occurs. However, they observe that the singularity disappears for large β, which suggests "the regularizing effect of the parameter β". We believe that their conclusion that the gyroscopic term has a damping effect is a statement that is valid only for the specific initial data they choose. Their study is designed to treat the LLG as a perturbation of the harmonic map heat flow and hence gives little insight into the more interesting question concerning the manner in which the gyroscopic term contributes to blowup. In this paper, we report on our numerical findings of singularity formation for LLG in two dimensions, with particular emphasis on the regime of small λ. We introduce a class of initial data, which is seen in our experiments to generate discrete blowup under Dirichlet boundary conditions.

2. Approximation scheme and main result

In this section we describe the approximation scheme and state the main result of this paper.

2.1. **Preliminaries.** Given a regular triangulation $\mathcal{T}$ of the polygonal or polyhedral domain $\Omega \subseteq \mathbb{R}^n$ into triangles or tetrahedra for $n = 2$ or $n = 3$, respectively, we let $h := \max\{\mathrm{diam}(K) : K \in \mathcal{T}\}$ be the maximal mesh size of $\mathcal{T}$. The set of nodes in $\mathcal{T}$ is denoted by $\mathcal{N}$, and the function space $\mathcal{S}^1(\mathcal{T}) \subseteq W^{1,2}(\Omega)$ consists of all continuous, $\mathcal{T}$-elementwise affine functions. For each $z \in \mathcal{N}$ the function $\varphi_z \in \mathcal{S}^1(\mathcal{T})$ satisfies $\varphi_z(z) = 1$ and $\varphi_z(y) = 0$ for all $y \in \mathcal{N} \setminus \{z\}$. Throughout this paper we set $(v; w) := \int_\Omega v \cdot w \,\mathrm{d}x$ for $v, w \in L^2(\Omega)$. We write $H^1(A)$ instead of $H^1(A; \mathbb{R}^\ell)$ for $\ell = 1, 3$.

2.2. **Approximation scheme.** We follow ideas of Alouges and Jaisson in [1] to derive an approximation scheme for the Landau-Lifshitz-Gilbert equation. Testing (1.5) with a function ϕ and using $(u \times \partial_t u) \cdot (u \times \phi) = \partial_t u \cdot \phi$ (owing to $|u| = 1$), $(u \times \partial_t u) \cdot \phi = -\partial_t u \cdot (u \times \phi)$, and $(u \times \Delta u) \cdot \phi = -\Delta u \cdot (u \times \phi)$ we infer

$$(u \times \partial_t u; u \times \phi) - \lambda(\partial_t u; u \times \phi) = -(1 + \lambda^2)(\Delta u; u \times \phi).$$

We replace $w = u \times \phi$ and integrate by parts to verify

$$\lambda(\partial_t u; w) - \big(u \times \partial_t u; w\big) = -(1 + \lambda^2)(\nabla u; \nabla w).$$

The fact that $w \cdot u = 0$ and $u_t \cdot u = 0$ almost everywhere in Ω motivates an explicit numerical scheme in which an approximation v_h of u_t is computed in each time step. The updated approximation $u_h + k v_h$ of u is then projected in order to satisfy the constraint $|u| = 1$ in an appropriate way.

Algorithm (A). *Input: a time-step size $k > 0$, a positive integer J, a regular triangulation $\mathcal{T}$ of Ω, and $u_h^{(0)} \in \mathcal{S}^1(\mathcal{T})^3$ such that $|u_h^{(0)}(z)| = 1$ for all $z \in \mathcal{N}$.*

(a) *Set $j := 0$.*

(b) *Compute $v_h^{(j+1)} \in L^{(j)} := \{w_h \in \mathcal{S}^1(\mathcal{T})^3 : w_h(z) \cdot u_h^{(j)}(z) = 0$ for all $z \in \mathcal{N}\}$ such that*

$$\lambda\big(v_h^{(j+1)}; w_h\big) - \big(u_h^{(j)} \times v_h^{(j+1)}; w_h\big) = -(1 + \lambda^2)\big(\nabla u_h^{(j)}; \nabla w_h\big) \qquad \text{for all } w_h \in L^{(j)}.$$

(c) *Set*

$$u_h^{(j+1)} := \sum_{z \in \mathcal{N}} \frac{u_h^{(j)}(z) + k\, v_h^{(j+1)}(z)}{|u_h^{(j)}(z) + k\, v_h^{(j+1)}(z)|} \varphi_z$$

and $j := j + 1$.

(d) *Stop if $j = J$; go to (b) otherwise.*

Remarks 2.1. (i) The variational formulation in (b) can be recast as $a(v_h^{(j+1)}; w_h) + b(v_h^{(j+1)}; w_h) = \ell(w_h)$ with a continuous, elliptic, symmetric bilinear form a on $L^{(j)} \times L^{(j)}$, a continuous, skew-symmetric bilinear form b on $L^{(j)} \times L^{(j)}$, and a continuous linear form ℓ on $L^{(j)}$. By the Lax-Milgram theorem there exists a unique solution $v_h^{(j+1)} \in L^{(j)}$ in (b).

(ii) Suppose that $|u_h^{(j)}(z)| = 1$ for some $j \geq 0$ and all $z \in \mathcal{N}$. Since $u_h^{(j)}(z) \cdot v_h^{(j+1)}(z) = 0$ for all $z \in \mathcal{N}$, it follows that $|u_h^{(j)}(z) + k v_h^{(j+1)}(z)| \geq 1$ for all $z \in \mathcal{N}$ so that (c) in Algorithm (A) is well defined and $|u_h^{(j+1)}(z)| = 1$ for all $z \in \mathcal{N}$.

2.3. **Approximation result.** Convergence for the above scheme to a solution of the Landau-Lifshitz-Gilbert equation has been proved in [1] if $k \to 0$ and subsequently $h \to 0$. Here, we present a refined convergence result. The following definition is due to [2].

Definition 2.1. Given $u_0 \in H^1(\Omega)$ such that $|u_0| = 1$ almost everywhere in Ω, a function u is called a weak solution of (1.3) if for all $T > 0$ we have that (i) $u \in H^1((0,T)\times\Omega)$, $u(0,\cdot) = u_0$ in the sense of traces, (ii) $|u| = 1$ almost everywhere in $(0,T) \times \Omega$, (iii) for almost all $T' \in (0,T)$ we have

$$\frac{\lambda}{1+\lambda^2} \int_{(0,T')\times\Omega} |\partial_t u|^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\int_\Omega |\nabla u(T')|^2 \,\mathrm{d}x \leq \frac{1}{2}\int_\Omega |\nabla u_0|^2 \,\mathrm{d}x,$$

and (iv) for all $\phi \in C^\infty(\Omega_T)$ with $\Omega_T = (0,T) \times \Omega$, we have

$$\int_{\Omega_T} \partial_t u \cdot \phi \,\mathrm{d}x\,\mathrm{d}t + \lambda \int_{\Omega_T} \big(u \times \partial_t u\big) \cdot \phi \,\mathrm{d}x\,\mathrm{d}t = -(1+\lambda^2) \int_{\Omega_T} \nabla u \cdot \nabla(u \times \phi) \,\mathrm{d}x\,\mathrm{d}t\,.$$

Theorem A. *Given $0 \leq t \leq T \leq Jk$ such that $t \in [jk, (j+1)k]$ for some $0 \leq j \leq J-1$ and $x \in \Omega$ let*

$$\hat{u}_{h,k}(t,x) := \frac{t-jk}{k} u_h^{(j+1)}(x) + \frac{(j+1)k - t}{k} u_h^{(j)}(x).$$

Let $u_0 \in H^1(\Omega)$ and suppose $u_h^{(0)} \to u_0$ in $H^1(\Omega)$ for $h \to 0$. If $\mathcal{T}$ is quasi-uniform and $(k,h) \to 0$ such that $kh^{-1-n/2} \to 0$, then there exists a subsequence of $\big(\hat{u}_{h,k}\big)$ that weakly converges in $H^1((0,T)\times\Omega)$ to a weak solution of (1.3).

We refer to Section 3 for a proof of Theorem A and to Lemma 3.2 and Theorem 3.1 for more precise statements, in particular, a priori bounds with explicit dependence on the possibly small parameter λ; tracing this parameter is motivated from [2, Prop. 5.1], where solutions to the Cauchy problem (1.1) are constructed as certain limits of weak solutions u_λ ($\lambda \to 0$) to (1.5). Throughout Section 3 we use several ideas from [1]. Section 4 discusses a modification of Algorithm (A) which leads to simpler linear systems in (b) but still allows for weak subconvergence to a solution.

3. Proof of Theorem A

Throughout this section we assume that $\mathcal{T}$ is quasi-uniform and make repeated use of the following inverse estimates: There exists an h-independent constant $c_0 > 0$ such that for all $1 \leq p \leq \infty$ and $\phi_h \in \mathcal{S}^1(\mathcal{T})$,

$$(3.1) \qquad ||\nabla\phi_h||_{L^p(\Omega)} \leq c_0 h^{-1} ||\phi_h||_{L^p(\Omega)} \quad \text{and} \quad ||\phi_h||_{L^4(\Omega)} \leq c_0 h^{-n/4} ||\phi_h||_{L^2(\Omega)}.$$

We let $\mathcal{I}_h$ denote the nodal interpolation operator on $\mathcal{T}$. Given a sequence $(a^{(j)} : j = 0, 1, \ldots, J)$ we set $d_t a^{(j+1)} := k^{-1}(a^{(j+1)} - a^{(j)})$ for $j = 0, 1, \ldots, J-1$. We abbreviate $\mu := 1 + \lambda^2$.

Lemma 3.1. *For each $j = 0, 1, 2, \ldots, J$, let $r_h^{(j+1)} := k\big(d_t u_h^{(j+1)} - v_h^{(j+1)}\big)$. There exists an (h, k, λ, μ, j)-independent constant $c_1 > 0$ such that for all $j = 0, 1, 2, \ldots, J-1$,*

$$\begin{aligned}
||v_h^{(j+1)}||_{L^2(\Omega)} &\leq c_0(\mu/\lambda)h^{-1}||\nabla u_h^{(j)}||_{L^2(\Omega)},\\
||r_h^{(j+1)}||_{L^1(\Omega)} &\leq c_1 k^2 ||v_h^{(j+1)}||^2_{L^2(\Omega)},\\
||r_h^{(j+1)}||_{L^2(\Omega)} &\leq c_1 k^2 h^{-n/2} ||v_h^{(j+1)}||^2_{L^2(\Omega)},\\
||d_t u_h^{(j+1)}||_{L^2(\Omega)} &\leq \big(1 + c_1 k h^{-n/2} ||v_h^{(j+1)}||_{L^2(\Omega)}\big)||v_h^{(j+1)}||_{L^2(\Omega)}.
\end{aligned}$$

Proof. Choosing $w_h = v_h^{(j+1)}$ in (b) of Algorithm (A) and using (3.1) yields

$$\frac{\lambda}{\mu}||v_h^{(j+1)}||^2_{L^2(\Omega)} \leq ||\nabla u_h^{(j)}||_{L^2(\Omega)}||\nabla v_h^{(j+1)}||_{L^2(\Omega)} \leq c_0 h^{-1}||\nabla u_h^{(j)}||_{L^2(\Omega)}||v_h^{(j+1)}||_{L^2(\Omega)}.$$

For all $z \in \mathcal{N}$, we have

$$\begin{aligned}
|r_h^{(j+1)}(z)| &= |u_h^{(j+1)}(z) - u_h^{(j)}(z) - k v_h^{(j+1)}(z)|\\
&= \left| \frac{u_h^{(j)}(z) + k v_h^{(j+1)}(z)}{|u_h^{(j)}(z) + k v_h^{(j+1)}(z)|} - \big(u_h^{(j)}(z) + k v_h^{(j+1)}(z)\big) \right|\\
&= \big|1 - |u_h^{(j)}(z) + k v_h^{(j+1)}(z)|\big|.
\end{aligned}$$

Since $u_h^{(j)}(z) \cdot v_h^{(j+1)}(z) = 0$ we find $1 \leq |u_h^{(j)}(z) + k v_h^{(j+1)}(z)| = \sqrt{1 + k^2 |v_h^{(j+1)}(z)|^2} \leq 1 + \frac{1}{2}k^2|v_h^{(j+1)}(z)|^2$ and hence

$$|r_h^{(j+1)}(z)| \leq \frac{1}{2}k^2 |v_h^{(j+1)}(z)|^2$$

for all $z \in \mathcal{N}$. Since there exists $c > 0$ such that for all $1 \leq p \leq \infty$ and all $\phi_h \in \mathcal{S}^1(\mathcal{T})$,

$$c^{-1}||\phi_h||^p_{L^p(\Omega)} \leq h^n \sum_{z \in \mathcal{N}} |\phi_h(z)|^p \leq c||\phi_h||^p_{L^p(\Omega)},$$

we verify the second assertion of the lemma and

$$||r_h^{(j+1)}||_{L^2(\Omega)} \leq c k^2 ||v_h^{(j+1)}||^2_{L^4(\Omega)} \leq c k^2 h^{-n/2} ||v_h^{(j+1)}||^2_{L^2(\Omega)},$$

where we used (3.1). We then verify

$$\begin{aligned}
||d_t u_h^{(j+1)}||_{L^2(\Omega)} &\leq ||v_h^{(j+1)}||_{L^2(\Omega)} + k^{-1}||r_h^{(j+1)}||_{L^2(\Omega)}\\
&\leq \big(1 + c k h^{-n/2}||v_h^{(j+1)}||_{L^2(\Omega)}\big)||v_h^{(j+1)}||_{L^2(\Omega)},
\end{aligned}$$

which finishes the proof of the lemma. $\square$

Lemma 3.2. *For all $0 \le J' \le J$ we have*

$$\lambda\Big(\frac{1-\Gamma_1}{1+\Gamma_2}\Big)k\sum_{j=0}^{J'-1}||d_t u_h^{(j+1)}||^2_{L^2(\Omega)} + \frac{\mu}{2}||\nabla u_h^{(J')}||^2_{L^2(\Omega)}$$
$$\le \lambda(1-\Gamma_1)k\sum_{j=0}^{J'-1}||v_h^{(j+1)}||^2_{L^2(\Omega)} + \frac{\mu}{2}||\nabla u_h^{(J')}||^2_{L^2(\Omega)} \le \frac{\mu}{2}||\nabla u_h^{(0)}||^2_{L^2(\Omega)},$$

where $\Gamma_1 := c_0^2\big(c_1 + (1+C_0)^2\big)(\mu/\lambda)kh^{-2}$ *and* $\Gamma_2 := c_0c_1(\mu/\lambda)kh^{-1-n/2}C_0$ *for* $C_0 := ||\nabla u_h^{(0)}||_{L^2(\Omega)}$, *and provided that* $\Gamma_1 \le 1$ *and* $c_0c_1(\mu/\lambda)kh^{-1-n/2} \le 1$.

The inductive argument used in the subsequent proof is borrowed from [3].

Proof. We choose $w_h = v_h^{(j+1)}$ in (b) of Algorithm (A) and use $v_h^{(j+1)} = d_t u_h^{(j+1)} - k^{-1}r_h^{(j+1)}$ and $u_h^{(j)} = u_h^{(j+1)} - kd_tu_h^{(j+1)}$ to verify

$$\begin{aligned}\lambda||v_h^{(j+1)}||^2_{L^2(\Omega)} &= -\mu\big(\nabla u_h^{(j)};\nabla d_t u_h^{(j+1)}\big) + \mu k^{-1}\big(\nabla u_h^{(j)};\nabla r_h^{(j+1)}\big)\\ &= -\mu\big(\nabla u_h^{(j+1)};\nabla d_t u_h^{(j+1)}\big) + \mu k||\nabla d_t u_h^{(j+1)}||^2_{L^2(\Omega)}\\ &\quad + \mu k^{-1}\big(\nabla u_h^{(j)};\nabla r_h^{(j+1)}\big).\end{aligned}$$

The identity

$$-\mu\big(\nabla u_h^{(j+1)};\nabla d_t u_h^{(j+1)}\big) = -\frac{\mu k}{2}||\nabla d_t u_h^{(j+1)}||^2_{L^2(\Omega)} - \frac{\mu}{2}d_t||\nabla u_h^{(j+1)}||^2_{L^2(\Omega)}$$

implies

$$\lambda||v_h^{(j+1)}||^2_{L^2(\Omega)} + \frac{\mu}{2}d_t||\nabla u_h^{(j+1)}||^2_{L^2(\Omega)} = \frac{\mu}{k}\big(\nabla u_h^{(j)};\nabla r_h^{(j+1)}\big) + \frac{\mu k}{2}||\nabla d_t u_h^{(j+1)}||^2_{L^2(\Omega)}.$$

Hölder's inequality, (3.1), $||u_h^{(j)}||_{L^\infty(\Omega)} \le 1$, and the second assertion of Lemma 3.1 lead to

$$\begin{aligned}&\lambda||v_h^{(j+1)}||^2_{L^2(\Omega)} + \frac{\mu}{2}d_t||\nabla u_h^{(j+1)}||^2_{L^2(\Omega)}\\ &\quad\le c_0^2h^{-2}\frac{\mu}{k}||r_h^{(j+1)}||_{L^1(\Omega)} + c_0^2h^{-2}\mu k||d_tu_h^{(j+1)}||^2_{L^2(\Omega)}\\ &\quad\le c_0^2c_1\mu kh^{-2}||v_h^{(j+1)}||^2_{L^2(\Omega)} + c_0^2\mu kh^{-2}||d_tu_h^{(j+1)}||^2_{L^2(\Omega)}.\end{aligned}\tag{3.2}$$

Suppose that $||\nabla u_h^{(j)}||_{L^2(\Omega)} \le C_0$ (which holds for $j=0$ and $C_0 = ||\nabla u_h^{(0)}||_{L^2(\Omega)}$). Since $c_1k \le c_0^{-1}(\lambda/\mu)h^{1+n/2}$, the first assertion of Lemma 3.1 yields

$$c_1kh^{-n/2}||v_h^{(j+1)}||_{L^2(\Omega)} \le c_0^{-1}(\lambda/\mu)h||v_h^{(j+1)}||_{L^2(\Omega)} \le C_0.\tag{3.3}$$

A combination of this bound with the fourth estimate of Lemma 3.1 and (3.2) shows

$$\begin{aligned}\lambda||v_h^{(j+1)}||^2_{L^2(\Omega)} + \frac{\mu}{2}d_t||\nabla u_h^{(j+1)}||^2_{L^2(\Omega)} &\le c_0^2c_1\mu kh^{-2}||v_h^{(j+1)}||^2_{L^2(\Omega)}\\ &\quad + c_0^2\mu kh^{-2}\big(1+C_0\big)^2||v_h^{(j+1)}||^2_{L^2(\Omega)} = \lambda\Gamma_1||v_h^{(j+1)}||^2_{L^2(\Omega)}.\end{aligned}\tag{3.4}$$

Since $\Gamma_1 \le 1$ this implies $||\nabla u_h^{(j+1)}||_{L^2(\Omega)} \le C_0$. Therefore, (3.4) holds for all $j = 0,1,2,\dots,J-1$ and multiplication of (3.4) with k and summation over $j = 0,1,2,\dots,J'-1$ prove

$$\lambda(1-\Gamma_1)k\sum_{j=0}^{J'-1}||v_h^{(j+1)}||^2_{L^2(\Omega)} + \frac{\mu}{2}||\nabla u_h^{(J')}||^2_{L^2(\Omega)} \le \frac{\mu}{2}||\nabla u_h^{(0)}||^2_{L^2(\Omega)}.$$

We combine the fourth estimate of Lemma 3.1 and (3.3) to verify

$$||d_t u_h^{(j+1)}||_{L^2(\Omega)} \le \big(1+c_0c_1(\mu/\lambda)kh^{-1-n/2}C_0\big)||v_h^{(j+1)}||_{L^2(\Omega)} = (1+\Gamma_2)||v_h^{(j+1)}||_{L^2(\Omega)}.$$

A combination of the last two estimates proves the lemma. □

Definition 3.1. For $x \in \Omega$ and $t \in [jk,(j+1)k)$, $0 \le j \le J-1$, define

$$\hat{u}_{h,k}(t,x) := \frac{t-jk}{k}u_h^{(j+1)}(x) + \frac{(j+1)k-t}{k}u_h^{(j)}(x),$$
$$u_{h,k}^-(t,x) := u_h^{(j)}(x), \quad v_{h,k}^+(t,x) := v_h^{(j+1)}(x), \quad r_{h,k}^+(t,x) := r_h^{(j+1)}(x).$$

Lemma 3.3. *Suppose that $\Gamma_1 \le 1/2$, assume that $T \in [0,Jk]$, and define $\Omega_T := (0,T)\times\Omega$. For all $w_h \in L^2(0,T;H^1(\Omega;\mathbb{R}^3))$ such that $w_h(t,\cdot) \in \mathcal{S}^1(\mathcal{T})^3$ for almost all $t \in (0,T)$ and $w_h(t,z)\cdot u_h^-(t,z) = 0$ for all $z \in \mathcal{N}$ and almost all $t \in (0,T)$ it follows that*

$$\Big|\lambda\int_{\Omega_T}\partial_t\hat{u}_{h,k}\cdot w_h\,\mathrm{d}x\,\mathrm{d}t - \int_{\Omega_T}\big(u_{h,k}^-\times\partial_t\hat{u}_{h,k}\big)\cdot w_h\,\mathrm{d}x\,\mathrm{d}t + \mu\int_{\Omega_T}\nabla u_{h,k}^-\cdot\nabla w_h\,\mathrm{d}x\,\mathrm{d}t\Big|$$
$$\le C_0(\mu/\lambda)^{1/2}\Lambda\Big(\int_0^T ||w_h||_{L^2(\Omega)}^2\,\mathrm{d}t\Big)^{1/2}$$

where $\Lambda := c_0c_1C_0(1+\lambda)(\mu/\lambda)kh^{-1-n/2}$.

Proof. Replacing $v_{h,k}^+ = \partial_t\hat{u}_{h,k} - k^{-1}r_{h,k}^+$ in (b) of Algorithm (A) we find for almost all $t \in (0,T)$ that

$$LHS(t) := \lambda\big(\partial_t\hat{u}_{h,k};w_h\big) - \big((u_{h,k}^-\times\partial_t\hat{u}_{h,k});w_h\big) + \mu\big(\nabla u_{h,k}^-;\nabla w_h\big)$$
$$= \frac{1}{k}\Big(\lambda r_{h,k}^+ - \big(u_{h,k}^-\times r_{h,k}^+\big);w_h\Big) =: RHS(t).$$

Hölder's inequalities, the estimates of Lemma 3.1, and (3.3) prove for almost all $t \in (jk,(j+1)k)$ that

$$\begin{aligned} k|RHS(t)| &\le \lambda||r_h^{(j+1)}||_{L^2(\Omega)}||w_h||_{L^2(\Omega)} + ||u_h^{(j)}||_{L^\infty(\Omega)}||r_h^{(j+1)}||_{L^2(\Omega)}||w_h||_{L^2(\Omega)} \\ &\le (\lambda+1)c_1k^2h^{-n/2}||v_h^{(j+1)}||_{L^2(\Omega)}^2||w_h||_{L^2(\Omega)} \\ &\le (\lambda+1)c_0c_1(\mu/\lambda)C_0k^2h^{-1-n/2}||v_h^{(j+1)}||_{L^2(\Omega)}||w_h||_{L^2(\Omega)}. \end{aligned}$$

An integration over $(0,T)$ shows

$$\Big|\int_0^T LHS(t)\,\mathrm{d}t\Big| \le \int_0^T|RHS(t)|\,\mathrm{d}t \le \Lambda\Big(\int_0^T||v_{h,k}^+||_{L^2(\Omega)}^2\Big)^{1/2}\Big(\int_0^T||w_h||_{L^2(\Omega)}^2\Big)^{1/2}$$

and the bound $\int_0^T||v_{h,k}^+||_{L^2(\Omega)}^2\,\mathrm{d}t \le k\sum_{j=0}^{J-1}||v_h^{(j+1)}||_{L^2(\Omega)}^2 \le (\mu/\lambda)C_0^2$ of Lemma 3.2 finishes the proof. □

Theorem 3.1. *Suppose that $(k,h) \to 0$ such that $kh^{-1-n/2} \to 0$ and $u_h^{(0)} \to u_0$ in $H^1(\Omega)$. Given any $T \in [0,Jk]$ and $\Omega_T := (0,T)\times\Omega$ there exists $u \in H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;H^1(\Omega))$ such that (after extraction of a subsequence) $\hat{u}_{h,k} \rightharpoonup u$ in $H^1(\Omega_T)$. It follows that $|u(t,x)| = 1$ for almost all $(t,x) \in \Omega_T$, $u(0,\cdot) = u_0$ in the sense of traces,*

$$\lambda\int_{(0,T')\times\Omega}|\partial_t u|^2\,\mathrm{d}x\,\mathrm{d}t + \frac{\mu}{2}\int_\Omega|\nabla u(x,T')|^2\,\mathrm{d}x \le \frac{\mu}{2}\int_\Omega|\nabla u_0(x)|^2\,\mathrm{d}x \tag{3.5}$$

for almost all $T' \in (0,T)$, and for all $\phi \in C^\infty(\Omega_T)$ we have

$$\int_{\Omega_T} \partial_t u \cdot \phi \,\mathrm{d}x\,\mathrm{d}t + \lambda \int_{\Omega_T} (u \times \partial_t u) \cdot \phi \,\mathrm{d}x\,\mathrm{d}t = -\mu \int_{\Omega_T} \nabla u \cdot \nabla(u \times \phi)\,\mathrm{d}x\,\mathrm{d}t. \tag{3.6}$$

Proof. Lemma 3.2 and the estimate $||\hat{u}_{h,k} - u^-_{h,k}||_{L^2(\Omega)} \le k||\partial_t \hat{u}_{h,k}||_{L^2(\Omega)}$ yield the existence of some $u \in H^1(\Omega_T)$ such that, after extraction of a subsequence,

$$\hat{u}_{h,k} \rightharpoonup u \quad \text{in } H^1(\Omega_T), \qquad u^-_{h,k} \to u \quad \text{in } L^2(\Omega_T),$$
$$\hat{u}_{h,k},\, u^-_{h,k} \rightharpoonup^* u \quad \text{in } L^\infty(0,T;H^1(\Omega)).$$

Notice that $|u^-_{h,k}(z,t)| = 1$ for all $z \in \mathcal{N}$ and almost all $t \in (0,T)$. Hence, for all $K \in \mathcal{T}$,

$$\begin{aligned}\big\||u^-_{h,k}|^2 - 1\big\|_{L^2(K)} &\le ch\big\|\nabla[|u^-_{h,k}|^2 - 1]\big\|_{L^2(K)}\\ &= ch||2(\nabla u^-_{h,k})u^-_{h,k}||_{L^2(K)} \le 2ch\|\nabla u^-_{h,k}||_{L^2(K)}.\end{aligned}$$

This implies that $|u^-_{h,k}| \to 1$ in $L^2(\Omega_T)$ and hence $|u(x,t)| = 1$ for almost all $(x,t) \in \Omega_T$. Continuity of the trace operator and $\hat{u}_{h,k}(0,\cdot) \to u_0$ in $H^1(\Omega)$ imply $u(0,\cdot) = u_0$ in Ω in the sense of traces. Passing to the limits in the estimate of Lemma 3.2 we verify (3.5). Given $\phi \in C^\infty(\Omega_T)$ let $w := u\times\phi$ and $w_h := \mathcal{I}_h(u^-_{h,k}\times\phi)$. An application of the triangle inequality shows

$$\begin{aligned}||w_h - w||_{L^2(\Omega)} &\le ||\mathcal{I}_h(u^-_{h,k} \times \phi) - u^-_{h,k} \times \phi||_{L^2(\Omega)} + ||u^-_{h,k} \times \phi - u \times \phi||_{L^2(\Omega)}\\ &\le ch||\nabla(u^-_{h,k} \times \phi)||_{L^2(\Omega)} + ||\phi||_{L^\infty(\Omega)}||u^-_{h,k} - u||_{L^2(\Omega)}\end{aligned}$$

and proves that $w_h \to w$ in $L^2(\Omega_T)$. Since $\partial_t \hat{u}_{h,k} \rightharpoonup \partial_t u$ in $L^2(\Omega_T)$ we verify that

$$\int_{\Omega_T} \partial_t \hat{u}_{h,k} \cdot w_h \,\mathrm{d}x\,\mathrm{d}t \to \int_{\Omega_T} \partial_t u \cdot w \,\mathrm{d}x\,\mathrm{d}t. \tag{3.7}$$

The bound $|u^-_{h,k}| \le 1$ almost everywhere in Ω_T yields

$$\begin{aligned}\int_{\Omega_T} \big(u^-_{h,k} \times \partial_t \hat{u}_{h,k}\big) \cdot w_h \,\mathrm{d}x\,\mathrm{d}t &= \int_{\Omega_T} \big(u^-_{h,k} \times \partial_t \hat{u}_{h,k}\big) \cdot (w_h - w)\,\mathrm{d}x\,\mathrm{d}t\\ &+ \int_{\Omega_T} \big(u^-_{h,k} \times \partial_t \hat{u}_{h,k}\big) \cdot w \,\mathrm{d}x\,\mathrm{d}t \to \int_{\Omega_T} \big(u \times \partial_t u\big) \cdot w \,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{3.8}$$

It follows that

$$\begin{aligned}\int_{\Omega_T} \nabla u^-_{h,k} \cdot \nabla w_h \,\mathrm{d}x\,\mathrm{d}t &= \int_{\Omega_T} \nabla u^-_{h,k} \cdot \nabla \mathcal{I}_h(u^-_{h,k} \times \phi)\,\mathrm{d}x\,\mathrm{d}t\\ = \int_{\Omega_T} &\nabla u^-_{h,k} \cdot \nabla\big(\mathcal{I}_h(u^-_{h,k} \times \phi) - u^-_{h,k} \times \phi\big)\,\mathrm{d}x\,\mathrm{d}t\\ &+ \int_{\Omega_T} \nabla u^-_{h,k} \cdot \nabla(u^-_{h,k} \times \phi)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Notice that $u^-_{h,k}$ is $\mathcal{T}$-elementwise affine and $\phi \in C_0^\infty(\Omega_T)$ so that for each $K \in \mathcal{T}$ we have

$$\begin{aligned}||\nabla\big(\mathcal{I}_h(u^-_{h,k} \times \phi) - u^-_{h,k} \times \phi\big)||_{L^2(K)} &\le ch||D^2(u^-_{h,k} \times \phi)||_{L^2(K)}\\ &\le c'h||\phi||_{W^{2,\infty}(K)}\big(||\nabla u^-_{h,k}||_{L^2(K)} + 1\big)\end{aligned} \tag{3.9}$$

and hence that $\nabla\big(\mathcal{I}_h(u_{h,k}^-\times\phi)-u_{h,k}^-\times\phi\big)\to 0$ in $L^2(\Omega_T)$. We use $\nabla u_{h,k}^-\cdot\nabla\big(u_{h,k}^-\times\phi\big)$ $=\nabla u_{h,k}^-\cdot\big(u_{h,k}^-\times\nabla\phi\big)$ and $\nabla u\cdot(u\times\nabla\phi)=\nabla u\cdot\nabla(u\times\phi)$ to verify

$$\begin{aligned}\int_{\Omega_T}\nabla u_{h,k}^-\cdot\nabla(u_{h,k}^-\times\phi)\,\mathrm{d}x\,\mathrm{d}t &= \int_{\Omega_T}\nabla u_{h,k}^-\cdot(u_{h,k}^-\times\nabla\phi)\,\mathrm{d}x\,\mathrm{d}t\\ &\to\int_{\Omega_T}\nabla u\cdot(u\times\nabla\phi)\,\mathrm{d}x\,\mathrm{d}t=\int_{\Omega_T}\nabla u\cdot\nabla(u\times\phi)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

A combination of the previous assertions shows

$$\int_{\Omega_T}\nabla u_{h,k}^-\cdot\nabla w_h\,\mathrm{d}x\,\mathrm{d}t\to\int_{\Omega_T}\nabla u\cdot\nabla w\,\mathrm{d}x\,\mathrm{d}t. \tag{3.10}$$

Since $w_h\to w$ in $L^2(\Omega_T)$ we obtain a uniform bound for $\int_0^T||w_h||^2_{L^2(\Omega)}\,\mathrm{d}t$. Using (3.7)-(3.10) to pass to the limits in the estimate of Lemma 3.3 we verify that

$$\lambda\int_{\Omega_T}\partial_t u\cdot(u\times\phi)\,\mathrm{d}x\,\mathrm{d}t-\int_{\Omega_T}\big(u\times\partial_t u\big)\cdot(u\times\phi)\,\mathrm{d}x\,\mathrm{d}t=-\mu\int_{\Omega_T}\nabla u\cdot\nabla(u\times\phi)\,\mathrm{d}x\,\mathrm{d}t.$$

Therefore, $\partial_t u\cdot(u\times\phi)=-(u\times\partial_t u)\cdot\phi$ and $(u\times\partial_t u)\cdot(u\times\phi)=\partial_t u\cdot\phi$ (since $|u|=1$), which implies (3.6) and finishes the proof of the theorem. $\square$

4. Increased efficiency through reduced integration

In order to increase the efficiency of our approximation scheme we employ reduced integration, i.e., we use a modified Algorithm (A′), which is obtained by replacing (b) in Algorithm (A) by the following:

(b′) Compute $v_h^{(j+1)}\in L^{(j)}:=\{w_h\in\mathcal{S}^1(\mathcal{T})^3: w_h(z)\cdot u_h^{(j)}(z)=0 \text{ for all } z\in\mathcal{N}\}$ such that

$$\lambda\big(v_h^{(j+1)};w_h\big)_h-\big(u_h^{(j)}\times v_h^{(j+1)};w_h\big)_h=-(1+\lambda^2)\big(\nabla u_h^{(j)};\nabla w_h\big)\qquad\text{for all } w_h\in L^{(j)}.$$

Here, given any $\eta,\psi\in C(\overline{\Omega};\mathbb{R}^\ell)$ we set $\big(\eta;\psi\big)_h:=\int_\Omega\mathcal{I}_h(\eta\cdot\psi)\,\mathrm{d}x$. Since $||\phi_h||^2_{L^2(\Omega)}$ $\le(\phi_h;\phi_h)_h$ for all $\phi_h\in\mathcal{S}^1(\mathcal{T})$, Lemma 3.1 and Lemma 3.2 remain unchanged for Algorithm (A′). A modified version of Lemma 3.3 holds. Using that

$$\big|(\partial_t\hat{u}_{h,k};w_h)-(\partial_t\hat{u}_{h,k};w_h)_h\big|\le c_2h||\partial_t\hat{u}_{h,k}||_{L^2(\Omega)}||\nabla w_h||_{L^2(\Omega)}$$

and

$$\begin{aligned}&\big|(u_{h,k}^-\times\partial_t\hat{u}_{h,k};w_h)-(u_{h,k}^-\times\partial_t\hat{u}_{h,k};w_h)_h\big|\\ &\quad=\big|(\partial_t\hat{u}_{h,k};u_{h,k}^-\times w_h)-(\partial_t\hat{u}_{h,k};u_{h,k}^-\times w_h)_h\big|\\ &\quad\le\big|(\partial_t\hat{u}_{h,k};u_{h,k}^-\times w_h)-(\partial_t\hat{u}_{h,k};\mathcal{I}_h[u_{h,k}^-\times w_h])\big|\\ &\qquad+\big|(\partial_t\hat{u}_{h,k};\mathcal{I}_h[u_{h,k}^-\times w_h])-(\partial_t\hat{u}_{h,k};\mathcal{I}_h[u_{h,k}^-\times w_h])_h\big|\\ &\quad\le ch||\partial_t\hat{u}_{h,k}||_{L^2(\Omega)}||\nabla\mathcal{I}_h[u_{h,k}^-\times w_h]||_{L^2(\Omega)}\\ &\quad\le c_2h||\partial_t\hat{u}_{h,k}||_{L^2(\Omega)}\big(||\nabla w_h||_{L^2(\Omega)}+||w_h||_{L^\infty(\Omega)}||\nabla u_{h,k}^-||_{L^2(\Omega)}\big)\end{aligned}$$

we verify with the bounds of Lemma 3.2 that

$$\begin{aligned}\Big|\lambda\int_{\Omega_T}\partial_t\hat{u}_{h,k}\cdot w_h\,\mathrm{d}x\,\mathrm{d}t-\int_{\Omega_T}\big(u^-_{h,k}\times\partial_t\hat{u}_{h,k}\big)\cdot w_h\,\mathrm{d}x\,\mathrm{d}t+\mu\int_{\Omega_T}\nabla u^-_{h,k}\cdot\nabla w_h\,\mathrm{d}x\,\mathrm{d}t\Big|\\ \le C_0(\mu/\lambda)^{1/2}\Lambda\Big(\int_0^T||w_h||^2_{L^2(\Omega)}\,\mathrm{d}t\Big)^{1/2}\\ +c_2C_0(1+\lambda)(\mu/\lambda)^{1/2}h\Big(\int_0^T||\nabla w_h||^2_{L^2(\Omega)}\,\mathrm{d}t\Big)^{1/2}\\ +c_2C_0^2(\mu/\lambda)^{1/2}T^{1/2}h||w_h||_{L^\infty(\Omega_T)}\end{aligned}$$

for all w_h, as in Lemma 3.3. The proof of Theorem 3.1 then requires bounds for $\int_0^T||\nabla w_h||^2_{L^2(\Omega)}\,\mathrm{d}t$ and $||w_h||_{L^\infty(\Omega_T)}$ with $w_h=\mathcal{I}_h(u^-_{h,k}\times\phi)$. The first bound can be deduced from (3.9) and the second one follows immediately from $||u^-_{h,k}||_{L^\infty(\Omega_T)}=1$.

Remark 4.1. Reduced integration not only leads to simpler systems of equations but also has a stabilising effect. Indeed, for $\Omega\subset\mathbb{R}$ and a uniform triangulation $\mathcal{T}$ of Ω into intervals of length h it follows that

$$\big(v_h;v_h\big)_h=||v_h||^2_{L^2(\Omega)}+\frac{h^2}{6}||v_h'||^2_{L^2(\Omega)}$$

for all $v_h\in\mathcal{S}^1(\mathcal{T})^3$. Then, the choice $w_h=v_h^{(j+1)}$ in (b′) above yields to the identity

$$||v_h^{(j+1)}||^2_{L^2(\Omega)}+||\nabla v_h^{(j+1)}||^2_{L^2(\Omega)}=-\mu\big(\nabla u_h^{(j)};\nabla v_h^{(j+1)}\big),$$

which allows for slightly better estimates than the proof of Lemma 3.2. However, this does not seem to lead to a significantly improved estimate than the one given in Lemma 3.2.

5. Experiments seeking blowup

We report the results of our experiments on singularity formation of (1.3) for $\Omega=(-1/2,1/2)^2$. Since we will work in the equivariant setting, a setting that has been considered in work on singularity formation for the harmonic map heat flow problem, e.g., [4, 5], we begin with some notation. Let (α,r) denote domain polar coordinates and (θ,ϕ) spherical coordinates on S^2. A point (θ,ϕ) corresponds to the point $(\cos\theta\sin\phi,\sin\theta\sin\phi,\cos\phi)$. An equivariant map is given by

$$(\alpha,r)\to(\theta,\phi(r)),$$

where $\theta=l\alpha+\tilde{\theta}(r),l\in\mathbb{Z}$.

Our choice of initial data is influenced by strong evidence that static solutions play a crucial role in singularity formation of LLG, as they do in the harmonic map heat flow. From the formulation of LLG given in (1.6), we see that the static solutions of LLG are exactly those of the harmonic map heat flow, namely, *harmonic maps* that are solutions to

$$-\Delta u=|\nabla u|^2u.$$

There are plenty of nontrivial equivariant harmonic maps. A family of solutions $\phi:D\to S^2$ results from the observation that $\phi(r)=2\tan^{-1}r$ is a solution. By scaling $r\to r/\rho$, $\rho>0$, the maps

$$\phi_\rho(r)=2\tan^{-1}(r/\rho)$$

are also solutions. In the construction of the Struwe solution carried out in [8], as a singular time is approached, energy concentration occurs and after appropriate rescaling, a harmonic map separates. Using the energy bound

$$E(u) \le E(u_0)$$

and the observation that, for u harmonic and conformal,

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 = \text{Area of Image}(u),$$

an immediate consequence of the Struwe solution construction in the equivariant setting is that if $E[u_0] < 4\pi$, no singularities can form. Recently [9] reports orbital stability of LLG about harmonic maps before blowup. Even though the blowup time has not analytically been shown to be finite, our candidate initial data supports this picture that near the time of blowup, a harmonic map is approached.

The initial data u_0 is prescribed as follows:

$$\theta = \alpha,\ \phi(r) = \begin{cases} 2\tan^{-1}\rho(r), & r \le 1/2 \\ \pi, & r \ge 1/2 \end{cases},\ \rho(r) = \frac{r}{A(r)}.$$

$A(r)$ is chosen with the following properties:

(i) $A(1/2) = 0$.
(ii) The energy density $E(u_0) = \phi_r^2 + r^{-2}\sin\phi^2$ has a decreasing profile, peaked at $r = 0$ and 0 at $r = 1/2$.

A function that satisfies these conditions is $A = (1-2r)^4/s$, where increasing s sharpens the concentration of the energy density about the origin. Elementary calculations show that

$$u_0(r) = (\cos\theta\sin\phi(r), \sin\theta\sin\phi(r), \cos\phi(r))$$

is then given by

$$u_0(r) = (2xA, 2yA, A^2 - r^2)/(A^2 + r^2).$$

Notice that $u_0(r) = (0, 0, -1)$ for $r \ge 1/2$ so that for $\Omega = (-1/2, 1/2)^2$, this initial data then wraps around the sphere once.

6. Numerical experiments

In this section we report on the practical performance of Algorithms (A) and (A′) in some numerical experiments and study finite time blowup of weak solutions. Moreover, we investigate the dependence of numerical approximations upon the parameter λ. The implementation of the algorithm was performed in MATLAB with an assembly of the stiffness matrices in C. The constraints included in the subspace $L^{(j)}$ were directly incorporated in the linear systems, which were solved using the backslash operator in MATLAB. We remark that also the scheme defined by reduced integration in Section 4 leads to linear systems of equations with nondiagonal system matrices in each time step.

Example 6.1. Let $\Omega := (-1/2, 1/2)^2$ and let $u_0 : \Omega \to \mathbb{R}^3$ be defined by

$$u_0(x) := \begin{cases} (0,0,-1) & \text{for } |x| \geq 1/2, \\ \big(2xA, A^2 - |x|^2\big)/\big(A^2 + |x|^2\big) & \text{for } |x| \leq 1/2, \end{cases}$$

where $A := (1-2|x|)^4/s$ for some $s > 0$. The triangulations $\mathcal{T}_\ell$ used in the numerical simulations are defined through a positive integer ℓ and consist of $2^{2\ell+1}$ halved squares with edge length $h := 2^{-\ell}$. Motivated by Lemma 3.2 we use $k = (\mu/\lambda)h^{5/2}/10$ (unless otherwise stated), where the additional power $h^{1/2}$ guarantees that $kh^{-2},\ kh^{-1-n/2} \to 0$ (for $n = 2$). As discrete initial data we employed the nodal interpolant of u_0, i.e., we set $u_h^{(0)} := \mathcal{I}_{\mathcal{T}_\ell} u_0$ in all experiments.

We ran Algorithm (A$'$) in Example 6.1 with $s = 1$, $\ell = 4$, and $\lambda = 1$. Figure 1 shows snapshots of the numerical solution for various times; displayed are the orthogonal projection of the vector field $\hat{u}_{h,k}(t,\cdot)$ onto the plane $\{(x,y,0) : x, y \in \mathbb{R}\}$. We observe that for $t \approx 0.0586$ the vector $\hat{u}_{h,k}(t,0)$ changes its direction from $(1,0,0)$ to $-(1,0,0)$. Figure 2 magnifies this change of direction.

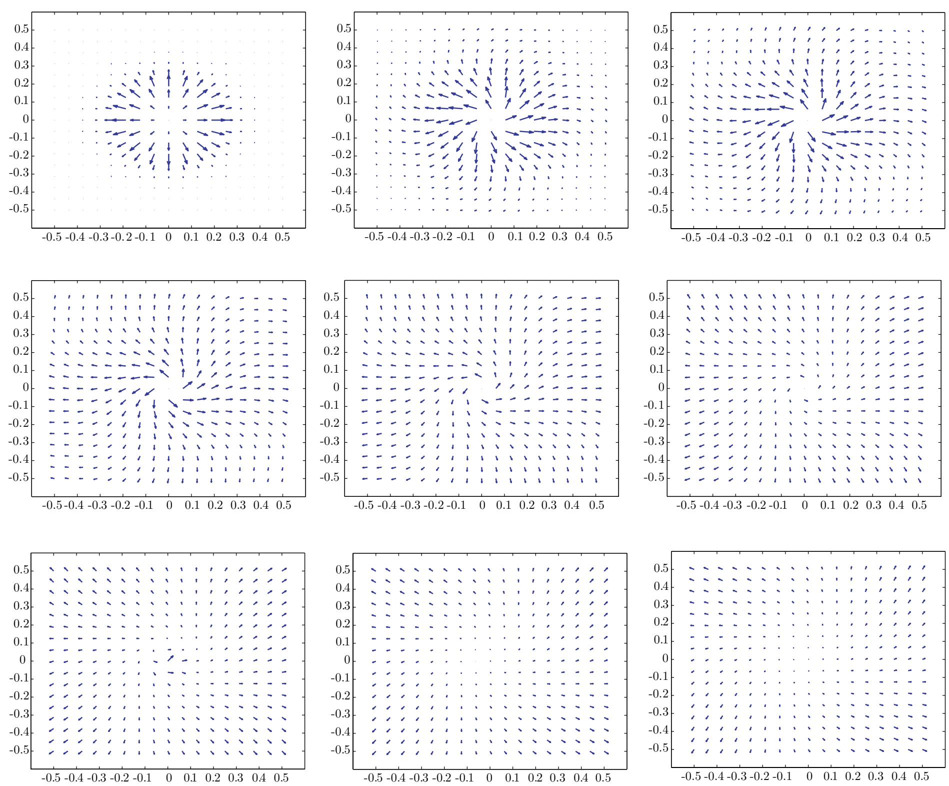

FIGURE 1. Numerical approximation $\hat{u}_{h,k}(t,\cdot)$ in Example 6.1 with $s = 1$, $\ell = 4$, and $\lambda = 1$ for $t = 0, 0.0098,$ $0.0195, 0.0293, 0.0391, 0.0488, 0.0586, 0.0684, 0.0781.$

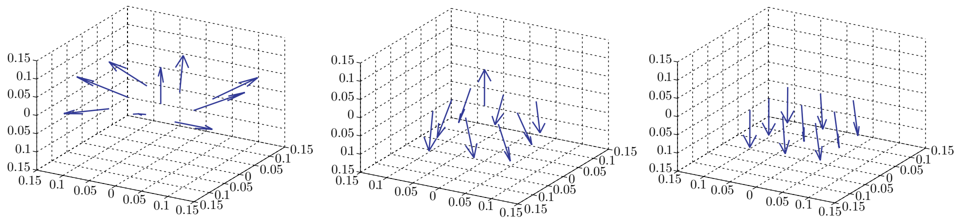

FIGURE 2. Nodal values $\hat{u}_{h,k}(t,z)$ for nodes $z \in \mathcal{N}$ close to the origin in Example 6.1 with $s = 1$, $\ell = 4$, and $\lambda = 1$ for $t = 0.0098, 0.0488, 0.0684$.

6.1. **Instability of the numerical scheme for $k = O(h^2)$ and stabilizing effect of reduced integration.** Our first numerical experiment reveals that the relation $k \sim h^2$ is not sufficient to guarantee stability and convergence of our approximation scheme. We ran Algorithms (A) and (A$'$) in Example 6.1 with $\lambda = 1$, $s = 1$ and using the triangulations $\mathcal{T}_\ell$ for $\ell = 4, 5$. For both Algorithms we tried the time step sizes $k_1 = (\mu/\lambda)h^{5/2}/10$ and $k_2 = (\mu/\lambda)h^2/10$. Figure 3 displays the energy

$$E(\hat{u}_{h,k}(t)) = \frac{1}{2}\int_\Omega |\nabla \hat{u}_{h,k}(t)|^2 \,\mathrm{d}x$$

as a function of time in the interval $(0,1)$. The energy is not decreasing for k_2 in Algorithm (A) which indicates instability of Algorithm (A) if the time-step size violates the conditions of Lemma 3.2. The results also show that reduced integration stabilizes the scheme as no instability is observable when Algorithm (A$'$) is used with the large time-step size k_2. We remark that reduced integration significantly increased the efficiency of our scheme, e.g., in the above experiments the CPU time for Algorithm (A$'$) was about 10% of that of Algorithm (A).

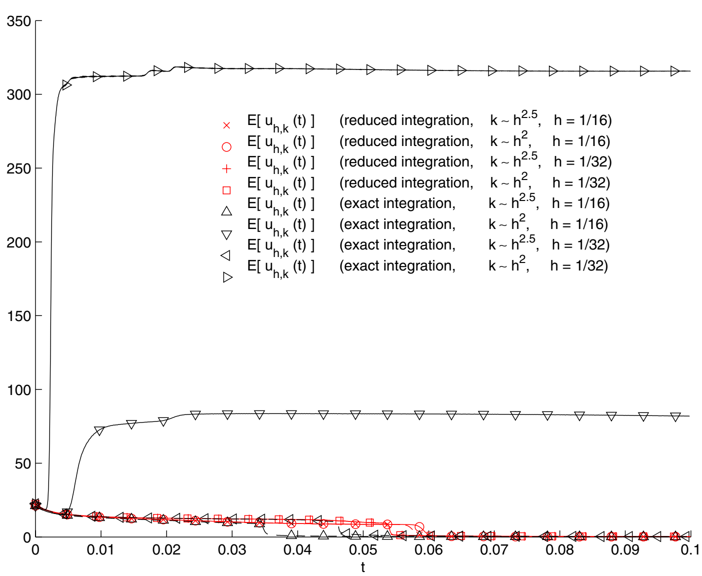

FIGURE 3. Energy for different discretization parameters in Algorithms (A) (exact integration) and (A$'$) (reduced integration).

6.2. **Breakdown of blowup for higher resolution and comparison to Dirichlet boundary conditions.** For fixed $\lambda = 1$ and $s = 1$ we tried $\ell = 4, 5, 6$ in Example 6.1, and in Figure 4 we displayed the energy $E(\hat{u}_{h,k}(t))$ and the $W^{1,\infty}$ semi-norm $||\nabla \hat{u}_{h,k}(t)||_{L^\infty(\Omega)}$ as functions of t for $t \in (0, 6/100)$ for $\ell = 4, 5, 6$. For each $\ell = 4, 5$, $||\nabla \hat{u}_{h,k}(t)||_{L^\infty(\Omega)}$ assumes the maximum value $2\sqrt{2}h^{-1}$ (among functions $v_h \in \mathcal{S}^1(\mathcal{T}_\ell)^3$ with $|v_h(z)| = 1$ for all $z \in \mathcal{N}$). Surprisingly, this is not the case for $\ell = 6$, indicating a breakdown of the (discrete) finite-time blowup for sufficiently high resolutions. For Dirichlet boundary conditions, (discrete) finite-time blowup still occurs for $\ell = 6$. We remark that our algorithm and analysis can be used for time-independent Dirchlet boundary conditions by choosing the initial $u_h^{(0)}$ appropriately and employing

$$L^{(j)} := \{w_h \in \mathcal{S}^1(\mathcal{T})^3 : w_h(z) \cdot u_h^{(j)}(z) = 0 \text{ for all } z \in \mathcal{N},\, w_h|_{\partial\Omega} = 0\}.$$

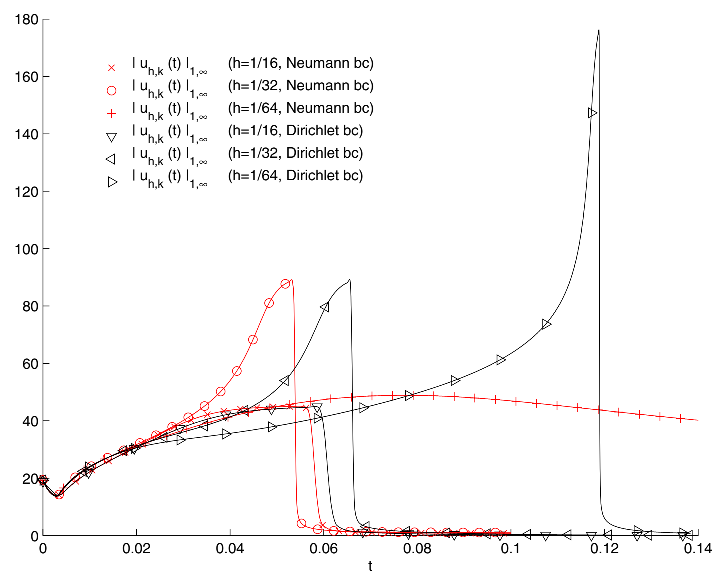

FIGURE 4. $W^{1,\infty}$ semi-norm for decreasing mesh sizes in Example 6.1 with $\lambda = 1$ and $s = 1$ for Neumann and Dirichlet type boundary conditions.

ACKNOWLEDGMENTS

Part of the work was written when S.B. visited Forschungsinstitut für Mathematik (ETH Zürich) in January 2005 and Brown University in March 2005. S.B. gratefully acknowledges hospitality by the Department of Mathematics of the University of Maryland at College Park. S.B. was partially funded by NSF grant DMS-0405853.

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