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EQUILIBRATED RESIDUAL ERROR ESTIMATOR FOR EDGE ELEMENTS

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ABSTRACT. Reliable a posteriori error estimates without generic constants can be obtained by a comparison of the finite element solution with a feasible function for the dual problem. A cheap computation of such functions via equilibration is well known for scalar equations of second order. We simplify and modify the equilibration such that it can be applied to the curl-curl equation and edge elements. The construction is more involved for edge elements since the equilibration has to be performed on subsets with different dimensions. For this reason, Raviart–Thomas elements are extended in the spirit of distributions.

1. INTRODUCTION

Recently, a posteriori error estimates without constants have attracted much interest; see Ainsworth and Oden [1], Neittaanmäki and Repin [17], Vejchodský [20], Luce and Wohlmuth [13] and also Ladevèze and Leguillon [12]. At first glance they look like estimators which use local Neumann problems as introduced by Bank and Weiser [5], but they are based on a comparison of primal and dual forms of the variational problems. Following Prager and Synge (1949) in the special case of the Poisson equation $-\Delta u = f$ in Ω, one compares a finite element approximation $u_h \in H^1(\Omega)$ and a function $\sigma \in H(\text{div})$ that satisfies the equilibrium condition $\operatorname{div} \sigma + f = 0$. In principle, the latter can be obtained via mixed methods, but in practical computations a feasible function σ is constructed by an equilibration of ∇u_h.

The equilibration can be done by solving local problems; see [1, Chapter 6.4]. The solution of local problems by polynomials of sufficiently high order is avoided in [20] by the combination with a variant of the hypercircle method and in [13] by the introduction of a dual mesh. We will go a different way in order to avoid generic constants in the main term of the upper estimate. A small portion of the error that results from the data oscillation is estimated in the classical manner. So the local problems can be solved on finite-dimensional spaces, and no generic constant enters via approximation arguments. Moreover, the local problems are solved on patches around vertices of the mesh in order to avoid nonlocal auxiliary quantities. The procedure becomes transparent, and a generalization to other types

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of elliptic problems is now natural (although not trivial). In the 2D case there is even a simple geometrical interpretation of the resulting equilibration procedure.

In the present paper we also establish a posteriori error estimators with similar properties for edge elements and the equations of magnetostatics. There is an analogue to the result of Prager and Synge although we have to deal with different Sobolev spaces and edge elements. The equilibration in $H(\mathrm{curl})$, however, is more involved since the splitting of the residual currents into local divergence-free currents has to be done with more constraints. Moreover, the constraints refer to currents on geometrical objects with different dimensions.

To overcome these obstacles we proceed as we have shown for the Poisson equation. We extend the Raviart–Thomas elements and Nédélec elements to finite element spaces such that the differential operators curl and div act on distributions. We show that the differential operators and the extended spaces still form exact sequences. The sequences generalize the de Rham sequences and the discrete analogs that were frequently used in the last years for constructing and understanding new finite element spaces [2, 3, 11].

In Section 2 we write the equilibration procedure for the (scalar) Poisson equation in our setting in order to make the reader familiar with the modifications for avoiding generic constants. The resulting local problems will differ from those in the literature. The distributional Raviart–Thomas elements and the corresponding Nédélec elements are introduced in Section 3. Details are provided for the 2-dimensional case while the discussion of the 3-dimensional case is treated more briefly. Section 4 contains the application to a posteriori error estimators for the curl-curl equation.

The Sobolev spaces $H^1(\Omega)$, $H^1_0(\Omega)$, $H(\mathrm{div},\Omega)$ and $H(\mathrm{curl},\Omega)$ are defined as usual. The specification of the domain will often be suppressed when there is no danger of ambiguity.

2. Equilibrated residual error estimates for scalar equations

In this section we consider the scalar equation

$$-\Delta u = f \tag{2.1}$$

on a polygonal domain Ω in 2-space or 3-space. Moreover, let $u = 0$ on a nonempty subset $\Gamma_D \subset \partial\Omega$ and $\partial u/\partial n = 0$ on $\Gamma_N := \partial\Omega\backslash\Gamma_D$. The a posteriori error estimates in [1, 12, 17, 15, 13, 20] are related to a result of Prager and Synge [18] although their presentations are very different. We provide ideas for achieving estimates without generic constants and with simpler local problems. For convenience, we restrict ourselves to the Poisson equation. The generalization to equations with piecewise constant coefficients will be clear from the considerations in Section 4.

The finite element solutions of (2.1) are determined on a triangulation of Ω into triangles or tetrahedra, $\bar\Omega = \bigcup_T T$. Let

$$\mathcal{M}^k_{-1} := \{v \in L_2(\Omega);\ v|_T \in \mathcal{P}_k\}, \quad \mathcal{M}^k_0 := \mathcal{M}^k_{-1} \cap C^0(\Omega)$$

be the sets of polynomial Lagrange finite elements on the triangulation above, and let $u_h \in V_h := \mathcal{M}^1_0$ (with the essential boundary conditions incorporated) be the finite element solution for linear elements, i.e.,

$$(\nabla u_h, \nabla v) = (f, v) \quad \text{for } v \in V_h\,. \tag{2.2}$$

The distribution

$$f_h := -\Delta u_h \tag{2.3}$$

is a functional on $H^1(\Omega)$ and evaluates to

$$\begin{aligned}\langle f_h, v\rangle &= (\nabla u_h, \nabla v) = \sum_T \left\{-\int_T \Delta u_h\, v + \int_{\partial T} \frac{\partial u_h}{\partial n}\, v\right\}\\ &=: \sum_F \int_F f_{h,F}\, v\,. \end{aligned}\tag{2.4}$$

Here F runs over the faces of the elements (and over the edges in the 2D case, resp.), and $f_{h,F} := [\frac{\partial u_h}{\partial n}] := \frac{\partial u_{h,l}}{\partial n_l} + \frac{\partial u_{h,r}}{\partial n_r}$. In particular, the right-hand side of (2.4) is understood as the face contributions of the divergence of ∇u_h.

Since we treat the d-dimensional case for $d = 2$ and $d = 3$ simultaneously, we use the letter F for $(d-1)$-dimensional simplices. In particular, (2.4) describes the face contribution of the divergence of ∇u_h. In general, contributions of tetrahedral/triangular elements, faces, edges and vertices are distinguished by the labels T, F, E, and V, respectively.

2.1. **The theorem of Prager and Synge.**

Theorem 1 (Theorem of Prager and Synge)**.** *Let $\sigma \in H(\mathrm{div})$, $\sigma \cdot n = 0$ on Γ_N while $v \in H^1(\Omega)$, $v = 0$ on Γ_D and assume that*

$$\operatorname{div} \sigma + f = 0. \tag{2.5}$$

Furthermore, let u be the solution of the Poisson equation (2.1)*. Then,*

$$\|\nabla u - \nabla v\|^2 + \|\nabla u - \sigma\|^2 = \|\nabla v - \sigma\|^2. \tag{2.6}$$

A proof can be found in [18, 8]; see also Theorem 10.

We are looking for a cheap construction of a function σ that satisfies (2.5). First we assume that f is constant on each element. Then there is such a function σ in the Raviart–Thomas space $\mathcal{RT}$:

$$\begin{aligned}\mathcal{RT}_{-1} &:= \{\tau \in [L_2(\Omega)]^d;\ \tau|_T = a_T + b_T x,\ a_T \in \mathbb{R}^d,\ b_T \in \mathbb{R}\ \forall T\},\\ \mathcal{RT} &:= \mathcal{RT}_{-1} \cap H(\mathrm{div}).\end{aligned}$$

If we solve the original equation (2.1) by the mixed method with the Raviart–Thomas element [8, pp. 148, 181], then we would yield the function $\sigma \in \mathcal{RT}$ with (2.5) for which

$$\|\nabla u_h - \sigma\|$$

is minimal. Indeed, it follows from (2.6) that this is equivalent to the minimization of $\|\nabla u - \sigma\|$, and here the minimum is attained at the solution of the mixed method of Raviart–Thomas. This procedure, however, would be too expensive for computing a posteriori error estimates.

We rather construct a function σ satisfying (2.5) from the given finite element solution u_h by a local procedure usually called *equilibration*. We perform the construction for the difference $\sigma - \nabla u_h =: \sigma^\Delta$. Obviously, σ^Δ belongs to the *broken Raviart–Thomas space* $\mathcal{RT}_{-1}$ defined above. So we proceed in $\mathcal{RT}_{-1}$ and not on the continuous level. The conditions $\sigma^\Delta + \nabla u_h \in H(\text{div})$ and $\text{div}(\sigma^\Delta + \nabla u_h) = -f$ are rewritten as

$$\begin{aligned} \text{div}\, \sigma^\Delta &= -f && \text{in } T, \\ \left[\sigma^\Delta \cdot n\right] &= -\left[\nabla u_h \cdot n\right] && \text{on } F. \end{aligned} \tag{2.7}$$

2.2. **Equilibration.** Given a vertex V of the mesh, we assign to it the patch $\omega_V := \bigcup\{T,\ V \in \partial T\}$. The correction σ^Δ will be constructed from the solutions σ_{ω_V} of local problems on the patches:

$$\sigma^\Delta = \sum_V \sigma_{\omega_V}\,. \tag{2.8}$$

Here V runs over all vertices of the triangulation, and $\text{supp}\, \sigma_{\omega_V} \subset \omega_V$.

We recall that f is assumed to be constant on each element. Let V be a node of the mesh and ψ_V be the linear nodal function with $\psi_V(V) = 1$ and $\psi_V(x) = 0$ for $x \in \Omega\backslash\omega_V$. From the characterization (2.2) of u_h as a finite element solution and by partial integration we obtain

$$\sum_{T \subset \omega_V} \int_T f\psi_V = \sum_{T \subset \omega_V} \int_T \nabla u_h \nabla \psi_V = \sum_{F \subset \omega_V} \int_F \left[\frac{\partial u_h}{\partial n}\right] \psi_V\,.$$

Since ψ_V is piecewise linear, we have $\int_F \psi_V dx = \frac{1}{d}\int_F dx = \frac{1}{d}|F|$, and we obtain the central relation of this section:

$$\sum_{T \subset \omega_V} \int_T f\psi_V = \frac{1}{d} \sum_{F \subset \omega_V} \int_F f_{h,F}\,. \tag{2.9}$$

Now, we fix the functions of the decomposition (2.8) by

$$\begin{aligned} \text{div}\, \sigma_{\omega_V} &= -\tfrac{1}{|T|}\textstyle\int_T f\psi_V && \text{in } T \subset \omega_V, \\ [\sigma_{\omega_V} \cdot n] &= -\tfrac{1}{d}\left[\nabla u_h \cdot n\right] && \text{on } F \subset \omega_V, \\ \sigma_{\omega_V} \cdot n &= 0 && \text{on } \partial\omega_V. \end{aligned} \tag{2.10}$$

Remark 2. There are some modifications at the vertices on the boundary of Ω. If $\partial\omega_V \cap \partial\Omega \subset \Gamma_N$, there is no change apart from the geometry. If $V \in \Gamma_D$, then there is no test function associated with the vertex, and (2.9) does not hold for the vertex V. In this case, however, there is no boundary condition on $\partial\omega_V \cap \Gamma_D$ when we construct σ_{ω_V}. There is not a problem. Thus we will ignore adaptations at boundaries in the sequel.

The existence of solutions of (2.10) follows from the following lemma when it is applied to the patches ω_V. The assumption in the lemma is guaranteed by (2.9). Since $\sum_V \psi_V = 1$ and each face has d vertices, indeed, the sum (2.8) yields a solution of (2.7).

Lemma 3. *Let $\omega = \bigcup_T T \subset \mathbb{R}^d$, $d = 2$ or 3, be simply connected, and let $\bigcup_F F = \bigcup_T \partial T \backslash \partial\omega$ be a decomposition of the interelement boundaries. If the distribution g,*

$$\langle g, v\rangle := \sum_T \int_T g_T v + \sum_F \int_F g_F v \tag{2.11}$$

with piecewise constant functions $g_T,\ g_F$ *satisfies* $\langle g,1\rangle = 0$*, then there exists* $\sigma \in \mathcal{RT}_{-1}$ *such that*

$$\begin{array}{rcll} \sigma\cdot n &=& 0 & \text{on } \partial\omega,\\ \operatorname{div}\sigma &=& g_T & \text{in } T\subset\omega,\\ [\sigma\cdot n] &=& -g_F & \text{on } F\subset\omega. \end{array} \tag{2.12}$$

Moreover, there exists a constant c *depending only on the shape parameter of the mesh such that*

$$\|\sigma\|_0^2 \le c\Big(\sum_T h_T^2\|g_T\|_{L_2(T)}^2 + \sum_F h_F\|g_F\|_{L_2(F)}^2\Big).$$

Proof. First we reduce the given equations to a problem without face terms. We choose $\sigma^1 \in \mathcal{RT}_{-1}$ by setting

$$\sigma^1\cdot n = -\frac{1}{2}\, g_F \quad \text{at internal interfaces}$$

and $\sigma^1\cdot n = 0$ on $\partial\omega$. Thus, the face contributions of $\operatorname{div}\sigma^1$ coincide with the face contributions of g, and the difference is the regular function. Moreover, by Gauss' theorem $\big\langle \operatorname{div}\sigma^1, 1\big\rangle = \sum_T\int_T \operatorname{div}\sigma^1 - \sum_F\int_F[\sigma^1\cdot n] = 0$. Hence,

$$g-\operatorname{div}\sigma^1 \in \mathcal{M}^0_{-1}\subset L_2(\Omega) \quad \text{and} \quad \big\langle g-\operatorname{div}\sigma^1, 1\big\rangle = 0.$$

From Remark 2.1 in [3] we know that the sequence

$$\mathcal{RT}_{0,0} \xrightarrow{\operatorname{div}} \mathcal{M}^0_{-1} \xrightarrow{\int 1} \mathbb{R} \tag{2.13}$$

is exact, where $\mathcal{RT}_{0,0} := \{\tau\in\mathcal{RT}_{-1},\ \tau\cdot n = 0 \text{ on } \partial\omega\}$ and the second mapping is defined by $\int 1 : g\mapsto \langle g,1\rangle$. Thus there exists $\sigma^0\in\mathcal{RT}$ such that

$$\begin{array}{rcll} \operatorname{div}\sigma^0 &=& g-\operatorname{div}\sigma^1 & \text{in } \omega,\\ \sigma^0\cdot n &=& 0 & \text{on } \partial\omega. \end{array}$$

Setting $\sigma := \sigma^0+\sigma^1$ we obtain a solution of (2.12).

The stability estimate will be proven in Section 3.4 for more general cases. □

There is also a constructive proof if the analysis is restricted to the 2-dimensional case as in [8, p. 183].

The first result of the lemma can be understood as an extension of (2.13) being an exact sequence. Details will be given in the next section.

2.3. **Data oscillation.** Eventually, we want to abandon the assumption that f is piecewise constant. Let $\bar f$ be the L_2-projection of f onto piecewise constant functions. Since $\sum_V\int_T f\psi_V = \int_T \bar f$, the preceding investigation applies to the error if the right-hand side of (2.1) is replaced by $\bar f$. Now the difference between the solutions for f and $\bar f$ can be bounded by

$$ch\|f-\bar f\|. \tag{2.14}$$

This effect of the *data oscillation* is well known [8, p. 174]. We emphasize that the constant c depends on the shape of the elements, but it does not depend on the domain Ω. Since (2.14) is a term of higher order, we can admit a generic constant here.

2.4. **Efficiency.** By construction, the error estimate

$$\|\nabla(u-u_h)\| \le \|\sigma^\Delta\| + ch\|f-\bar{f}\|$$

is reliable. By Lemma 3, $\|\sigma^\Delta\|$ can be bounded by the terms $h_T\|f_T\|$ and $h_f^{1/2}\|f_{h,F}\|$, i.e., by the ingredients of the well-known residual error estimator. Thus $\|\sigma^\Delta\|$ is bounded by a multiple of that estimator. Since the residual error estimator is efficient, the same holds for the estimates determined by equilibration.

The results of this section are summarized for the Poisson equation as follows.

Theorem 4. *For each node V there exists a broken RT-function σ_{ω_V} with support in ω_V and satisfying* (2.10). *Choose σ_{ω_V} with (quasi-) minimal L_2-norm, and let $\sigma^\Delta := \sum_V \sigma_{\omega_V}$. Then we have the a posteriori error estimate*

$$c_0\|\sigma^\Delta\| - ch\|f-\bar{f}\| \;\le\; \|\nabla(u-u_h)\| \;\le\; \|\sigma^\Delta\| + ch\|f-\bar{f}\|. \tag{2.15}$$

3. Distributional finite element de Rham sequences

In the treatment of the scalar equation we already encountered distributional finite elements. In this section, we introduce and study exact sequences of finite elements which contain more distributional terms and are suitable for the equation of magnetostatics.

We start with the two-dimensional case and continue with three-dimensional finite elements.

Let Ω be a simply connected domain in $\mathbb{R}^2$. In 2D, we write curl for the differential operator $(\frac{\partial}{\partial y}, -\frac{\partial}{\partial x})$. Then, the de Rham sequence

$$\mathbb{R} \longrightarrow H^1 \xrightarrow{\text{curl}} H(\text{div}) \xrightarrow{\text{div}} L_2 \longrightarrow 0 \tag{3.1}$$

is an exact sequence [2]. This means that

- the operator curl has a trivial kernel in $H^1/\mathbb{R}$;
- the kernel $\{\sigma \in H(\text{div}) : \text{div}\,\sigma = 0\}$ of the operator div is exactly the range of the operator curl;
- the range of the operator div is exactly L_2.

An analogous property holds for the spaces with zero boundary conditions H_0^1 and $H_0(\text{div}) := \{\sigma \in H(\text{div}) : \sigma\cdot n = 0 \text{ on } \partial\Omega\}$:

$$0 \longrightarrow H_0^1 \xrightarrow{\text{curl}} H_0(\text{div}) \xrightarrow{\text{div}} L_2 \xrightarrow{\int 1} \mathbb{R} \longrightarrow 0. \tag{3.2}$$

As usual, the space $L_{2,0} := \{f \in L_2 : \int_\Omega f = 0\}$ of functions with zero mean values is identified with $L_2/\mathbb{R}$. We focus on sequences without boundary conditions, i.e., on sequences of type (3.1) in the following introductory discussion although we will deal later also with generalizations of (3.2).

Note that we find the right-hand part of the last exact sequence on the discrete level in (2.13).

3.1. **First distributional triangular elements.** The exact sequence property is inherited on the discrete level when we choose piecewise linear and continuous Lagrangian elements $\mathcal{M}_0^1$ for modeling H^1, the Raviart–Thomas elements $\mathcal{RT}$ for $H(\text{div})$, and piecewise constant, noncontinuous elements $\mathcal{M}_{-1}^0$ for L_2, [7, p. 175]:

$$\mathbb{R} \longrightarrow \mathcal{M}_0^1 \xrightarrow{\text{curl}} \mathcal{RT} \xrightarrow{\text{div}} \mathcal{M}_{-1}^0 \longrightarrow 0. \tag{3.3}$$

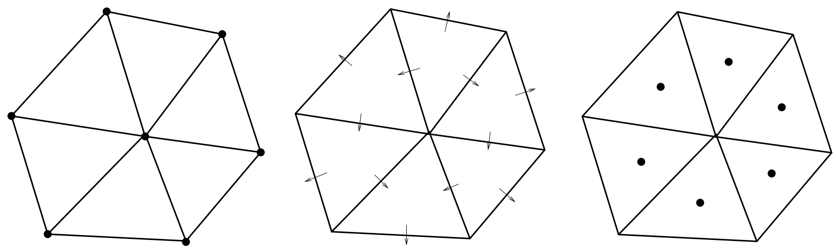

FIGURE 1. Classical finite element spaces in the sequence (3.3)

Let $u \in \mathcal{M}_0^1$, $\sigma \in \mathcal{RT}$, and $f \in \mathcal{M}_{-1}^0$. Their natural degrees of freedom are nodal values $\widehat{u}^{\,V} := u(V)$, edge integrals of the normal components $\widehat{\sigma}^{\,E} := \int_E \sigma \cdot n$, and element integrals $\widehat{f}^{\,T} := \int_T f$, respectively. Note that an orientation is associated with each edge for defining the normal components of $\mathcal{RT}$ functions.

Here and below, symbols with a hat refer to the integral over the geometrical object specified by the superscript. The representation of the differential operators with respect to these degrees of freedom depends only on the element topology and is independent of the shape of the elements. In terms of degrees of freedom we find

$$\sigma = \operatorname{curl} u \quad \text{as} \quad \widehat{\sigma}^{\,E} = \widehat{u}^{\,V_{E,1}} - \widehat{u}^{\,V_{E,2}},$$

where $V_{E,1}$ and $V_{E,2}$ are the two vertices of the edge E, ordered consistently with the previously defined normal vector. Similarly, the expression

$$f = \operatorname{div} \sigma \quad \text{reads as} \quad \widehat{f}^{\,T} = \sum_{E \subset T} \pm \widehat{\sigma}^{\,E},$$

where the sign depends on the orientation of the normal vector. Specifically, the sign is positive for normal vectors pointing to the outside of the triangle.

An element f in $\mathcal{M}_{-1}^0$ generates the regular distribution

$$\langle f, v \rangle = \sum_T \int_T f_T v.$$

For our purposes we introduce the space $\mathcal{M}_{-3}^0$ of distributions involving element, edge, and vertex terms:

$$\langle f, v \rangle = \sum_T \int_T f_T v + \sum_E \int_E f_E v + \sum_V f_V v(V). \tag{3.4}$$

The functions f_T and f_E are constant on each triangle and edge, respectively. The subspace of distributions of the form (3.4) with vanishing vertex terms is denoted as $\mathcal{M}_{-2}^0$.

First, we restrict ourselves to those distributions with element and vertex terms, i.e., with $f_V = 0$ for all V; see also (2.11). In this context we recall the extension of the Raviart–Thomas space to the broken Raviart–Thomas space and obtain the *first distributional de Rham sequence*

$$\mathbb{R} \longrightarrow \mathcal{M}_0^1 \xrightarrow{\operatorname{curl}} \mathcal{RT}_{-1} \xrightarrow{\operatorname{div}} \mathcal{M}_{-2}^0 \longrightarrow 0. \tag{3.5}$$

The sequence (3.5) is well defined. This is clear for the curl operator. To verify it

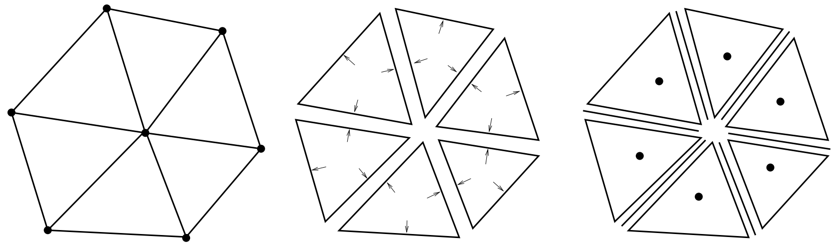

FIGURE 2. Distributional finite element spaces in the sequence (3.5). The middle lines of the edges represent the edge terms in (3.4)

for the divergence, let $\sigma \in \mathcal{R}T_{-1}$, and define $f = \operatorname{div}\sigma$ in the distributional sense by

$$\langle f, v\rangle := -\langle \sigma, \nabla v\rangle \quad \text{for } v \in C_0^\infty.$$

Integration by parts leads to

$$\begin{aligned}\langle f, v\rangle &= -\sum_T \int_T \sigma \cdot \nabla v \\ &= \sum_T \int_T \operatorname{div}_T \sigma\, v - \int_{\partial T} \sigma \cdot n\, v \\ &= \sum_T \int_T \operatorname{div}_T \sigma\, v - \sum_E \int_E \sum_{T: E\subset T} \sigma_T \cdot n_E\, v.\end{aligned}$$

Here the normal vectors are defined element by element and as usual in the outgoing direction; cf. Figure 2. Thus the image $\operatorname{div}\sigma$ belongs to $\mathcal{M}_{-2}^0$, and the relation $f = \operatorname{div}\sigma$ evaluates to two relations

$$f_T = \operatorname{div}_T \sigma_T \qquad \text{and} \qquad f_E = -\sum_{T:E\subset T} \sigma_T \cdot n_E\,. \tag{3.6}$$

Since $\sigma \in \mathcal{R}T_{-1}$ is determined by the fluxes on each side of the edges, we have in terms of degrees of freedom

$$\widehat{f}^{\,T} = \sum_{E\subset T} \widehat{\sigma_T}^{\,E} \qquad \text{and} \qquad \widehat{f}^{\,E} = -\sum_{T:E\subset T} \widehat{\sigma_T}^{\,E}.$$

Theorem 5. *The sequence* (3.5) *is exact.*

Proof. We recall that the classical sequence (3.3) is exact [2, 7, 9, 11, 14].

Due to (3.6) the properties $\sigma \in \mathcal{R}T_{-1}$ and $\operatorname{div}\sigma = 0$ imply that $\sigma \in \mathcal{R}T$. Hence, the divergence is defined as usual in $H(\operatorname{div})$ and vanishes. Now the exactness of (3.3) guarantees that Raviart–Thomas elements with vanishing divergence are curls of functions in $\mathcal{M}_0^1$.

The surjectivity of the divergence onto $\mathcal{M}_{-2}^0$ is also obtained from the exactness (3.3) by similar arguments as for the reduction in the proof of Lemma 3 (cf. also the reduction in the proof of the next theorem). $\square$

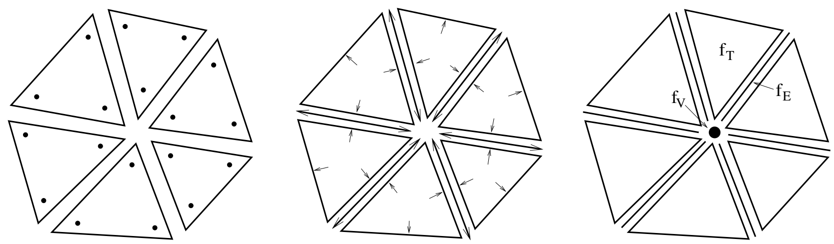

FIGURE 3. Distributional finite element spaces in the sequence (3.7). The arrows along the edges represent the edge and the vertex terms in (3.8)

3.2. **Second distributional triangular elements.** For the treatment of the curl-curl equations, we need another extension of the sequence. Distributional elements on edges are added to the finite elements that model $H(\mathrm{div})$, and the entire set $\mathcal{M}^0_{-3}$ of distributions of the form (3.4) enters into the theory.

The associated sequence will be called the *second distributional de Rham sequence*:

$$\mathbb{R} \longrightarrow \mathcal{M}^1_{-1} \xrightarrow{\text{curl}} \mathcal{R}T_{-2} \xrightarrow{\text{div}} \mathcal{M}^0_{-3} \longrightarrow 0. \tag{3.7}$$

The space $\mathcal{M}^1_{-1}$ consists of piecewise linear and noncontinuous finite elements. The degrees of freedom are the values $\widehat{u_T}^{\,V}$ at the three vertices of each triangle; see Figure 3.

The corresponding Raviart–Thomas distributions are of the form

$$\langle \sigma, v\rangle = \sum_T \int_T \sigma_T \cdot v + \sum_E \int_E \sigma_E \cdot v,$$

where $\sigma_T = \vec{a} + b\vec{x}$, and $\sigma_E = (a+bx)\vec{\tau}_E$ are 1D Raviart–Thomas elements mapped to the edge where $\vec{\tau}_E$ is a tangential vector. The degrees of freedom are

$$\widehat{\sigma_T}^{\,E} = \int_E \sigma_T \cdot n_E \qquad \text{and} \qquad \widehat{\sigma_E}^{\,V} = \sigma_E(V) \cdot n_V. \tag{3.8}$$

Here n_V is the vector pointing outwards at the vertex V of an edge E.

The representation of the operation $f = \operatorname{div} \sigma$ in terms of the degrees of freedom is

$$\begin{aligned} \widehat{f}^{\,T} &= \sum_{E\subset T} \widehat{\sigma_T}^{\,E}, \\ \widehat{f}^{\,E} &= \sum_{V\in E} \widehat{\sigma_E}^{\,V} - \sum_{T:E\subset T} \widehat{\sigma_T}^{\,E}, \\ \widehat{f}^{\,V} &= -\sum_{E:V\in E} \widehat{\sigma_E}^{\,V}. \end{aligned} \tag{3.9}$$

Remark 6. In contrast to the previous case, $\operatorname{div} \sigma = 0$ is now possible for elements σ that are not in the classical Raviart–Thomas space $\mathcal{R}T$. The distributional parts of $\operatorname{div} \sigma$ may add to zero in (3.9). Nevertheless, there is a geometrical understanding. We may blow up the edges to *slim rectangles* (in an exploded mesh); see Figure 4. If

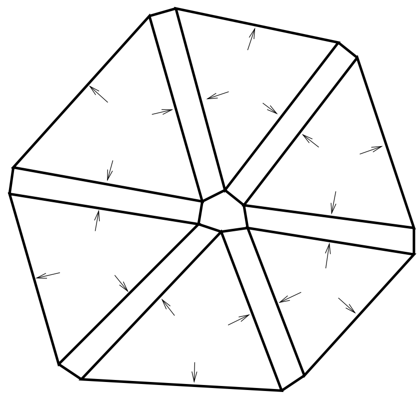

FIGURE 4. Slim rectangles in the sense of Remark 6

the divergence vanishes in the distributional sense, the total flow over the boundary of a slim rectangle (and not only over the boundary of the triangles) is zero.

The imagination with the slim rectangles has another advantage. The (classical) Raviart–Thomas elements in 2-space are given by the fluxes on the edges. Now all the degrees of freedom of the distributional Raviart–Thomas elements are fluxes on edges, i.e., they live on 1-dimensional objects. The terms on the right-hand side of (3.9) are fluxes over boundaries of triangles, slim rectangles, or the central area in Figure 4.

The differential operation $\sigma = \operatorname{curl} u$ reads as

$$\begin{aligned} \widehat{\sigma_T}^{\,E} &= \widehat{u_T}^{\,V_{E,1}} - \widehat{u_T}^{\,V_{E,2}}, \\ \widehat{\sigma_E}^{\,V} &= \widehat{u_{T_1}}^{\,V} - \widehat{u_{T_2}}^{\,V}, \end{aligned} \tag{3.10}$$

where T_1 is the left and T_2 is the right triangle when looking into the direction of n_v.

Theorem 7. *The second distributional de Rham sequence is exact.*

Proof. We start with proving that the operator div is a mapping onto $\mathcal{M}^0_{-3}$. Given $f \in \mathcal{M}^0_{-3}$, we first choose σ^2 such that the vertex terms of $\operatorname{div} \sigma^2$ coincide with the vertex terms of f. To this end we set

$$\widehat{\sigma^2{}_E}^{\,V} = -\frac{1}{N_V}\widehat{f}^{\,V},$$

where N_V is the number of edges sharing the vertex V. From $(3.9)_3$ we know that

$$f - \operatorname{div} \sigma^2 \in \mathcal{M}^1_{-2}.$$

By the first distributional exact sequence, there exists $\sigma^1 \in \mathcal{RT}_{-1}$ such that $\operatorname{div} \sigma^1 = f - \operatorname{div} \sigma^2$. Hence,

$$\operatorname{div}(\sigma^1 + \sigma^2) = f.$$

This proves that the divergence operator is surjective and proves the exactness of the second operator.

Next, consider $\sigma \in \mathcal{RT}_{-2}$ with $\operatorname{div} \sigma = 0$. We construct a function $u^2 \in \mathcal{M}^1_{-1}$ such that the edge terms of $\operatorname{curl} u^2$ coincide with the edge terms of the given σ.

This is done by a local construction for each vertex. Given a vertex V, we conclude from the vertex part of $\operatorname{div}\sigma$ that

$$\sum_{E:V\in E} \widehat{\sigma_E}^{\,V} = 0.$$

Now, enumerate the triangles sharing the vertex V from 1 to N_V. Enumerate the edges such that E_i is between triangle T_i and $T_{i+1 \bmod N_V}$. We set

$$u^2_{T_1}(V) = 0 \qquad \text{and} \qquad u^2_{T_{i+1}}(V) = u^2_{T_i}(V) + \widehat{\sigma_{E_i}}^{\,V}.$$

Since the vertex currents sum to 0, it follows that

$$u^2_{T_1}(V) = u^2_{T_{N_V}}(V) + \widehat{\sigma_{E_{N_V}}}^{\,V}.$$

By construction, the edge terms of $\operatorname{curl} u^2$ coincide with the edge terms of σ. Thus, the difference satisfies

$$\sigma - \operatorname{curl} u^2 \in \mathcal{R}T_{-1} \qquad \text{and} \qquad \operatorname{div}(\sigma - \operatorname{curl} u^2) = 0.$$

We know from Theorem 5 that there exists $u^1 \in \mathcal{M}^1_0$ such that $\operatorname{curl} u^1 = \sigma - \operatorname{curl} u^2$, and $u = u^1 + u^2$ is the desired function in $\mathcal{M}^1_{-1}$. $\square$

3.3. **Distributional tetrahedral elements.** The three-dimensional de Rham sequence contains an additional space. In the case of zero boundary conditions it reads

$$0 \longrightarrow H^1_0 \xrightarrow{\text{grad}} H_0(\text{curl}) \xrightarrow{\text{curl}} H_0(\text{div}) \xrightarrow{\text{div}} L_2 \xrightarrow{\int 1} \mathbb{R} \longrightarrow 0. \tag{3.11}$$

The canonical lowest order finite elements inherit the exact sequence property

$$0 \longrightarrow \mathcal{M}^1_0 \xrightarrow{\text{grad}} \mathcal{N}d \xrightarrow{\text{curl}} \mathcal{R}T \xrightarrow{\text{div}} \mathcal{M}^0_{-1} \xrightarrow{\int 1} \mathbb{R} \longrightarrow 0; \tag{3.12}$$

see [14]. Here, $\mathcal{N}d$ consists of the lowest order Nédélec elements.

We define the space $\mathcal{M}^0_{-4}$ of scalar distributions of the form

$$\langle f, v\rangle = \sum_T \int_T f_T v + \sum_F \int_F f_F v + \sum_E \int_E f_E v + \sum_V f_V v(V), \tag{3.13}$$

where f_T, f_F, and f_E are piecewise constant functions on tetrahedra, faces, and edges, respectively. The f_V are real numbers. The subspaces $\mathcal{M}^0_{-1} \subset \mathcal{M}^0_{-2} \subset \mathcal{M}^0_{-3}$ of lower distributional orders are defined to contain

only element terms,
element and face terms, and
element, face, and edge terms, respectively.

Moreover, we define the space $\mathcal{R}T_{-3}$ of $H(\text{div})$ distributions of the form

$$\langle \sigma, v\rangle = \sum_T \int_T \sigma_T \cdot v + \sum_F \int_F \sigma_F \cdot v + \sum_E \int_E \sigma_E \cdot v, \tag{3.14}$$

where σ_T, σ_F, and σ_E are in the Raviart–Thomas element space on tetrahedra T, triangular faces F in 3D space, and edges E in 3D. The degrees of freedom are the normal fluxes through the boundary. Specifically, we take the normal flux $\widehat{\sigma_T}^{\,F}$ of σ_T through the face $F \subset \partial T$, the normal flux $\widehat{\sigma_F}^{\,E}$ of σ_F through the edge $E \subset \partial F$, and the flux $\widehat{\sigma_E}^{\,V}$ of σ_E into the vertex V of E. The degrees $\widehat{\sigma_F}^{\,E}$ of a face flux are depicted in Figure 5.

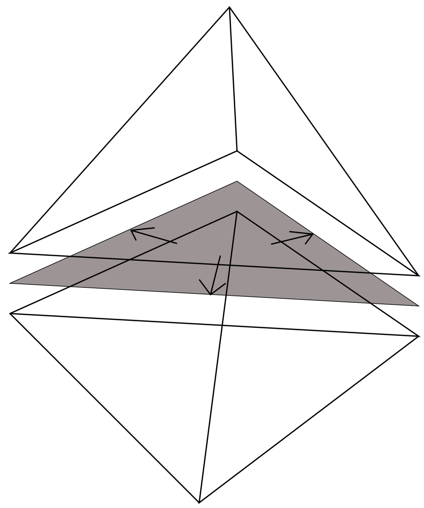

FIGURE 5. Raviart–Thomas face distribution

Only element and face distributions are required for modeling the space $H(\text{curl})$,

$$\langle H, v\rangle = \sum_T \int_T H_T \cdot v + \sum_F \int_F H_F \cdot v. \tag{3.15}$$

These distributions generate the space $\mathcal{N}d_{-2}$, and $\mathcal{N}d_{-1}$ is the subspace with vanishing face terms. The finite element functions are spanned in each tetrahedron and triangle by Nédélec shape functions. Their degrees of freedom are the tangential components along the tetrahedral and triangular edges. Note that there are jumps of the tangential components of a distributional Nédélec function between the tetrahedra, and individual values of the components are given on common edges.

Now we are ready to formulate three sequences for distributional finite element spaces in $\mathbb{R}^3$. They differ by the order of the distributions. We focus on the spaces with boundary conditions (but the sequences for the versions without boundary conditions are also exact):

$$0 \longrightarrow \mathcal{M}_0^1 \xrightarrow{\text{grad}} \mathcal{N}d_0 \xrightarrow{\text{curl}} \mathcal{R}T_{-1} \xrightarrow{\text{div}} \mathcal{M}_{-2}^0 \xrightarrow{\int 1} \mathbb{R} \longrightarrow 0, \tag{3.16}$$

$$0 \longrightarrow \mathcal{M}_0^1 \xrightarrow{\text{grad}} \mathcal{N}d_{-1} \xrightarrow{\text{curl}} \mathcal{R}T_{-2} \xrightarrow{\text{div}} \mathcal{M}_{-3}^0 \xrightarrow{\int 1} \mathbb{R} \longrightarrow 0, \tag{3.17}$$

$$0 \longrightarrow \mathcal{M}_{-1}^1 \xrightarrow{\text{grad}} \mathcal{N}d_{-2} \xrightarrow{\text{curl}} \mathcal{R}T_{-3} \xrightarrow{\text{div}} \mathcal{M}_{-4}^0 \xrightarrow{\int 1} \mathbb{R} \longrightarrow 0. \tag{3.18}$$

The first sequence (3.16) was already used for the construction of the equilibrated fluxes for the scalar equation in the previous section. The second sequence (3.17) will be used to construct the equilibrated magnetic fields for the curl-curl equation. The third sequence is formulated only for completeness.

Lemma 8. *The sequences* (3.16), (3.17), *and* (3.18) *are exact.*

Proof. We start with the first sequence. The exactness of $\mathcal{R}T_{-1} \xrightarrow{\text{div}} \mathcal{M}_{-2}^0 \xrightarrow{\int 1} \mathbb{R}$ was already proven in Lemma 3. Since $\operatorname{div}\sigma = 0$ for $\sigma \in \mathcal{R}T_{-1}$ implies that $\sigma \in \mathcal{R}T$, the exactness of the rest of the sequence follows from the exactness of the standard finite element sequence.

We continue with the sequence (3.17). Given $f \in \mathcal{M}^0_{-3}$ that contains element, face, and edge terms, we construct a $\sigma \in \mathcal{R}T_{-2}$ such that $\operatorname{div}\sigma = f$. We define the edge degrees of freedom for an auxiliary σ^1 by

$$\widehat{\sigma^1{}_F}^{\,E} := \frac{1}{N_E}\widehat{f}^{\,E},$$

where N_E is the number of faces sharing the edge E. Thus, the edge terms of $\operatorname{div}\sigma^1$ are equal to the edge terms of f, and thus $f - \operatorname{div}\sigma \in \mathcal{M}^0_{-2}$. The first distributional sequence yields the existence of a $\sigma^2 \in \mathcal{R}T_{-1}$ satisfying $\operatorname{div}\sigma^2 = f - \operatorname{div}\sigma^1$, and we have

$$\operatorname{div}(\sigma^1 + \sigma^2) = f.$$

We turn to the middle part of (3.17). Given $\sigma \in \mathcal{R}T_{-2}$ with $\operatorname{div}\sigma = 0$, we construct a function $H \in \mathcal{N}d_{-1}$ such that $\operatorname{curl} H = \sigma$. From the edge part of the divergence it follows that $\sum_{F:E\subset F} \widehat{\sigma_F}^{\,E} = 0$. We fix an edge E and enumerate the tetrahedra and faces around the edges such that face F_i is between T_i and T_{i+1}. Also here, element indices are taken modulo N_E. We define an H^1 by

$$\widehat{H^1{}_{T_1}}^{\,E} := 0 \qquad \text{and} \qquad \widehat{H^1{}_{T_{i+1}}}^{\,E} := \widehat{H^1{}_{T_i}}^{\,E} + \widehat{\sigma_{F_i}}^{\,E}.$$

Since $\operatorname{div}\sigma = 0$, we end up with $\widehat{H^1{}_{T_{N+1}}}^{\,E} = 0$ after a complete cycle. The residual $\sigma - \operatorname{curl} H^1$ is divergence free, and it is contained in $\mathcal{R}T_{-1}$. We apply the first distributional sequence to ensure the existence of an $H^2 \in \mathcal{N}d_0$ such that

$$\operatorname{curl}(H^1 + H^2) = \sigma.$$

To complete the second part, we pick an $H \in \mathcal{N}d_{-1}$ such that $\operatorname{curl} H = 0$. By definition $H \in L_2$ holds as well as $\operatorname{curl} H = 0 \in L_2$; thus $H \in H(\operatorname{curl})$. This implies that the tangential components of H are continuous, i.e., $H \in \mathcal{N}d$. By the exactness of the standard sequence, there exists a $\varphi \in \mathcal{M}^1_0$ such that $\operatorname{grad}\varphi = H$.

We skip the proof of the third sequence, since it follows the same lines, and it was added only for completeness. ☐

3.4. **Stability of inverses.** For $f \in \mathcal{M}^0_{-4}$, $\sigma \in \mathcal{R}T_{-3}$, $H \in \mathcal{N}d_{-2}$, and $u \in \mathcal{M}^1_{-1}$ we define the mesh-dependent norms

$$\begin{aligned}
\|f\|^2_{0,h} &:= \sum_T \|f_T\|^2_{L_2(T)} + \sum_F h_F \|f_F\|^2_{L_2(F)} + \sum_E h_E^2 \|f_E\|^2_{L_2(E)} \\
&\quad + \sum_V h_V^3 |f_V|^2, \\
\|\sigma\|^2_{0,h} &:= \sum_T \|\sigma_T\|^2_{L_2(T)} + \sum_F h_F \|\sigma_F\|^2_{L_2(F)} + \sum_E h_E^2 \|\sigma_E\|^2_{L_2(E)}\,, \\
\|H\|^2_{0,h} &:= \sum_T \|H_T\|^2_{L_2(T)} + \sum_F h_F \|H_F\|^2_{L_2(F)}\,, \\
\|u\|^2_{0,h} &:= \sum_T \|u_T\|^2_{L_2(T)}\,.
\end{aligned}$$

Lemma 9. *The right inverses of the differential operators constructed above satisfy the norm estimates*

$$\begin{aligned} \|\sigma\|_{0,h} &\leq ch\,\|f\|_{0,h}, \quad \textit{where} \quad \operatorname{div}\sigma = f,\\ \|H\|_{0,h} &\leq ch\,\|\sigma\|_{0,h}, \quad \textit{where} \quad \operatorname{curl} H = \sigma,\\ \|u\|_{0,h} &\leq ch\,\|H\|_{0,h}, \quad \textit{where} \quad \nabla u = H. \end{aligned}$$

Proof. By transformation to the reference element (using standard, covariant, or the Piola transformation), one easily proves that

$$\begin{aligned} \|f\|_{0,h}^2 &\simeq h^{-3}\Big\{\sum_T \big(\widehat{f_T}^{\,T}\big)^2 + \sum_F \big(\widehat{f_F}^{\,F}\big)^2 + \sum_E \big(\widehat{f_E}^{\,E}\big)^2 + \sum_V \big(f_V\big)^2\Big\},\\ \|\sigma\|_{0,h}^2 &\simeq h^{-1}\Big\{\sum_T\sum_{F\subset T} \big(\widehat{\sigma_T}^{\,F}\big)^2 + \sum_F\sum_{E\subset F} \big(\widehat{\sigma_F}^{\,E}\big)^2 + \sum_E\sum_{V\in E} \big(\widehat{\sigma_E}^{\,V}\big)^2\Big\},\\ \|H\|_{0,h}^2 &\simeq h^{1}\Big\{\sum_T\sum_{E\subset T} \big(\widehat{H_T}^{\,E}\big)^2 + \sum_F\sum_{E\subset F} \big(\widehat{H_F}^{\,E}\big)^2\Big\},\\ \|u\|_{0,h}^2 &\simeq h^{3}\Big\{\sum_T\sum_{V\in T} \big(\widehat{u_T}^{\,V}\big)^2\Big\}. \end{aligned}$$

Define $\widehat{f}$ as the vector containing all degrees of freedom. The relation

$$\operatorname{div}\sigma = f$$

can be written as a singular, but consistent matrix equation for the coefficient vectors

$$B_{\mathrm{div}}\widehat{\sigma} = \widehat{f},$$

where the matrix B_{div} is defined according to (3.9). All matrix elements are either $+1$, -1, or 0. The matrix depends only on the topology of the mesh. Assuming a patch of shape regular elements, there is only a finite number of possible topologies, and thus there exists a common constant c such that

$$\|\widehat{\sigma}\|_{\mathbb{R}^n} \leq c\,\|\widehat{f}\|_{\mathbb{R}^n}.$$

Together with the norm equivalences the statement follows. □

Similar arguments on matrices with entries $+1$, -1, and 0 in this context can be found in [13].

4. Equilibration in $H(\mathrm{curl})$

We consider the curl-curl equation for the vector potential: *Find* $u \in H(\mathrm{curl})$ *such that*

$$(\mu^{-1}\operatorname{curl} u, \operatorname{curl} v) = (j, v) \quad \text{for } v \in H(\mathrm{curl}).$$

The given current density j is supposed to be divergence free. Moreover, we assume that j is element-wise constant. Thus, j can be represented by means of Raviart–Thomas functions.

We are interested in a posteriori error estimates of the finite element discretization u_h with Nédélec elements of lowest order,

$$(\mu^{-1}\operatorname{curl} u_h, \operatorname{curl} v) = (j, v) \quad \text{for } v \in \mathcal{N}d.$$

The magnetic field H defined as

$$H := \mu^{-1}\operatorname{curl} u \tag{4.1}$$

satisfies Ampère's law

$$\operatorname{curl} H = j. \tag{4.2}$$

The magnetic field H_h obtained from the finite element discretization,

$$H_h := \mu^{-1} \operatorname{curl} u_h,$$

leads in general to a different current density

$$j_h := \operatorname{curl} H_h. \tag{4.3}$$

For piecewise linear vector potentials u_h, the magnetic flux H_h is piecewise constant, and the discrete curl, i.e. j_h, is a face-based $\mathcal{RT}$ distribution.

4.1. **An equation of Prager–Synge type.** The following result will be the basis of the error estimate. It is the analogue to Theorem 1.

Theorem 10. *Assume that $v \in H(\operatorname{curl})$ satisfies the boundary conditions and that $\tilde{H} \in H(\operatorname{curl})$ satisfies Ampère's law $\operatorname{curl} \tilde{H} = j$. Then*

$$\|\mu^{-1/2} \operatorname{curl}(u-v)\|_0^2 + \|\mu^{1/2}(H - \tilde{H})\|_0^2 = \|\mu^{-1/2}(\operatorname{curl} v - \mu \tilde{H})\|_0^2 . \tag{4.4}$$

Proof. Integration by parts yields the orthogonality relation

$$\begin{aligned}
&\int_\Omega \operatorname{curl}(u-v)(H-\tilde{H}) \\
&\quad = \int_\Omega (u-v) \operatorname{curl}(H-\tilde{H}) + \int_{\partial\Omega} [(u-v) \times n] \cdot (H-\tilde{H}) \\
&\quad = \int_\Omega (u-v)(j-j) = 0.
\end{aligned}$$

By applying the binomial formula to $\mu^{-1/2} \operatorname{curl}(u-v) + \mu^{1/2}(\tilde{H} - H)$ and noting that $\mu H = \operatorname{curl} u$ we obtain (4.4). □

The lemma provides error estimates for $\operatorname{curl} u$, and the estimate is independent of the gauge.

The lemma above will be applied to $v := u_h$. In order to achieve a good candidate for $\tilde{H}$ we have to solve $\operatorname{curl}(\tilde{H} - H_h) = j - j_h$. For this reason we are going to construct a correction H^Δ such that

$$j - j_h = \operatorname{curl} H^\Delta.$$

Again, we construct H_{ω_V} locally on the vertex patch ω_V such that we obtain a decomposition

$$H^\Delta = \sum H_{\omega_V}.$$

The construction will be independent of the material parameter μ, and μ will enter only at the final end when Theorem 10 will be applied. In particular, the coefficient μ may be piecewise constant on the mesh.

4.2. **The discrete current.** The distribution j_h is evaluated by using partial integration and recalling that H_h is piecewise constant:

$$\begin{aligned}\langle j_h, v\rangle &= \langle \operatorname{curl} H_h, v\rangle = (H_h, \operatorname{curl} v)\\ &= \sum_T \int_T \operatorname{curl} H_h \cdot v\, dx + \sum_F \int_F [H_h \times n]\cdot v\, ds\\ &= \sum_F ([H_h \times n], v)_F\, .\end{aligned}$$

The discrete current distributions are

$$j_{h,F} = [H_h \times n]. \tag{4.5}$$

Both currents, the prescribed current j as well as the discrete current j_h can be represented by distributional Raviart–Thomas elements of order 1. Both currents are divergence free.

We utilize the properties of the Galerkin orthogonality, namely

$$\left\langle j - j_h, \varphi^E\right\rangle = 0 \tag{4.6}$$

for each Nédélec basis function φ^E associated with the generic edge E. Let V_1 and V_2 be its two vertices. Given an edge E of an element T, the basis function can be expressed on the simplex T in terms of the barycentric coordinates

$$\varphi^E = \lambda_1 \nabla\lambda_2 - \lambda_2 \nabla \lambda_1\, ;$$

see [14, (5.47)]. We recall that j as well as $\nabla\lambda_i$ is constant on the element and evaluate the contribution of j on an element T sharing the edge E:

$$\begin{aligned}\int_T j\cdot\varphi^E &= \int_T j\cdot(\lambda_1\nabla\lambda_2 - \lambda_2\nabla\lambda_1)\\ &= (j\cdot\nabla\lambda_2)\int_T \lambda_1 - (j\cdot\nabla\lambda_1)\int_T\lambda_2\\ &= \frac{|T|}{4}\left\{j\cdot\nabla\lambda_2 - j\cdot\nabla\lambda_1\right\}.\end{aligned}$$

Now, observe that $\nabla\lambda_i$ is proportional to the normal vector on the face F_i that lies opposite to vertex V_i, and the factor is the inverse of the height of the element over the face F_i:

$$\nabla\lambda_i = -h_i^{-1} n_i = -\frac{|F_i|}{3|T|} n_i.$$

Thus, the element contributions evaluate to

$$\begin{aligned}\int_T j\cdot\varphi^E &= \frac{1}{12}\left\{|F_1|\, j\cdot n_1 - |F_2|\, j\cdot n_2\right\}\\ &= \frac{1}{12}\left\{\int_{F_1} j\cdot n - \int_{F_2} j\cdot n\right\} = \frac{1}{12}\left\{\widehat{j_T}^{F_1} - \widehat{j_T}^{F_2}\right\}.\end{aligned} \tag{4.7}$$

Note that the fluxes through element faces are the degrees of freedom of the Raviart–Thomas elements.

Similarly, the contribution of j_h on a face F is determined by

$$\begin{aligned}\int_F j_{h,F}\cdot\varphi^E &= \frac{1}{6}\left\{\int_{E_1} j_h\cdot n - \int_{E_2} j_h\cdot n\right\}\\ &= \frac{1}{6}\left\{\widehat{j_{h,F}}^{E_1} - \widehat{j_{h,F}}^{E_2}\right\}\end{aligned}$$

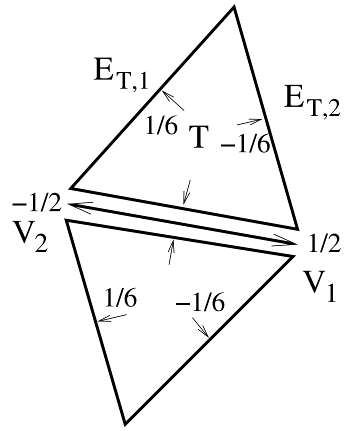

FIGURE 6. Factors in relation (4.10) referring to an edge and adjacent triangles. The edge terms refer to j and the vertex terms to j_h

where V_1, V_2 are the endpoints of E, and E_1, E_2 are the edges of the face F which lie opposite to the vertices above. The normal vectors to the edges refer to the plane F and are vectors in F.

The integrals above are inserted in (4.6) to derive a relation between the original and the discrete current:

$$\begin{aligned}\frac{1}{12}\sum_{T:E\subset T}&\left\{\textstyle\int_{F_{T,1}} j\cdot n-\int_{F_{T,2}} j\cdot n\right\}\\ &=\frac{1}{6}\sum_{F:E\subset F}\left\{\textstyle\int_{E_{F,1}} j_h\cdot n-\int_{E_{F,2}} j_h\cdot n\right\}\end{aligned} \tag{4.8}$$

or

$$\frac{1}{12}\sum_{T:E\subset T}\left\{\widehat{j_T}^{\,F_{T,1}}-\widehat{j_T}^{\,F_{T,2}}\right\}-\frac{1}{6}\sum_{F:E\subset F}\left\{\widehat{j_{h,F}}^{\,E_{F,1}}-\widehat{j_{h,F}}^{\,E_{F,1}}\right\}=0. \tag{4.9}$$

4.3. **Equilibration in 2D.** The basic relation for the 2D model that corresponds to (4.9) can be established in the same way:

$$\frac{1}{6}\sum_{T:E\subset T}\left\{\widehat{j_T}^{\,E_{T,1}}-\widehat{j_T}^{\,E_{T,2}}\right\}+\frac{1}{2}\left\{\widehat{j_{h,E}}^{\,V_1}-\widehat{j_{h,E}}^{\,V_2}\right\}=0. \tag{4.10}$$

As above, V_1, V_2 are the endpoints of the edge E under consideration, and $E_{T,1}$, $E_{T,2}$ are the edges of the triangle T which lie opposite to them; see Figure 6. (The sign of the second term in (4.10) differs from that in (4.9), since V_1, V_2 refer directly to the points and not to objects opposite to them.)

We proceed with the 2D case and are going to decompose the residual current into local, divergence free currents, i.e.,

$$j-j_h=\sum_V j_{\omega_V}\,.$$

We consider a generic node V and the patch $\omega_V:=\bigcup\{T,\ V\in\partial T\}$. Let T be a triangle in ω_V and E be an edge of the triangle sharing the vertex V. The edge of

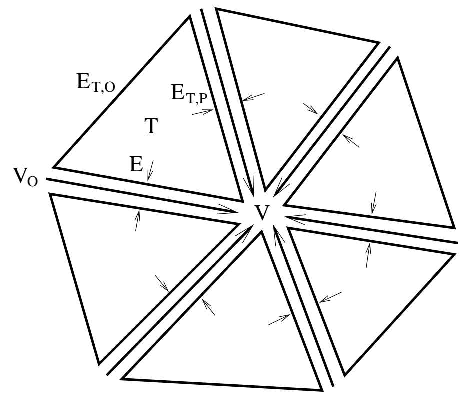

FIGURE 7. Some notation for the local current j_{ω_V}

the triangle T opposite to V is denoted as $E_{T,O}$. It is located on the boundary of ω_V. The third edge is denoted as $E_{T,P}$. We define the local current j_{ω_V} on T by

$$
\begin{aligned}
\widehat{j_{\omega_V,T}}^{\,E} &:= \frac{1}{2}\widehat{j_T}^{\,E} + \frac{1}{6}(\widehat{j_T}^{\,E_{T,O}} - \widehat{j_T}^{\,E_{T,P}}),\\
\widehat{j_{\omega_V,T}}^{\,E_P} &:= \frac{1}{2}\widehat{j_T}^{\,E_{T,P}} + \frac{1}{6}(\widehat{j_T}^{\,E_{T,O}} - \widehat{j_T}^{\,E}),\\
\widehat{j_{\omega_V,T}}^{\,E_O} &:= 0.
\end{aligned}
\tag{4.11}
$$

Obviously, the setting is symmetric with respect to the two edges that share the vertex V, but the representation with respect to a given edge E will be more useful in the sequel. Moreover, the flow is fixed such that the flow on the boundary of ω_V is zero.

Next, let E be an edge in the patch ω_V that connects V with a point V_O on $\partial\omega_V$. The vertex distributional parts are now fixed and evaluated from the fluxes on E via

$$
\begin{aligned}
\widehat{j_{\omega_V,E}}^{\,V} &:= -\widehat{j_{h,E}}^{\,V},\\
\widehat{j_{\omega_V,E}}^{\,V_O} &:= 0\,.
\end{aligned}
\tag{4.12}
$$

By definition, this current also has zero flow on $\partial\omega_V$.

Lemma 11. *If j_{ω_V} is defined by* (4.11) *and* (4.12)*, then* $\operatorname{div} j_{\omega_V} = 0$ *and we have a decomposition*

$$
j - j_h = \sum_V j_{\omega_V}\,. \tag{4.13}
$$

Proof. Let T be a triangle with edge E whose endpoints are V_1 and V_2. When we sum over all patches, only the patches with centers V_1 and V_2 contribute to the sum of $\widehat{j_{\omega_V,T}}^{\,E}$. Recalling (4.11) we have

$$
\begin{aligned}
\sum_V \widehat{j_{\omega_V,T}}^{\,E} &= \widehat{j_{\omega_{V_1},T}}^{\,E} + \widehat{j_{\omega_{V_2},T}}^{\,E}\\
&= \frac{1}{2}\widehat{j_T}^{\,E} + \frac{1}{2}\widehat{j_T}^{\,E} = \widehat{j_T}^{\,E}
\end{aligned}
$$

since the terms with the factor 1/6 in (4.11) cancel in the sum. Moreover, only the patch with center V contributes to the sum of $\widehat{j_{\omega_V,E}}^{\,V}$, Hence,

$$\sum_{V'} \widehat{j_{\omega_{V'},E}}^{\,V} = \widehat{j_{\omega_V,E}}^{\,V} = -\widehat{j_{h,E}}^{\,V}.$$

The last two equations show that (4.13) holds.

We consider now the divergence of j_{ω_V} and do it recalling (3.9). By adding the terms in (4.11) it follows that

$$\begin{aligned}
\widehat{\operatorname{div} j_{\omega_V}}^{\,T} &= \sum_{E:E\subset T} \widehat{j_{\omega_V,T}}^{\,E} = \widehat{j_{\omega_V,T}}^{\,E} + \widehat{j_{\omega_V,T}}^{\,E_{T,P}} + \widehat{j_{\omega_V,T}}^{\,E_{T,O}} \\
&= \frac{1}{2}\widehat{j_T}^{\,E} + \frac{1}{6}(\widehat{j_T}^{\,E_{T,O}} - \widehat{j_T}^{\,E_{T,P}}) \\
&\quad + \frac{1}{2}\widehat{j_T}^{\,E_{T,P}} + \frac{1}{6}(\widehat{j_T}^{\,E_{T,O}} - \widehat{j_T}^{\,E}) + 0 \\
&= \frac{1}{3}\big\{\widehat{j_T}^{\,E} + \widehat{j_T}^{\,E_{T,O}} + \widehat{j_T}^{\,E_{T,P}}\big\} \\
&= \frac{1}{3}\int_{\partial T} j\cdot n = \frac{1}{3}\int_T \operatorname{div} j = 0. \qquad (4.14)
\end{aligned}$$

We obtain the edge terms from (4.11) and (4.12):

$$\begin{aligned}
\widehat{\operatorname{div} j_{\omega_V}}^{\,E} &= \widehat{j_{\omega_V,E}}^{\,V} + \widehat{j_{\omega_V,E}}^{\,V_{E,O}} - \sum_{T:E\subset T} \widehat{j_{\omega_V,T}}^{\,E} \\
&= -\widehat{j_{h,E}}^{\,V} + 0 - \sum_{T:E\subset T} \Big\{\frac{1}{2}\widehat{j_T}^{\,E} + \frac{1}{6}\big(\widehat{j_T}^{\,E_{T,O}} - \widehat{j_E}^{\,E_{T,P}}\big)\Big\}. \qquad (4.15)
\end{aligned}$$

Since the normal components of j are continuous, we have $\sum_{T:E\subset T} \frac{1}{2}\widehat{j_T}^{\,E} = 0$. From $\operatorname{div} j_h = 0$ it follows that $\widehat{j_{h,E}}^{\,V} + \widehat{j_{h,E}}^{\,V_O} = 0$, and we continue with

$$\widehat{\operatorname{div} j_{\omega_V}}^{\,E} = -\frac{1}{2}\big(\widehat{j_{h,E}}^{\,V} - \widehat{j_{h,E}}^{\,V_O}\big) - \sum_{T:E\subset T} \frac{1}{6}\big(\widehat{j_T}^{\,E_{T,O}} - \widehat{j_E}^{\,E_{T,P}}\big) = 0.$$

Here we applied the Galerkin equation (4.10) to $V_1 = V$ and to $V_2 = V_O$.

Finally, the vertex terms are given by the flow into the center of the patch. From the definition (4.12) we have

$$\widehat{\operatorname{div} j_{\omega_V}}^{\,V} = \sum_{E:V\in E} \widehat{j_{\omega_V,E}}^{\,V} = \sum_{E:V\in E} \widehat{-j_{h,E}}^{\,V} = -\widehat{\operatorname{div} j_h}^{\,V} = 0. \qquad (4.16)$$

This concludes the proof of $\operatorname{div} j_{\omega_V} = 0$. □

We note that (4.16) can be obtained from (4.14) and (4.15) by virtue of arguments in the spirit of Remark 6. Since the current vanishes on $\partial\omega_V$ and the divergence on the triangles and the slim rectangles is zero, the total flux into V must also be zero. (This argument is also helpful in the 3-dimensional case.)

Since j_{ω_V} is in $\mathcal{RT}_{-2}$ with vanishing boundary values, we can apply the second distributional de Rham sequence to find an H_{ω_V} in the scalar noncontinuous P^1 space $\mathcal{M}^1_{-1}$ with vanishing boundary values such that

$$\operatorname{curl} H_{\omega_V} = j_{\omega_V}.$$

Recalling (3.10) we see that H_{ω_V} is easily determined.

4.4. **Equilibration in 3D.** We consider the construction in the 3-dimensional case very briefly. We construct the local current on the patch around a generic vertex V. Regard a tetrahedron T, a face F and an edge E such that $V \in E \subset F \subset T$. Let $F_{T,O}$ be the face opposite to V, let $F_{T,P}$ be the face containing V and opposite to E, and let $F_{T,Q}$ be the remaining face containing E. Similarly, let $E_{F,O}$ be the edge of the face opposite to V and $E_{F,P}$ the edge of F containing the vertex V. The face terms on a tetrahedron depend only on j in T; cf. (4.11). We set

$$\widehat{j_{\omega_V,T}}^{F} := \frac{1}{3}\widehat{j_T}^{F} + \frac{1}{12}\widehat{j_T}^{F_{T,O}} - \frac{1}{24}\left\{\widehat{j_T}^{F_{T,P}} + \widehat{j_T}^{F_{T,Q}}\right\}. \tag{4.17}$$

By symmetry, this defines also the fluxes through $F_{T,P}$ and $F_{T,Q}$. Moreover, the flux on the boundary of the patch is set to zero:

$$\widehat{j_{\omega_V,T}}^{F_{T,O}} := 0.$$

Contrary to the 2D case, the fluxes through faces depend not only on fluxes in faces, but involve also element terms. We set

$$\begin{aligned} \widehat{j_{\omega_V,F}}^{E} \quad &:= \quad -\left\{\frac{1}{2}\widehat{j_{h,F}}^{E} + \frac{1}{6}\widehat{j_{h,F}}^{E_{F,O}} - \frac{1}{6}\widehat{j_{h,F}}^{E_{F,P}}\right\} \\ &\qquad + \sum_{T:F\subset T} \frac{1}{24}\left\{\widehat{j_T}^{F_{T,O}} - \widehat{j_T}^{F_{T,P}}\right\}. \end{aligned} \tag{4.18}$$

Again, fluxes through the outer face are set to zero, i.e. $\widehat{j_{\omega_V,F}}^{E_{F,O}} = 0$.

Lemma 12. *This is a local, divergence free decomposition of the residual, i.e.,*

$$j - j_h = \sum_V j_{\omega_V}$$

and

$$\operatorname{div} j_{\omega_V} = 0.$$

We abandon the proof that proceeds along the lines of the proof of Lemma 11.

The results of this section are summarized for the Maxwell equation in 3-space as follows.

Theorem 13. *For each node V there exists a broken Nédélec function H_{ω_V} with support in ω_V such that*

$$\operatorname{curl} H_{\omega_V} \quad = \quad j_{\omega_V}$$

holds in the distributional sense, where j_{ω_V} is defined by (4.17) *and* (4.18)*. Choose H_{ω_V} with (quasi-) minimal L_2-norm, and let $H^\Delta := \sum_V H_{\omega_V}$.*

Then the postprocessed magnetic flux $\tilde{H} = \mu^{-1} \operatorname{curl} u_h + H^\Delta$ satisfies Ampère's law $\operatorname{curl} \tilde{H} = j$, and we have the a posteriori error estimate

$$c_0 \|\mu^{1/2} H^\Delta\| \;\le\; \|\mu^{-1/2} \operatorname{curl}(u - u_h)\| \;\le\; \|\mu^{1/2} H^\Delta\|. \tag{4.19}$$

Proof. The reliability follows from Lemma 12, the exactness of the second distributional de Rham sequence (3.17), and Theorem 10. The efficiency estimate follows from the stability of the right inverse, Lemma 9, and the efficiency of the residual error estimator analyzed in [6]. □

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