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RATIONALITY PROBLEM OF THREE-DIMENSIONAL PURELY MONOMIAL GROUP ACTIONS: THE LAST CASE

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ABSTRACT. A k-automorphism σ of the rational function field $k(x_1,\ldots,x_n)$ is called *purely monomial* if σ sends every variable x_i to a monic Laurent monomial in the variables $x_1,\ldots,x_n$. Let G be a finite subgroup of purely monomial k-automorphisms of $k(x_1,\ldots,x_n)$. The rationality problem of the G-action is the problem of whether the G-fixed field $k(x_1,\ldots,x_n)^G$ is k-rational, i.e., purely transcendental over k, or not. In 1994, M. Hajja and M. Kang gave a positive answer for the rationality problem of the three-dimensional purely monomial group actions *except one case*. We show that the remaining case is also affirmative.

1. INTRODUCTION

Let K be a field and L a finite Galois extension of k. Let Π be the Galois group of L/K and $\mathcal{L}$ a Π-module with a $\mathbb{Z}$-free basis $\{l_1,\ldots,l_n\}$. Then an integral representation $\rho:\ \Pi \longrightarrow \mathbf{GL}_n(\mathbb{Z})$ is defined by $\sigma \longmapsto (a_{ij})$ with

$$l_j{}^{\sigma} = \sum_{i=1}^{n} a_{ij} l_i \qquad (1 \le j \le n). \tag{1.1}$$

We now assume that Π acts on $L(x_1,\ldots,x_n)$, the rational function field over L with n variables $x_1,\ldots,x_n$, from the right by the following manner:

(1) Π acts on L as the Galois group,
(2) $x_j{}^{\sigma} = \prod_{i=1}^{n} x_i{}^{a_{ij}}$ with $\rho(\sigma) = (a_{ij})$ for $1 \le j \le n$.

We know that there is a duality between the category of all Π-modules and the category of all algebraic L/K-tori, algebraic tori over K which split over L. Then the fixed subfield $L(x_1,\ldots,x_n)^{\Pi}$ of $L(x_1,\ldots,x_n)$ can be identified with the function field of the algebraic L/K-torus T corresponding to the Π-module $\mathcal{L}$ by the duality above. We say that the algebraic L/K-torus T is rational when the Π-fixed field $L(x_1,\ldots,x_n)^{\Pi}$ is K-rational. One-dimensional algebraic tori are trivially rational. Voskresenskiĭ [17, 18] showed that all two-dimensional algebraic tori are rational. The birational classification of three-dimensional algebraic tori was given by Kunyavskiĭ [8]. We note that there are many irrational algebraic tori of dimension ≥ 3 (cf. [19]).

The rationality problem of a purely monomial group action is defined as a restricted version of "rationality questions" mentioned above. Let k be a field and

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$k(x_1, \dots, x_n)$ the function field over k with n variables $x_1, \dots, x_n$. Let G be a finite subgroup of $\mathbf{GL}_n(\mathbb{Z})$ which acts from the right on $k(x_1, \dots, x_n)$ as follows:

(1) G acts *trivially* on k,
(2) ${x_j}^A = \prod_{i=1}^n {x_i}^{a_{ij}}$ with $A = (a_{ij}) \in G$ for $1 \le j \le n$.

We call the G-action *purely monomial*. The rationality problem of the purely monomial G-action is the problem of whether the fixed subfield $k(x_1, \dots, x_n)^G$ of $k(x_1, \dots, x_n)$ is k-rational or not.

The fixed field of a purely monomial group action generally cannot be identified with a function field of any algebraic torus. But the rationality problem of purely monomial group actions has a special meaning in constructive aspects of inverse Galois theory. Let Γ be a finite group acting on the rational function field $k(x_g \mid g \in \Gamma)$ via the regular representation. The k-rationality problem of this Γ-action is called *Noether's problem* of Γ over k. If this problem has a positive answer, we can construct a regular Galois Γ-extension over $k(x_g \mid g \in \Gamma)^\Gamma$. This is known as *Noether's strategy* for constructing a *generic* Galois Γ-extension over k. When Γ is abelian, Lenstra [9] gave a necessary and sufficient condition that the Noether's problem of Γ over k has a positive answer. We, however, know very little for non-abelian cases. The rationality problem of purely monomial group actions is crucial in studying Noether's problem of non-abelian groups. The reader may consult [5, 6, 12, 13, 14, 15] about Noether's problem.

The rationality problem of one-dimensional purely monomial group actions is trivially affirmative. For two-dimensional cases, Hajja [2] gave the following result:

Theorem 1.1 (Hajja). *Let k be a field and G be a finite subgroup of $\mathbf{GL}_2(\mathbb{Z})$. Then $k(x_1, x_2)^G$ is k-rational.*

The three-dimensional cases are much more difficult than the two-dimensional ones. Tahara [16] proved that $\mathbf{GL}_3(\mathbb{Z})$ has 73 conjugacy classes of finite subgroups. Hajja-Kang [3, 4] obtained affirmative answers for 72 classes of them. Let G_0 be the finite subgroup of $\mathbf{GL}_3(\mathbb{Z})$ generated by

$$\begin{pmatrix} 1 & 1 & 0 \\ -2 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{1.2}$$

Theorem 1.2 (Hajja-Kang). *Let k be a field and G be a finite subgroup of $\mathbf{GL}_3(\mathbb{Z})$. Then $k(x_1, x_2, x_3)^G$ is k-rational if G is not conjugate to G_0 in $\mathbf{GL}_3(\mathbb{Z})$.*

A three-dimensional algebraic torus corresponding to G_0 (or its congugate in $\mathbf{GL}_3(\mathbb{Z})$) is not rational. For this reason, it might have been considerd that the remaining case is negative. But it is also a fact that there are irrational three-dimensional algebraic tori corresponding to purely monomial group actions whose fixed fields are k-rational. In this paper, we show that the remaining case is also affirmative.

Theorem 1.3 (Main result). *For an arbitrary field k, the fixed field $k(x_1, x_2, x_3)^{G_0}$ is k-rational. Consequently, the rationality problem of the three-dimensional purely monomial group actions has a positive answer.*

Finally, we note that this result can also be expressed from a viewpoint of multiplicative invariant theory. The lattice which is treated in the main result is isomorphic to the signed root lattice $\mathbb{Z}^- \otimes_{\mathbb{Z}} A_3$. The rationality problem for this lattice is introduced as an interesting open problem in [10, Problem 14]. Let $\mathfrak{S}_n$ be the symmetric group on n letters $\{1,\dots,n\}$. The group $\mathfrak{S}_n$ acts multiplicatively on $\mathbb{Z}$ via the sign homomorphism. We denote the non-trivial $\mathfrak{S}_n$-lattice with this action by $\mathbb{Z}^-$, and we regard $\mathbb{Z}$ as the trivial lattice. For $n \geq 2$, $\mathfrak{S}_n$ permutes a $\mathbb{Z}$-basis of the lattice $\mathbb{Z}[\mathfrak{S}_n/\mathfrak{S}_{n-1}]$. The kernel A_{n-1} of the augmentation map of the permutation $\mathfrak{S}_n$-lattice $\mathbb{Z}[\mathfrak{S}_n/\mathfrak{S}_{n-1}]$ also has a $\mathfrak{S}_n$-lattice structure. Thus we obtain a signed root lattice $\mathbb{Z}^- \otimes_{\mathbb{Z}} A_{n-1}$. The k-rationality problem of $k(\mathbb{Z}^- \otimes_{\mathbb{Z}} A_3)^{\mathfrak{S}_4}$ is equivalent to Hajja-Kang's "the exceptional case" treated as $W_{10}(198)$ in [4].

Corollary 1.4. *For an arbitrary field k, the $\mathfrak{S}_4$-invariant field $k(\mathbb{Z}^- \otimes_{\mathbb{Z}} A_3)^{\mathfrak{S}_4}$ is k-rational.*

2. Strategy

Our purpose is to show $k(x_1,x_2,x_3)^{G_0}$, where $G_0 = \langle A_0, B_0\rangle$ with

$$A_0 := \begin{pmatrix} 1 & 1 & 0 \\ -2 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \qquad B_0 := \begin{pmatrix} -1 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \tag{2.1}$$

is rational over an arbitrary field k. From a relation ${A_0}^4 = {B_0}^2 = (A_0B_0)^3 = I_3$, where I_3 is the identity matrix, G_0 is isomorphic to the symmetric group $\mathfrak{S}_4$. Here we put

$$A_1 := B_0{A_0}^2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad B_1 := A_0B_0{A_0}^2 = \begin{pmatrix} 1 & 1 & 1 \\ -2 & -1 & -1 \\ 0 & -1 & 0 \end{pmatrix}; \tag{2.2}$$

then G_0 is also generated by A_1 and B_1. To simplify our calculations, we take $G := \left\langle P^{-1}A_1P, P^{-1}B_1P\right\rangle$ where

$$P := \begin{pmatrix} -1 & -1 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \in \mathbf{GL}_3(\mathbb{Z}). \tag{2.3}$$

Because G_0 and G are conjugate in $\mathbf{GL}_3(\mathbb{Z})$, it is enough to show the k-rationality of the G-action to prove Theorem 1.3.

Denote $P^{-1}A_1P$ and $P^{-1}B_1P$ by A and B respectively:

$$A := \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \qquad B := \begin{pmatrix} 0 & -1 & 0 \\ 1 & 1 & 2 \\ -1 & 0 & -1 \end{pmatrix}. \tag{2.4}$$

They satisfy $A^4 = B^3 = (B^{-1}A^2B)^2 = (AB)^2 = I_3$. Hence G has the following normal series:

$$1 \quad \lhd \quad \left\langle A^2\right\rangle \quad \lhd \quad \left\langle A^2, B^{-1}A^2B\right\rangle \quad \lhd \quad \langle A, B\rangle = G. \tag{2.5}$$

We first choose appropriate $\left\langle A^2\right\rangle$-invariant functions s_1, s_2, s_3 which generate the $\left\langle A^2\right\rangle$-fixed field $k(x_1,x_2,x_3)^{\left\langle A^2\right\rangle}$ over k. To do this, we use the following lemma concerning two-dimensional weighted diagonal involutions which was obtained by Hajja-Kang [4]:

Lemma 2.1 (Hajja-Kang)**.** *Let k be a field and $\sigma \in \mathbf{Aut}_k k(x_1, x_2)$ be an involution defined by $(x_1, x_2) \longmapsto (m_1/x_1, m_2/x_2)$ with $m_1, m_2 \in k^\times$. Then the fixed field $k(x_1, x_2)^{\langle\sigma\rangle}$ is*

$$k\left(\frac{{x_1}^2{x_2}^2 - m_1 m_2}{x_1({x_2}^2 - m_2)}, \frac{x_2({x_1}^2 - m_1)}{x_1({x_2}^2 - m_2)}\right). \tag{2.6}$$

Since $\langle A^2\rangle$ has index two in $\langle A^2, B^{-1}A^2B\rangle$, we can find $\langle A^2, B^{-1}A^2B\rangle$-invariant functions t_1, t_2, t_3 which generate the $\langle A^2, B^{-1}A^2B\rangle$-fixed field. Finally, we show that the purely monomial $\langle A, B\rangle$-action on $k(t_1, t_2, t_3)$ is k-rational. We can find new generators u_1, u_2, u_3 of $k(t_1, t_2, t_3)$ to apply the following lemma given by Ahmad-Hajja-Kang [1, Theorem. 3.1].

Lemma 2.2 (Ahmad-Hajja-Kang)**.** *Let L be an arbitrary field and $L(x)$ be the rational function field with one variable over L. Let H be a group of automorphisms acting on $L(x)$. Suppose that, for any $\sigma \in H$, $\sigma(L) \subset L$, $x^\sigma = a_\sigma x + b_\sigma$ for some $a_\sigma \in L \setminus \{0\}$ and $b_\sigma \in L$. Then $L(x) = L^H$ or $L^H(f(x))$ for some polynomial $f(x) \in L[x]$ with positive degree. In particular, if L^H is rational over some subfield M, so is $L(x)^H$ over M.*

The final step is easier when the characteristic of k is two.

3. Proof of Theorem 1.3

The action of $G = \langle A, B\rangle$ on $k(x_1, x_2, x_3)$ is described by

$$\begin{cases} A: & (x_1, x_2, x_3) \longmapsto (x_2, x_3/x_1, x_3), \\ B: & (x_1, x_2, x_3) \longmapsto \big(x_2/x_3, x_2/x_1, {x_2}^2/x_3\big). \end{cases} \tag{3.1}$$

3.1. **The case when the characteristic of k is not two.** The action of A^2 on $k(x_1, x_2, x_3)$ is

$$(x_1, x_2, x_3) \longmapsto (x_3/x_1, x_3/x_2, x_3). \tag{3.2}$$

From Lemma 2.1, the fixed field $k(x_1, x_2, x_3)^{\langle A^2\rangle}$ is $k({s_1}', {s_2}', {s_3}')$, where

$${s_1}' := \frac{{x_1}^2{x_2}^2 - {x_3}^2}{x_1({x_2}^2 - x_3)}, \qquad {s_2}' := \frac{x_2({x_1}^2 - x_3)}{x_1({x_2}^2 - x_3)}, \qquad {s_3}' := x_3. \tag{3.3}$$

Then $B^{-1}A^2B$ acts on $k({s_1}', {s_2}', {s_3}')$ by

$$({s_1}', {s_2}', {s_3}') \longmapsto \left(\frac{(1 - {s_2}')(1 + {s_2}')}{{s_1}'}, -{s_2}', \frac{1}{{s_3}'}\right). \tag{3.4}$$

To linearize this action, we take the following birational transformation over k:

$$s_1 := \frac{{s_1}' + (1 + {s_2}')}{{s_1}' - (1 + {s_2}')}, \qquad s_2 := \frac{{s_1}' + {s_3}'(1 + {s_2}')}{{s_1}' - {s_3}'(1 + {s_2}')}, \qquad s_3 := {s_2}'. \tag{3.5}$$

Then we have $k({s_1}', {s_2}', {s_3}') = k(s_1, s_2, s_3)$ and

$$B^{-1}A^2B: \quad (s_1, s_2, s_3) \longmapsto (-s_1, -s_2, -s_3). \tag{3.6}$$

We have $k(s_1, s_2, s_3)^{\langle B^{-1}A^2B\rangle} = k({t_1}', {t_2}', {t_3}')$ where

$${t_1}' := s_1 s_3, \qquad {t_2}' := s_2 s_3, \qquad {t_3}' := {s_3}^2. \tag{3.7}$$

The action of B on $k({t_1}',{t_2}',{t_3}')$ is described as

$$\tag{3.8} \begin{cases} {t_1}' \longmapsto -\dfrac{{t_1}'({t_1}'-{t_2}')}{({t_1}'+{t_2}')({t_1}'+{t_3}')}, \quad {t_2}' \longmapsto \dfrac{{t_2}'({t_1}'-{t_2}')}{({t_1}'+{t_2}')({t_2}'+{t_3}')}, \\ {t_3}' \longmapsto \dfrac{({t_1}'-{t_2}')^2}{({t_1}'+{t_2}')(({t_1}'+{t_2}')(1+{t_3}')+2{t_1}'{t_2}'+2{t_3}')}. \end{cases}$$

We observe that (3.8) has a symmetry with respect to ${t_1}'$ and ${t_2}'$. By using this property, we put

$$\tag{3.9} t_1 := \frac{{t_1}'-{t_2}'}{{t_1}'+{t_2}'}, \quad t_2 := \frac{2{t_1}'{t_2}'+({t_1}'+{t_2}'){t_3}'}{({t_1}'-{t_2}'){t_3}'}, \quad t_3 := \frac{{t_1}'+{t_2}'+2{t_1}'{t_2}'}{{t_1}'-{t_2}'}$$

to linearize the $\langle A,B\rangle$-action on $k({t_1}',{t_2}',{t_3}')$. This is a birational transformation, because we have

$$\tag{3.10} {t_1}' = \frac{1-t_1t_3}{1-t_1t_2}, \qquad {t_2}' = \frac{1-t_1t_3}{-1+t_1}, \qquad {t_3}' = \frac{-1+t_1t_3}{1+t_1}.$$

Hence $k(t_1,t_2,t_3)=k(t_1',t_2',t_3')$, and the $\langle A,B\rangle$-action is described as follows:

$$\tag{3.11} \begin{cases} A: \quad (t_1,t_2,t_3) \longmapsto (-t_1,-t_3,-t_2)\,, \\ B: \quad (t_1,t_2,t_3) \longmapsto (t_2,t_3,t_1)\,. \end{cases}$$

We finally put

$$\tag{3.12} u_1 := t_2/t_1, \qquad u_2 := t_3/t_1, \qquad u_3 := t_1,$$

and hence $k(t_1,t_2,t_3)=k(u_1,u_2,u_3)$,

$$\tag{3.13} \begin{cases} A: \quad (u_1,u_2,u_3) \longmapsto (u_2,u_1,-u_3)\,, \\ B: \quad (u_1,u_2,u_3) \longmapsto (u_2/u_1,1/u_1,u_1u_3)\,. \end{cases}$$

For $L := k(u_1,u_2)$, we can easily check the following properties:

(1) $\sigma(L)\subset L$ for every $\sigma\in\langle A,B\rangle$.
(2) For any $\sigma\in\langle A,B\rangle$, ${u_3}^\sigma$ has degree one in $L[u_3]$.
(3) $L(u_3)^{\langle A,B\rangle} \neq L^{\langle A,B\rangle}$.

Therefore we can apply Lemma 2.2 to $L(u_3)^{\langle A,B\rangle}$. This follows that the G-fixed field $k(x_1,x_2,x_3)^G$ is rational over k.

3.2. **The case when the characteristic of k is two.** We recall (3.3); then $k(x_1,x_2,x_3)^{\langle A^2\rangle}$ is generated by ${s_1}',{s_2}',{s_3}'$ over k. Put

$$\tag{3.14} s_1 := \frac{1+{s_2}'}{{s_1}'}, \qquad s_2 := \frac{(1+{s_2}'){s_3}'}{{s_1}'}, \qquad s_3 := {s_2}'$$

so that $k({s_1}',{s_2}',{s_3}')=k(s_1,s_2,s_3)$ and

$$\tag{3.15} B^{-1}A^2B: \quad (s_1,s_2,s_3) \longmapsto (1/s_1,1/s_2,s_3)\,.$$

Applying Lemma 2.1 to $k(s_1,s_2,s_3)$, we have $k(s_1,s_2,s_3)^{\langle B^{-1}A^2B\rangle} = k({t_1}',{t_2}',{t_3}')$ where

$$\tag{3.16} {t_1}' := \frac{{s_1}^2{s_2}^2-1}{s_1({s_2}^2-1)}, \qquad {t_2}' := \frac{s_2({s_1}^2-1)}{s_1({s_2}^2-1)}, \qquad {t_3}' := s_3.$$

Then we obtain

$$(3.17)\qquad \begin{cases} A: & ({t_1}', {t_2}', {t_3}') \longmapsto ({t_1}'/{t_2}', 1/{t_2}', 1/{t_3}'), \\ B: & ({t_1}', {t_2}', {t_3}') \longmapsto \left(\dfrac{(1+{{t_2}'}^2){t_3}'}{(1+{t_2}')+{t_1}'(1+{t_3}')}, {t_2}', \dfrac{1+{t_2}'}{{t_1}'(1+{t_3}')} \right). \end{cases}$$

We here take

$$(3.18)\qquad t_1 := \frac{1+{t_2}'}{{t_1}'(1+{t_3}')}, \qquad t_2 := \frac{{t_1}'(1+{t_3}')}{{t_3}'(1+{t_2}')}, \qquad t_3 := {t_2}'$$

so that the $\langle A, B\rangle$-action is purely monomial. Then we can check $k({t_1}', {t_2}', {t_3}') = k(t_1, t_2, t_3)$ and that $\langle A, B\rangle$ acts on $k(t_1, t_2, t_3)$ by

$$(3.19)\qquad \begin{cases} A: & (t_1, t_2, t_3) \longmapsto (1/t_2, 1/t_1, 1/t_3), \\ B: & (t_1, t_2, t_3) \longmapsto (t_2, 1/t_1t_2, t_3). \end{cases}$$

This is a purely monomial $\mathfrak{S}_3$-action. Theorem 1.2 shows that $k(t_1, t_2, t_3)^{\langle A, B\rangle}$ is k-rational. This completes the proof of Theorem 1.3. □

Remark 3.1. It is possible to compute explicit generators of $k(x_1, x_2, x_3)^G$ over k with any characteristic by continuing the method above. To do this, one can use the explicit positive result about the Noether's problem of the cyclic group of order three in Kuniyoshi [7] and Masuda [11]. We omit displaying them because of their complicated expressions.

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