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NEW EXPANSIONS OF NUMERICAL EIGENVALUES FOR $-\Delta u = \lambda\rho u$ BY NONCONFORMING ELEMENTS

QUN LIN, HUNG-TSAI HUANG, AND ZI-CAI LI

Abstract. The paper explores new expansions of the eigenvalues for $-\Delta u = \lambda\rho u$ in S with Dirichlet boundary conditions by the bilinear element (denoted Q_1) and three nonconforming elements, the rotated bilinear element (denoted Q_1^{rot}), the extension of Q_1^{rot} (denoted EQ_1^{rot}) and Wilson's elements. The expansions indicate that Q_1 and Q_1^{rot} provide upper bounds of the eigenvalues, and that EQ_1^{rot} and Wilson's elements provide lower bounds of the eigenvalues. By extrapolation, the $O(h^4)$ convergence rate can be obtained, where h is the maximal boundary length of uniform rectangles. Numerical experiments are carried out to verify the theoretical analysis made.

1. Introduction

In this paper, we consider the eigenvalue problem

$$-\Delta u = \lambda\rho u \quad \text{in } S, \tag{1.1}$$

$$u = 0 \quad \text{in } \partial S, \tag{1.2}$$

where $S = [0,1]^2$, the function $\rho = \rho(x,y) > 0$ and $\rho \in C^2(S)$. Then Eqs. (1.1) and (1.2) can be written in a weak form: To seek $(\lambda, u) \in R \times H_0^1(S)$ with $u \neq 0$ such that

$$a(u,v) = \lambda(u,v),\ \forall v \in H_0^1(S), \tag{1.3}$$

where $H_0^1(S) = \{v | v \in H^1(S), v|_{\partial S} = 0\}$, and

$$a(u,v) = \iint_S \nabla u \nabla v, \tag{1.4}$$

$$(u,v) = \iint_S \rho uv. \tag{1.5}$$

We choose one conforming element, the bilinear element Q_1, and three nonconforming elements: the rotated Q_1 (denoted Q_1^{rot}), the extension of Q_1^{rot} (denoted EQ_1^{rot}) and Wilson's element. All the above elements are defined on rectangles $\square_{ij}$ (see Figure 1), and their admissible functions are defined as follows.

(1) **Bilinear element Q_1.** The piecewise interpolation functions $u_I \in Q_1 = \text{span}\{1, x, y, xy\}$ are formulated as

$$u(Z_i) = u_I(Z_i),\ i = 1,2,3,4, \tag{1.6}$$

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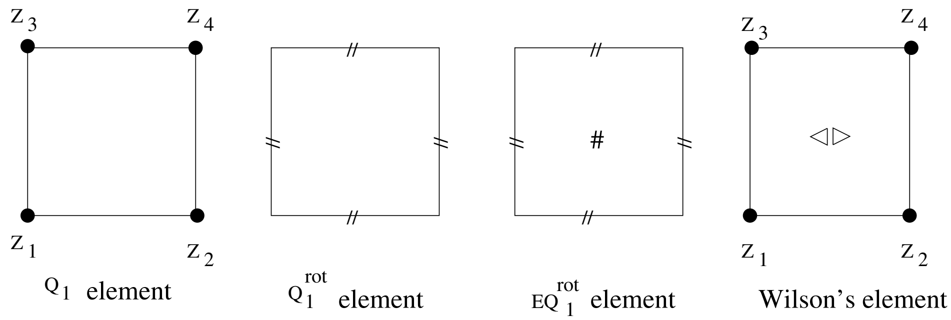

FIGURE 1. The rectangular elements, where // and # denote the line and the area elements, respectively, and ◁ and ▷ denote u_{xx} and u_{yy} at the center, respectively.

where Z_i are the four corners of $\Box_{ij}$, and $\Box_{ij} = \{(x,y)|x_i - h_i \le x \le x_i + h_i,\ y_j - k_j \le y \le y_j + k_j\}$.

(2) Rotated Q_1 element (Q_1^{rot}). The piecewise interpolation functions $u_I \in \text{span}\{1, x, y, x^2 - y^2\}$ are formulated by

$$\int_{\ell_k} u = \int_{\ell_k} u_I,\ k = 1, 2, 3, 4, \tag{1.7}$$

where ℓ_k are the edges of $\Box_{ij}$.

(3) Extension of Q_1^{rot} (EQ_1^{rot}). The piecewise interpolation functions $u_I \in \text{span}\{1, x, y, x^2, y^2\}$ are formulated by

$$\int_{\ell_k} u = \int_{\ell_k} u_I,\ k = 1, 2, 3, 4, \tag{1.8}$$

$$\iint_{\Box_{ij}} u = \iint_{\Box_{ij}} u_I. \tag{1.9}$$

(4) Wilson's element. The piecewise interpolation functions $u_I \in P_2 = \text{span}\{1, x, y, xy, x^2, y^2\}$ are formulated by

$$u(Z_i) = u_I(Z_i),\ i = 1, 2, 3, 4, \tag{1.10}$$

$$u_{xx}(O) = (u_I)_{xx}(O),\ u_{yy}(O) = (u_I)_{yy}(O), \tag{1.11}$$

where O is the center of $\Box_{ij}$.

Let $S = \bigcup_{ij} \Box_{ij}$, where $\Box_{ij}$ are quasi-uniform. Denote by $V_h^0 \in L^2(S)$ the finite-dimensional collection of the admissible functions defined in Q_1, Q_1^{rot}, EQ_1^{rot} and Wilson's elements. The conforming Q_1 element is used to seek the solution $(\lambda_h, u_h) \in R \times V_h^0$ ($V_h^0 \subset H_0^1(S)$) such that

$$a(u_h, v) = \lambda_h(u_h, v),\ \forall v \in V_h^0, \tag{1.12}$$

and the nonconforming elements, such as Q_1^{rot}, EQ_1^{rot} and Wilson's elements, are used to seek $(\lambda_h, u_h) \in R \times V_h^0$ [1] such that

$$a_h(u_h, v) = \lambda_h(u_h, v),\ \forall v \in V_h^0, \tag{1.13}$$

[1] Here V_h^0 is not a subset of $H_0^1(S)$.

where

$$a_h(u,v) = \sum_{ij} \iint_{\square_{ij}} \nabla u \nabla v. \tag{1.14}$$

In Q_1 and Wilson's elements, the nodal variables are used, but in Q_1^{rot} and EQ_1^{rot}, the line and the area variables are also chosen, which can be interpreted as the average values on the edges $\partial\square_{ij}$ and those in the area $\square_{ij}$. The line-area interpolation in Q_1^{rot} and EQ_1^{rot} is, rather than the nodal interpolation, advantageous in global superconvergence.

In this paper, we explore the expansions of the eigenvalues λ_h. When $\square_{ij}$ are uniform squares with the boundary length h, we obtain the following formulas:

$$\lambda_h - \lambda = \begin{cases} \dfrac{h^2}{3}\displaystyle\iint_S (u_{xx}^2 + u_{yy}^2) + O(h^4), & \text{for } Q_1 \text{ element}, \\ \dfrac{h^2}{6}\displaystyle\iint_S (u_{xx} - u_{yy})^2 + O(h^4), & \text{for } Q_1^{rot} \text{ element}, \\ -\dfrac{2h^2}{3}\displaystyle\iint_S u_{xy}^2 + O(h^4), & \text{for } EQ_1^{rot} \text{ element}, \\ -\dfrac{2h^2}{3}\displaystyle\iint_S u_{xx}u_{yy} - \dfrac{h^2}{3}\iint_S [u_{xx}(u_h)_{yy} + u_{yy}(u_h)_{xx}] + O(h^4), & \\ & \text{for Wilson's element}. \end{cases} \tag{1.15}$$

The detailed proof for Q_1^{rot} and EQ_1^{rot} elements is deferred to Section 3, and the proof for Q_1 and Wilson's elements will appear elsewhere. From the expansions of λ_h in (1.15), we may draw a few important conclusions:

(1) Both Q_1 and Q_1^{rot} provide an upper bound of λ, but in contrast, EQ_1^{rot} and Wilson's elements provide a lower bound of λ. The lower estimation of λ is particularly interesting, because all conforming FEMs can only provide an upper estimation on λ.

(2) Suppose that $\rho(x, y)$ is symmetric with respect to x and y. For the minimal eigenvalue $\lambda_{\min} = \lambda_1$, since the corresponding eigenfunction satisfies $u_{xx} = u_{yy}$, the Q_1^{rot} element yields the high $O(h^4)$ convergence rate. Such an ultraconvergence of Q_1^{rot} is retained for any eigenvalue whose corresponding eigenfunction is symmetric with respect to x and y.

(3) The errors of λ by Q_1, Q_1^{rot} and EQ_1^{rot} have the following relation:

$$E|_{Q_1^{rot}} - \frac{1}{2}(E|_{Q_1} + E|_{EQ_1^{rot}}) = O(h^4), \tag{1.16}$$

where $E = \lambda_h - \lambda$.

(4) By the extrapolation we may reach the high $O(h^4)$ convergence rates for Q_1, Q_1^{rot}, EQ_1^{rot}, and Wilson's elements.

In our numerical experiments, the $O(h^4)$ convergence rate has been confirmed by the extrapolation for all four elements, and the further extrapolation can be carried out for the Q_1 element to reach the $O(h^{2k})$ $(k \geq 2)$ convergence rates.

Let us mention the references related to this paper. Numerical eigenvalues are discussed in Babuska and Osborn [1, 2, 3], Chatelin [6], Koluta [10], Mercier et al.

[17], Pierce and Varga [18], Rannacher [19], Strang and Fix [20], Wu [22] and Yang [23, 24]. The nonconforming elements, such as the rotated bilinear element (i.e., Q_1^{rot}) and Wilson's element, are studied in Chen and Li [7], Hu et al. [9], Lua and Lin [16], and Lin and Lin [13], and the extrapolations for eigenvalues are explored in Blum et al. [4], Lin [12], Lin and Zhu [14], and Lü et al. [15].

It is worth pointing out that asymptotic lower bounds for eigenvalues have been obtained by the finite difference method (FDM) in Forsythe [8] and Weinberger [21]. In [8], for a convex S, the numerical eigenvalues by the standard five-node finite difference equations have lower bounds, and upper and lower bounds of numerical eigenvalues by FDM are also discussed in [21]. Since the FDM can be regarded as a special kind of FEM involving different integration rules in Li [11], the variational crimes, the terminology used in [20] for FEM with nonconforming elements and numerical integration, may produce the lower bounds of approximate eigenvalues.

2. Basic theorems

We rewrite (1.3) as:

$$a(u,v) = (f,v), \ \forall v \in H_0^1(S), \tag{2.1}$$

where $f = \lambda u$. Define the finite element projection R_h by

$$a_h(R_h u, v) = (f,v), \ \forall v \in V_h^0. \tag{2.2}$$

For simplicity, we assume the simple eigenvalues, and consider only a few leading eigenvalues

$$\lambda_1 < \lambda_2 \le ... \le \lambda_k, \tag{2.3}$$

where k is a small integer. Note that the minimal eigenvalue $\lambda_1 = \lambda_{\min}$ is of great interest in practical applications.

For the above elements, we cite the known results in [23, 24] as a lemma.

Lemma 2.1. *For the quasi-uniform $\square_{ij}$ with the maximal boundary length h, there exists the follwing bound for leading eigenvalues λ and their corresponding eigenfunctions u:*

$$|\lambda - \lambda_h| + \|u - u_h\|_{0,S} + \|u - R_h u\|_{0,S} \le Ch^2, \tag{2.4}$$

where C is a constant independent of h, and (λ_h, u_h) are the FEM solutions by Q_1, Q_1^{rot}, EQ_1^{rot} and Wilson's elements.

Below we give a new theorem.

Theorem 2.1 (Nonconforming)**.** *Let $\square_{ij}$ be quasi-uniform with the maximal boundary length h. For the nonconforming elements, there exists the error formula*

$$\lambda_h - \lambda = \lambda(u - u_I, u_h) - a_h(u - u_I, u_h) + a_h(u - R_h u, u_h) + O(h^4), \tag{2.5}$$

where u and u_I are the true solution (i.e., eigenfunction) and the FEM interpolation of u, respectively, and u_h and $R_h u$ are the FEM solution of (1.13) *and the FEM projection in* (2.2)*, respectively.*

Proof. For the eigenfunctions,

$$(u,u) = 1, \ (u_h, u_h) = 1. \tag{2.6}$$

We choose a different scale of u_h by $\bar{u}_h = \frac{u_h}{(u,u_h)}$. Then we have $(u, \bar{u}_h) = 1$, which yields

$$\lambda_h = \lambda_h(u, \bar{u}_h) = \lambda_h(R_h u, \bar{u}_h) + \lambda_h(u - R_h u, \bar{u}_h). \tag{2.7}$$

Moreover, from (1.13) and (2.2), we obtain

$$\lambda_h(R_h u, \bar{u}_h) = a_h(R_h u, \bar{u}_h) = \lambda(u, \bar{u}_h) = \lambda. \tag{2.8}$$

Since $\bar{u}_h$ has a small difference from u_h, we obtain from Lemma 2.1,

$$\|\bar{u}_h - u_h\|_{0,S} = \left\| \frac{(u, u - u_h)u_h}{(u, u_h)} \right\|_{0,S} \le Ch^2. \tag{2.9}$$

Hence by means of Lemma 2.1 again, a primary expansion from (2.7)−(2.9) is given by

$$\lambda_h = \lambda + \lambda_h(u - R_h u, \bar{u}_h) = \lambda + \lambda_h(u - R_h u, u_h) + O(h^4). \tag{2.10}$$

Finally, a further expansion can be obtained:

$$\begin{aligned} \lambda_h &= \lambda + \lambda_h(u - u_I, u_h) + \lambda_h(u_I - R_h u, u_h) + O(h^4) \\ &= \lambda + \lambda_h(u - u_I, u_h) + a_h(u_I - R_h u, u_h) + O(h^4) \\ &= \lambda + \lambda(u - u_I, u_h) + a_h(u_I - u, u_h) + a_h(u - R_h u, u_h) + O(h^4), \end{aligned} \tag{2.11}$$

where we have replaced λ_h by λ from Lemma 2.1. This is the desired result (2.5), and completes the proof of Theorem 2.1. □

In Theorem 2.1, in order to derive the errors $\lambda_h - \lambda$, we need to evaluate the following interpolation errors:

$$(u - u_I, v), \ a_h(u - u_I, v), \ \forall v \in V_h^0, \tag{2.12}$$

and the projection error

$$a_h(u - R_h u, v), \ \forall v \in V_h^0. \tag{2.13}$$

Note that the projection error (2.13) is null for the conforming elements[2] and that the estimation of (2.12) is similar to that for Poisson's equation. Hence the key analysis of the nonconforming elements is to derive the expansions of (2.13). In this paper, the detailed proof is provided only for Q_1^{rot} and EQ_1^{rot} (see the next section), and the proof for the Q_1 and Wilson's elements in (1.15) appears elsewhere.

In error estimates, we often use the Bramble-Hilbert lemma [5]: Denote by $B(u)$ a bounded linear function from $H^k(S)$ to R.[3] If for a polynomial P_k of degree k, $B(P_k) = 0$, then there exists a constant C independent of u such that

$$|B(u)| \le C|u|_{k+1,S}. \tag{2.14}$$

In this paper, we need more expansions of higher terms of degree $k+1$. We solicit the generalized Bramble-Hilbert Lemma. Let

$$B(u) = \sum_{|\alpha|=k+1} \frac{B(x^\alpha)}{\alpha!|S|} \iint_S D^\alpha u + H(u), \tag{2.15}$$

[2]For the conforming Q_1 element, the expansions of (2.12) will lead to those in (1.15), by using the same proof techniques in this paper.

[3]The bounded linear function $B(u)$ implies that it is continuous.

where $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2}$, $\alpha_1+\alpha_2=\alpha$, and $\alpha! = \alpha_1!\alpha_2!$. $H(u)$ in (2.15) is also a bounded linear function from $H^{k+1}(S)$ to R. We write the following lemma without proof, whose proof is given in Lin and Lin [13].

Lemma 2.2 (Generalized Bramble and Hilbert Lemma). *Let $u \in H^{k+2}(S)$ and $B(P_k)=0$. Suppose that $H(P_{k+1})=0$ in (2.15). There exists a bound,*

$$|H(u)| \le C|u|_{k+2,S}, \tag{2.16}$$

where C is a constant independent of u.

3. Q_1^{rot} and EQ_1^{rot} elements

In this paper, we will derive the expansions in (1.15) for Q_1^{rot} and EQ_1^{rot}. We merge their proofs together, because the main proof for both nonconforming elements has many features in common. Based on Theorem 2.1, the three terms in (2.5) need to be evaluated. For both Q_1^{rot} and EQ_1^{rot}, from their definition of u_I and by integration by parts, we can show the following equality easily:

$$a_h(u-u_I, v) = \iint_S \nabla(u-u_I)\nabla v = 0, \ \forall v \in Q_1^{rot} \text{ or } EQ_1^{rot}. \tag{3.1}$$

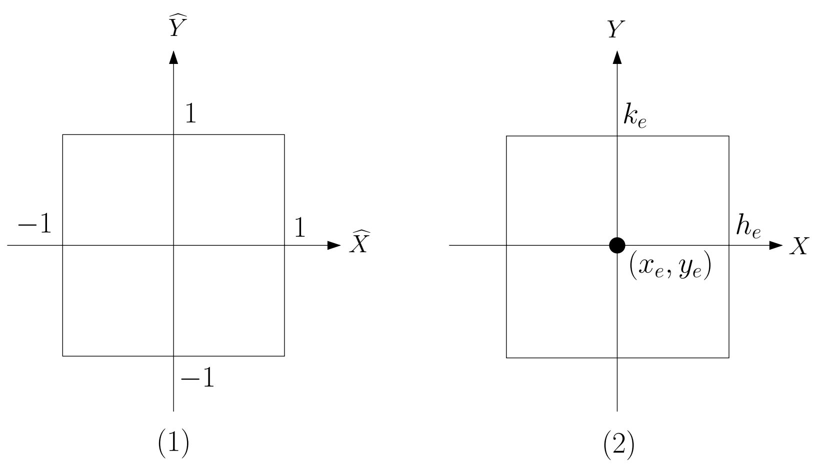

FIGURE 2. (1) $\widehat{e} = [-1,1]\times[-1,1]$. (2) $e = \Box_{ij} = [x_e - h_e, x_e + h_e]\times[y_e-k_e, y_e+k_e]$.

To obtain the expansions of the other two terms in (2.12) and (2.13), we need the following lemmas.

Lemma 3.1. *For $v \in EQ_1^{rot}$ or Q_1^{rot}, there exists the equality*

$$\begin{aligned} a_h(u-R_h u, v) \quad &= \quad \sum_e \Big[\frac{k_e^2}{3}\iint_e u_{xxy}v_y - \frac{4k_e^4}{45}\iint_e u_{xxyy}v_{yy} \\ &\quad + \frac{h_e^2}{3}\iint_e u_{yyx}v_x - \frac{4h_e^4}{45}\iint_e u_{yyxx}v_{xx}\Big] + O(h^5)|u|_5|v|_{2,h}, \end{aligned} \tag{3.2}$$

where $|v|_{m,h} = \sqrt{\sum_e |v|^2_{m,e}}$ $(m=1,2)$, and $e=\Box_{ij} = [x_e-h_e, x_e+h_e]\times[y_e-k_e, y_e+k_e]$ (see Figure 2). Moreover, for uniform rectangles $\Box_{ij}$ with $h_e = h$ and

$k_e = k$, there exists the following equality for $v \in EQ_1^{rot}$ or Q_1^{rot}:

$$a_h(u - R_h u, v) = -\frac{h^2 + k^2}{3} \iint_S u_{xxyy} v + O(h^4)\|u\|_5 \|v\|_{1,h}. \tag{3.3}$$

Lemma 3.2. *For $v \in EQ_1^{rot}(e)$,*

$$\iint_e (u - u_I)v = -\frac{h_e^4}{45} \iint_e u_{xxx} v_x - \frac{k_e^4}{45} \iint_e u_{yyy} v_y + O(h^5)|u|_{4,e}|v|_{1,e}. \tag{3.4}$$

Lemma 3.3. *For $v \in Q_1^{rot}(e)$,*

$$\begin{aligned} \iint_e (u - u_I)v &= -\frac{h^2}{6} \iint_e (u_{xx} + u_{yy})v + \frac{h^4}{30} \iint_e (u_{xxx} v_x + u_{yyy} v_y) \\ &\quad + \frac{h^4}{18} \iint_e (u_{xxy} v_y + u_{yyx} v_x) + O(h^5)|u|_{4,e}|v|_{1,e}. \end{aligned} \tag{3.5}$$

Lemma 3.4. *For $v \in EQ_1^{rot}(e)$ or $Q_1^{rot}(e)$, there exists the integral equality*

$$\begin{aligned} &\iint_e u_{xx}\Big((y - y_e)v_y(x, y_e) + ((y - y_e)^2 - \frac{k_e^2}{3})v_{yy}(x, y_e)\Big) \\ &= \frac{k_e^2}{3} \iint_e u_{xxy} v_y - \frac{4k_e^4}{45} \iint_e u_{xxyy} v_{yy} + O(h^5)|u|_{5,e}|v|_{2,e}. \end{aligned} \tag{3.6}$$

The proof of Lemmas 3.1–3.4 is deferred to Sections 3.1–3.4. For EQ_1^{rot}, we have the following theorem.

Theorem 3.1. *Let $\Box_{ij}$ be quasi-uniform. For EQ_1^{rot}, there exists the eigenvalue error*

$$\lambda_h - \lambda = \frac{1}{3} \sum_e \Big[\iint_e k_e^2 u_{xxy} u_y + h_e^2 \iint_e u_{xyy} u_x \Big] + O(h^3). \tag{3.7}$$

Moreover for uniform $\Box_{ij}$,

$$\lambda_h - \lambda = -\frac{h^2 + k^2}{3} \iint_S u_{xy}^2 + O(h^4). \tag{3.8}$$

Proof. From Lemma 3.2,

$$\begin{aligned} \lambda(u - u_I, u_h) &= \lambda \iint_S (u - u_I)\rho u_h \\ &= \lambda \sum_e \Big[\iint_e (u - u_I)(\rho u_h)_I - \iint_e (u - u_I)(\rho u_h - (\rho u_h)_I) \Big] \\ &= -\lambda \sum_e \Big[\frac{h_e^4}{45} \iint_e u_{xxx}((\rho u_h)_I)_x - \frac{k_e^4}{45} \iint_e u_{yyy}((\rho u_h)_I)_y \Big] + O(h^4) = O(h^4), \end{aligned} \tag{3.9}$$

where we have used

$$\iint_e (u - u_I)(\rho u_h - (\rho u_h)_I) = \iint_e (u - u_I)u_h(\rho - \rho_I) = O(h^4). \tag{3.10}$$

Also from Lemma 3.1,

$$\begin{aligned} a_h(u - R_h u, u_h) &= \sum_e \frac{k_e^2}{3} \iint_e u_{xxy}(u_h)_y + \sum_e \frac{h_e^2}{3} \iint_e u_{xyy}(u_h)_x + O(h^4) \\ &= \sum_e \frac{k_e^2}{3} \iint_e u_{xxy} u_y + \sum_e \frac{h_e^2}{3} \iint_e u_{xyy} u_x + O(h^3). \end{aligned} \tag{3.11}$$

Based on Theorem 2.1, combining (3.1), (3.9) and (3.11) yields the first desired result (3.7).

Next, we prove (3.8) for the uniform rectangles $\square_{ij}$. From Lemmas 3.1 and 2.1 and by integration by parts,

$$\begin{aligned} a_h(u - R_h u, u_h) &= \frac{h^2}{3} \sum_e \iint_e u_{xyy}(u_h)_x \\ &\quad + \frac{k^2}{3} \sum_e \iint_e u_{xxy}(u_h)_y + O(h^4)|u|_5 |u_h|_{2,h} \\ &= -\frac{h^2+k^2}{3} \sum_e \iint_e u_{xxyy}(u_h) + O(h^4) \\ &= -\frac{h^2+k^2}{3} \sum_e \iint_e u_{xxyy} u + O(h^4) \\ &= -\frac{h^2+k^2}{3} \iint_S u_{xy}^2 + O(h^4), \end{aligned} \tag{3.12}$$

where we have used the integration by parts again,

$$\iint_S u_{xxyy} u = -\iint_S u_{xyy} u_x = \iint_S u_{xy}^2, \tag{3.13}$$

and

$$\begin{aligned} |u_h|_{2,h} &\le |u_h - u_I|_{2,h} + |u_I - u|_{2,h} + |u|_2 \\ &\le C h^{-1} |u_h - u_I|_{1,h} + C|u|_2 \le C|u|_2. \end{aligned} \tag{3.14}$$

Based on Theorem 2.1, combining (3.1), (3.9) and (3.12) yield the second desired result (3.8) (i.e., (1.15) for EQ_1^{rot} with $k_e = h_e = h$). This completes the proof of Theorem 3.1. $\square$

Below, for Q_1^{rot}, we have the following theorem.

Theorem 3.2. *Let $\square_{ij}$ be uniform squares. For Q_1^{rot} there exists the eigenvalue error*

$$\lambda_h - \lambda = \frac{h^2}{6} \iint_S (u_{xx} - u_{yy})^2 + O(h^4). \tag{3.15}$$

Proof. For Q_1^{rot} on uniform square $\square_{ij}$ with $h = k$, we have from Lemmas 3.3 and 2.1,

$$\begin{aligned} \lambda(u - u_I, u_h) &= \lambda \iint_S (u - u_I)(\rho u_h)_I + O(h^4) \\ &= -\lambda \frac{h^2}{6} \iint_S (u_{xx} + u_{yy})(\rho u_h)_I + O(h^4) \\ &= -\lambda \frac{h^2}{6} \iint_S (u_{xx} + u_{yy}) \rho u + O(h^4) \\ &= \frac{h^2}{6} \iint_S (u_{xx} + u_{yy})^2 + O(h^4), \end{aligned} \tag{3.16}$$

where we have used (1.1). From integration by parts, there exists the equalilty

$$\iint_S u_{xx} u_{yy} = \iint_S u_{xy}^2. \tag{3.17}$$

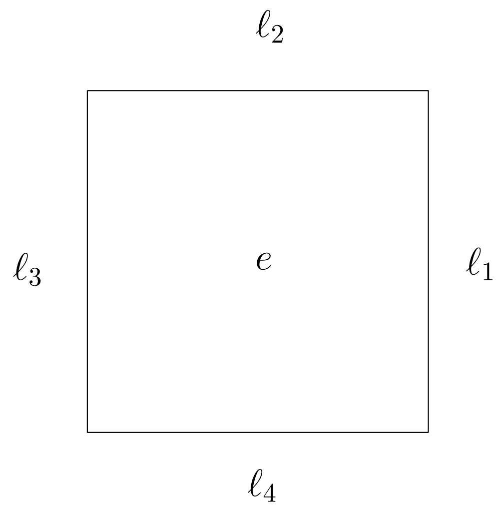

FIGURE 3. The rectangle.

For uniform squares $\square_{ij}$ with $h = k$, based on Theorem 2.1, combining (3.1), (3.12), (3.16) and (3.17) yields the desired result (3.15) (i.e., (1.15) for Q_1^{rot}). This completes the proof of Theorem 3.2. $\square$

Theorems 3.1 and 3.2 provide the desired expansions in (1.15) for EQ_1^{rot} and Q_1^{rot} elements. It is interesting to note that Q_1^{rot} and EQ_1^{rot} give the upper and the lower bounds of the leading eigenvalues, respectively.

3.1. **Proof of Lemma 3.1.** For the nonconforming errors of Q_1^{rot} and EQ_1^{rot}, we have from (2.1), (2.2) and the Green formula,

$$
\begin{aligned}
(3.18)\qquad a_h(u - R_h u, v) &= \sum_e \oint_{\partial e} \frac{\partial u}{\partial n} v \, ds \\
&= \sum_e \Big(\int_{\ell_1} - \int_{\ell_3}\Big) u_x v \, dy + \sum_e \Big(\int_{\ell_2} - \int_{\ell_4}\Big) u_y v \, dx,
\end{aligned}
$$

where ℓ_i are the edges in Figure 3. Since the average on ℓ_k is continuous based on the definitions in (1.7) and (1.8), we have

$$
(3.19)\qquad \sum_e \Big(\int_{\ell_1} - \int_{\ell_3}\Big) u_x \bar{v} \, dy = 0,
$$

where $\bar{v} = \int_{\ell_i} v \, ds / |\ell_i|$ is constant on ℓ_i. Hence we obtain

$$
(3.20)\qquad \sum_e \Big(\int_{\ell_1} - \int_{\ell_3}\Big) u_x v \, dy = \sum_e \Big(\int_{\ell_1} - \int_{\ell_3}\Big) u_x (v - \bar{v}) \, dy.
$$

Also since $v|_{\ell_1 \cup \ell_3} = \mathrm{span}\{1, y - y_e, (y - y_e)^2\}$, then $\bar{v}|_{\ell_1 \cup \ell_3} = \mathrm{span}\{1, 0, \frac{k_e^2}{3}\}$. Moreover, from Taylor's formula in the y variable for each x, we have

$$
(3.21)\qquad (v - \bar{v})|_{\ell_i} = (y - y_e) v_y(x, y_e) + \Big((y - y_e)^2 - \frac{k_e^2}{3}\Big) \frac{v_{yy}(x, y_e)}{2}, \ i = 1, 3.
$$

TABLE 1. The integration, $\iint_{\widehat{e}}(u-u_I)v$ for $u\in P_3\setminus EQ_1^{rot}$, $v\in EQ_1^{rot}=\text{span}\{1,x,y,x^2,y^2\}$, where $\widehat{e}=[-1,1]^2$ and the sign 0^+ denotes that the computed integrals are zero.

u	xy	x^3	x^2y	xy^2	y^3	Notes
u_I	0	x	$\frac{1}{3}y$	$\frac{1}{3}x$	y	/
$u-u_I$	xy	x^3-x	$x^2y-\frac{1}{3}y$	$xy^2-\frac{1}{3}x$	y^3-y	/
$\iint_{\widehat{e}}(u-u_I)$	0	0	0	0	0	$v=1$
$\iint_{\widehat{e}}(u-u_I)x$	0	$-\frac{8}{15}$	0	0^+	0	$v=x$
$\iint_{\widehat{e}}(u-u_I)y$	0	0	0^+	0	$-\frac{8}{15}$	$v=y$
$\iint_{\widehat{e}}(u-u_I)x^2$	0	0	0	0	0	$v=x^2$
$\iint_{\widehat{e}}(u-u_I)y^2$	0	0	0	0	0	$v=y^2$

Then

$$\begin{aligned}
&(\int_{\ell_1}-\int_{\ell_3})u_x(v-\bar{v})\,dy\\
&=(\int_{\ell_1}-\int_{\ell_3})u_x\Big[(y-y_e)v_y(x,y_e)+\Big((y-y_e)^2-\frac{k_e^2}{3}\Big)\frac{v_{yy}(x,y_e)}{2}\Big]dy\\
&=\iint_e u_{xx}\Big[(y-y_e)v_y(x,y_e)+\Big((y-y_e)^2-\frac{k_e^2}{3}\Big)\frac{v_{yy}(x,y_e)}{2}\Big]\,dxdy, \qquad (3.22)
\end{aligned}$$

where we have used that

$$\Big((y-y_e)v_y(x,y_e)+\Big[(y-y_e)^2-\frac{k_e^2}{3}\Big]\frac{v_{yy}(x,y_e)}{2}\Big)_x=0, \qquad (3.23)$$

based on $v_{xy}=v_{xyy}=0$ for Q_1^{rot} and EQ_1^{rot} elements.

Similarly, we have

$$\begin{aligned}
&(\int_{\ell_2}-\int_{\ell_4})u_y(v-\bar{v})\,dx \qquad (3.24)\\
&=\iint_e u_{yy}\Big[(x-x_e)v_x(x_e,y)+\Big((x-x_e)^2-\frac{h_e^2}{3}\Big)\frac{v_{xx}(x_e,y)}{2}\Big]\,dxdy.
\end{aligned}$$

Hence for both Q_1^{rot} and EQ_1^{rot}, we obtain from (3.18), (3.22) and (3.24),

$$\begin{aligned}
a_h(u-R_hu,v)&=\sum_e\iint_e u_{xx}\Big[(y-y_e)v_y(x,y_e)+\Big((y-y_e)^2-\frac{k_e^2}{3}\Big)\frac{v_{yy}(x,y_e)}{2}\Big]\\
&+\sum_e\iint_e u_{yy}\Big[(x-x_e)v_x(x_e,y)+\Big((x-x_e)^2-\frac{h_e^2}{3}\Big)\frac{v_{xx}(x_e,y)}{2}\Big]. \qquad (3.25)
\end{aligned}$$

The desired result (3.2) in Lemma 3.1 follows from Lemma 3.4. This completes the proof of Lemma 3.1. □

3.2. **Proof of Lemma 3.2.** Denote $B(u,v)=\iint_{\widehat{e}}(u-u_I)v$, where $\widehat{e}=[-1,1]^2$ in Figure 2. For $u\in EQ_1^{rot}$, we have $\iint_{\widehat{e}}(u-u_I)v=0$. For $u\in P_3\setminus EQ_1^{rot}$, the integration terms needed are given in Table 1. In Table 1 and other tables given below, the zero values can be easily seen by checking odd polynomials with respect to x or y, and the zero values with "+" in the tables are confirmed by real integral evaluation. Hence, we only examine those zeros with "+" and the nontrivial terms. First, take $u = x^2y$ and $v = y$ for example. We have

$$B(x^2y, y) = \iint_{\widehat{e}} (x^2y - \frac{y}{3})y = 0. \tag{3.26}$$

Similarly, for $u = xy^2$ and $v = x$,

$$B(xy^2, x) = 0. \tag{3.27}$$

Next, we examine the nontrivial terms in Table 1. When $u = x^3$ and $v = x$,

$$B(x^3, x) = \iint_{\widehat{e}} (x^3 - x)x = \iint_{\widehat{e}} (x^4 - x^2) = -\frac{8}{15} = -\frac{1}{45}\iint_{\widehat{e}} u_{xxx}v_x. \tag{3.28}$$

Similarly, when $u = y^3$ and $v = y$,

$$B(y^3, y) = \iint_{\widehat{e}} (y^3 - y)y = \iint_{\widehat{e}} (y^4 - y^2) = -\frac{8}{15} = -\frac{1}{45}\iint_{\widehat{e}} u_{yyy}v_y. \tag{3.29}$$

Define the new function

$$H(u, v) = B(u, v) + \frac{1}{45}\iint_{\widehat{e}} u_{xxx}v_x + \frac{1}{45}\iint_{\widehat{e}} u_{yyy}v_y. \tag{3.30}$$

Hence for $u \in P_3$, $H(u, v) = 0$, and then from Lemma 2.2,

$$|H(u, v)| \le C|u|_{4,\widehat{e}}|v|_{1,\widehat{e}}. \tag{3.31}$$

Denote $e = \square_{ij} = [x_e - h_e, x_e + h_e] \times [y_e - k_e, y_e + k_e]$ with the boundary lengths $2h_e$ and $2k_e$ (see Figure 2), where $h_e = O(h)$, $k_e = O(h)$ $\max\{\frac{h_e}{k_e}, \frac{k_e}{h_e}\} \le C_0$, and C_0 is a constant independent of h. Define an affine transformation $T : (x, y) \to (\widehat{x}, \widehat{y})$ with

$$\widehat{x} = \frac{x - x_e}{h_e},\ \widehat{y} = \frac{y - y_e}{k_e}. \tag{3.32}$$

Then, under T, we have that $e \to \widehat{e} = [-1, 1]^2$ and the following equations:

$$\begin{aligned} \widehat{u}(\widehat{x}, \widehat{y}) &= u(x, y), & \widehat{u}_I(\widehat{x}, \widehat{y}) &= u_I(x, y), \\ d\widehat{x} &= \frac{dx}{h_e}, & d\widehat{y} &= \frac{dy}{k_e}, \\ \widehat{u}_{\widehat{x}} &= h_e u_x, & \widehat{u}_{\widehat{y}} &= k_e u_y. \end{aligned}$$

By the affine transformation T in (3.32) we have[4]

$$\begin{aligned} \iint_e (u - u_I)v &= h_e k_e \iint_{\widehat{e}} (u - u_I)v \\ &= h_e k_e \Big[-\frac{1}{45}\iint_{\widehat{e}} u_{xxx}v_x - \frac{1}{45}\iint_{\widehat{e}} u_{yyy}v_y + O(1)|u|_{4,\widehat{e}}|v|_{1,\widehat{e}} \Big] \\ &= -\frac{1}{45}\Big[h_e^4 \iint_e u_{xxx}v_x + k_e^4 \iint_e u_{yyy}u_y \Big] + O(h^5)|u|_{4,e}|v|_{1,e}. \end{aligned} \tag{3.33}$$

This is the desired result (3.4) and completes the proof of Lemma 3.2. □

[4]For simplicity, we omit the hat notation on the top in the integral of $\widehat{e}$. For instance, the integration $\iint_{\widehat{e}} \widehat{u}_{\widehat{x}\widehat{x}\widehat{x}}\widehat{v}_{\widehat{x}}$ is simplified as $\iint_{\widehat{e}} u_{xxx}v_x$ in (3.33).

TABLE 2. The integration, $\iint_{\widehat{e}}(u-u_I)v$ for $u \in P_3 \setminus Q_1^{rot}$, $v \in Q_1^{rot} = \text{span}\{1, x, y, x^2-y^2\}$, where $\widehat{e} = [-1,1]^2$ and the sign 0^+ denotes that the computed integrals are zero.

u	x^2	xy	y^2	x^3	x^2y	xy^2	y^3	Notes
u_I	$\frac{2}{3}+\frac{1}{2}(x^2-y^2)$	0	$\frac{2}{3}-\frac{1}{2}(x^2-y^2)$	x	$\frac{1}{3}y$	$\frac{1}{3}x$	y	/
$u-u_I$	$\frac{1}{2}(x^2+y^2)-\frac{2}{3}$	xy	$\frac{1}{2}(x^2+y^2)-\frac{2}{3}$	x^3-x	$x^2y-\frac{1}{3}y$	$xy^2-\frac{1}{3}x$	y^3-y	/
$\iint_{\widehat{e}}(u-u_I)$	-$\frac{4}{3}$	0	-$\frac{4}{3}$	0	0	0	0	$v=1$
$\iint_{\widehat{e}}(u-u_I)x$	0	0	0	$-\frac{8}{15}$	0	0^+	0	$v=x$
$\iint_{\widehat{e}}(u-u_I)y$	0	0	0	0	0^+	0	$-\frac{8}{15}$	$v=y$
$\iint_{\widehat{e}}(u-u_I)(x^2-y^2)$	0^+	0	0^+	0	0	0	0	$v=x^2-y^2$

3.3. **Proof of Lemma 3.3.** Denote $B(u,v) = \iint_{\widehat{e}}(u-u_I)v$. For $u \in P_3 \setminus Q_1^{rot}$, the integration is given in Table 2. Let us check the terms with 0^+ and the nontrivial terms in Table 2. First for $u = x^2$ and $v = x^2 - y^2$, we have

$$\begin{aligned} \iint_{\widehat{e}}(u-u_I)v &= \iint_{\widehat{e}}\Big(\frac{1}{2}(x^2+y^2)-\frac{2}{3}\Big)(x^2-y^2) \\ &= \iint_{\widehat{e}}\Big[\frac{1}{2}(x^4-y^4)-\frac{2}{3}(x^2-y^2)\Big] = 0, \end{aligned} \tag{3.34}$$

where we have used the symmetry: $\iint_{\widehat{e}} x^2 = \iint_{\widehat{e}} y^2$ and $\iint_{\widehat{e}} x^4 = \iint_{\widehat{e}} y^4$. Similarly, for $u = y^2$ and $v = x^2 - y^2$,

$$\iint_{\widehat{e}}(u-u_I)v = 0. \tag{3.35}$$

Next, we examine the nontrivial terms. When $u = x^2$ and $v = 1$,

$$\iint_{\widehat{e}}(u-u_I)v = \iint_{\widehat{e}}\Big[\frac{1}{2}(x^2+y^2)-\frac{2}{3}\Big] = -\frac{4}{3} = -\frac{1}{6}\iint_{\widehat{e}} u_{xx}v. \tag{3.36}$$

Similarly, when $u = y^2$ and $v = 1$,

$$\iint_{\widehat{e}}(u-u_I)v = -\frac{4}{3} = -\frac{1}{6}\iint_{\widehat{e}} u_{yy}v. \tag{3.37}$$

Define a functional

$$H(u,v) = B(u,v) + \frac{1}{6}\iint_{\widehat{e}} u_{xx}v + \frac{1}{6}\iint_{\widehat{e}} u_{yy}v. \tag{3.38}$$

Hence for $u \in P_2$, $H(u,v) = 0$, $\forall v \in Q_1^{rot}$, and then from Lemma 2.2,

$$|H(u,v)| \le C|u|_{3,\widehat{e}}|v|_{0,\widehat{e}}. \tag{3.39}$$

Then we have

$$B(u,v) = -\frac{1}{6}\iint_{\widehat{e}} u_{xx}v - \frac{1}{6}\iint_{\widehat{e}} u_{yy}v + O(1)|u|_{3,\widehat{e}}|v|_{0,\widehat{e}}. \tag{3.40}$$

Below, we consider the additional terms in $P_3 \setminus P_2$, whose results are also listed in Table 2. First, when $u = x^2y$ and $v = y$, we have

$$\iint_{\widehat{e}}(u - u_I)v = \iint_{\widehat{e}}(x^2y - \frac{1}{3}y)y = 0, \tag{3.41}$$

and when $u = xy^2$ and $v = x$,

$$\iint_{\widehat{e}}(u - u_I)v = \iint_{\widehat{e}}(xy^2 - \frac{1}{3}x)x = 0. \tag{3.42}$$

Next, when $u = x^3$ and $v = x$,

$$\iint_{\widehat{e}}(u - u_I)v = \iint_{\widehat{e}}(x^3 - x)x = \iint_{\widehat{e}}(x^4 - x^2) = -\frac{8}{15}, \tag{3.43}$$

and when $u = y^3$ and $v = y$, similarly

$$\iint_{\widehat{e}}(u - u_I)v = -\frac{8}{15}. \tag{3.44}$$

Now we have to recount $H(u, v)$ for those extra nontrivial terms of $P_3 \setminus P_2$, and obtain from (3.38):

(1) When $u = x^2y$ and $v = y$,

$$H(u,v) = B(x^2y, y) + \frac{1}{6}\iint_{\widehat{e}} u_{xx}v + \frac{1}{6}\iint_{\widehat{e}} u_{yy}v = 0 + \frac{4}{9} + 0 = \frac{4}{9} = \frac{1}{18}\iint_{\widehat{e}} u_{xxy}v_y.$$

(2) When $u = xy^2$ and $v = x$, similarly

$$H(u,v) = \frac{4}{9} = \frac{1}{18}\iint_{\widehat{e}} u_{xyy}v_x.$$

(3) When $u = x^3$ and $v = x$,

$$\begin{aligned} H(u,v) &= B(x^3, x) + \frac{1}{6}\iint_{\widehat{e}} u_{xx}v = -\frac{8}{15} + \frac{1}{6}\iint_{\widehat{e}} 6x^2 \\ &= -\frac{8}{15} + \frac{4}{3} = \frac{4}{5} = \frac{1}{30}\iint_{\widehat{e}} u_{xxx}v_x. \end{aligned}$$

(4) When $u = y^3$ and $v = y$, similarly

$$H(u,v) = \frac{4}{5} = \frac{1}{30}\iint_{\widehat{e}} u_{yyy}v_y.$$

Hence we define a new functional

$$\begin{aligned} X(u,v) &= H(u,v) - \frac{1}{30}\iint_{\widehat{e}} u_{xxx}v_x - \frac{1}{30}\iint_{\widehat{e}} u_{yyy}v_y \\ &\quad -\frac{1}{18}\iint_{\widehat{e}} u_{xxy}v_y - \frac{1}{18}\iint_{\widehat{e}} u_{xyy}v_x. \end{aligned} \tag{3.45}$$

Obviously, for $u \in P_3$, $H(u, v) = 0$, $v \in Q_1^{rot}$, and then from Lemma 2.2,

$$X(u,v) \le C|u|_{4,\widehat{e}}|v|_{1,\widehat{e}}.$$

Then, we conclude that

$$\begin{aligned} B(u,v) &= -\frac{1}{6}\iint_{\widehat{e}}(u_{xx}+u_{yy})v+\frac{1}{30}\iint_{\widehat{e}}(u_{xxx}v_x+u_{yyy}v_y) \\ &\quad +\frac{1}{18}\iint_{\widehat{e}}(u_{xxy}v_y+u_{xyy}v_x)+O(1)|u|_{4,\widehat{e}}|v|_{1,\widehat{e}}. \end{aligned}$$

The desired result (3.5) follows by the proof techniques via the affine transformation T in (3.32). This completes the proof of Lemma 3.3. $\square$

TABLE 3. The integration, $\iint_{\widehat{e}} u_{xx}D(v)$ for $u \in P_4 \setminus EQ_1^{rot}$, $v \in EQ_1^{rot} = \mathrm{span}\{1, x, y, x^2, y^2\}$ and $D(v) \in \mathrm{span}\{1, 0, 1, 0, y^2 - \frac{1}{3}\}$, where $\widehat{e} = [-1, 1]^2$ and the sign "/" denotes the zero of integrals due to $u_{xx} = 0$.

u	1	x	y	x^2	y^2	xy	x^3	x^2y	xy^2	y^3	x^4	x^3y	xy^3	y^4	x^2y^2	$D(v)$
u_{xx}	0	0	0	2	0	0	$6x$	$2y$	0	0	$12x^2$	$6xy$	0	0	$2y^2$	
$\iint_{\widehat{e}} u_{xx}y$	/	/	/	0	/	/	0	$\frac{8}{3}$	/	/	0	0	/	/	0	$v = y$, $D(v) = y$
$\iint_{\widehat{e}} u_{xx}(y^2 - \frac{1}{3})$	/	/	/	0^+	/	/	0	0	/	/	0	0	/	/	$\frac{32}{45}$	$v = y^2$, $D(v) = y^2 - \frac{1}{3}$

3.4. **Proof of Lemma 3.4.** Denote $D(v) = yv_y(x,0) + (y^2 - \frac{1}{3})\frac{v_{yy}(x,0)}{2}$ on $\hat{e}$ and

$$B(u,v) = \iint_{\widehat{e}} u_{xx}\Big[yv_y(x,0) + (y^2 - \frac{1}{3})\frac{v_{yy}(x,0)}{2}\Big] = \iint_{\widehat{e}} u_{xx}D(v), \tag{3.46}$$

where $v \in \mathrm{span}\{1, x, y, x^2, y^2\}$ and $D(v) = \mathrm{span}\{0, 0, y, 0, y^2 - \frac{1}{3}\}$. We list in Table 3 the integration $\iint_{\widehat{e}} u_{xx}D(v)$ for $u \in P_4$ and $v \in EQ_1^{rot}$. Let us check the terms with 0^+ and the nontrivial terms in Table 3. First, when $u = x^2$, $v = y^2$ and $D(v) = y^2 - \frac{1}{3}$, the integral is zero:

$$\iint_{\widehat{e}} u_{xx}D(v) = 2\iint_{\widehat{e}}(y^2 - \frac{1}{3}) = 0.$$

Hence for $u \in P_2$, $B(u,v) = 0$, $v \in EQ_1^{rot}$, and then from Lemma 2.2,

$$|B(u,v)| \le C|u|_{3,\widehat{e}}|v|_{1,\widehat{e}}.$$

Next consider $u \in P_3/P_2$. When $u = x^2y$ and $v = y$,

$$\iint_{\widehat{e}} u_{xx}D(v) = \iint_{\widehat{e}} 2y^2 = \frac{8}{3} = \frac{1}{3}\iint_{\widehat{e}} u_{xxy}v_y. \tag{3.47}$$

Also $B(u,v) = 0$ for $u = x^3, xy^2, y^3$ and $v \in EQ_1^{rot}$ (see Table 3). Define a functional

$$H(u,v) = B(u,v) - \frac{1}{3}\iint_{\widehat{e}} u_{xxy}v_y.$$

For $u \in P_3$, $H(u,v) = 0$. From Lemma 2.2,

$$|H(u,v)| \le C|u|_{4,\widehat{e}}|v|_{1,\widehat{e}},$$

which yields

$$B(u,v) = \frac{1}{3}\iint_{\widehat{e}} u_{xxy}v_y + O(1)|u|_{4,\widehat{e}}|v|_{1,\widehat{e}}.$$

By the affine transformation (3.32), we have

$$\begin{aligned}
&\iint_e u_{xx}\Big[(y-y_e)v_y(x,y_e) + \Big((y-y_e)^2 - \frac{k_e^2}{3}\Big)\frac{v_{yy}(x,y_e)}{2}\Big] \\
&= \frac{k_e}{h_e}B(\widehat{u},\widehat{v}) = \frac{k_e}{h_e}\Big[\frac{1}{3}\iint_{\widehat{e}} \widehat{u}_{\widehat{x}\widehat{x}\widehat{y}}\widehat{v}_y + O(1)|\widehat{u}|_{4,\widehat{e}}|\widehat{v}|_{1,\widehat{e}}\Big] \\
&= \frac{k_e^2}{3}\iint_e u_{xxy}v_y + O(h^3)|u|_{4,e}|v|_{1,e}.
\end{aligned}$$

To discover the higher remainders of $O(h^4)$, we should also consider $v \in P_4 \setminus P_3$; the additional integrations are listed in Table 3. Below we consider the nontrivial terms only. For $u = x^2y^2$, $v = y^2$ and $D(v) = y^2 - \frac{1}{3}$, we have

$$\iint_{\widehat{e}} u_{xx}D(v) = \iint_{\widehat{e}} 2y^2(y^2 - \frac{1}{3}) = \frac{32}{45},$$

which gives

$$\begin{aligned}
H(u,v) = H(x^2y^2, y^2) &= B(x^2y^2,y^2) - \frac{1}{3}\iint_{\widehat{e}} u_{xxy}v_y = \frac{32}{45} - \frac{1}{3}\iint_{\widehat{e}} 4y\cdot 2y \\
&= \frac{32}{45} - \frac{32}{9} = -\frac{128}{45} = -\frac{4}{45}\iint_{\widehat{e}} u_{xxyy}v_{yy}.
\end{aligned}$$

Now we define a new functional

$$X(u,v) = H(u,v) + \frac{4}{45}\iint_{\widehat{e}} u_{xxyy}v_{yy}.$$

Hence for $u \in P_4$, $X(u,v) = 0$, and then from Lemma 2.2,

$$|X(u,v)| \le C|u|_{5,\widehat{e}}|v|_{2,\widehat{e}}.$$

This yields

$$B(u,v) = \frac{1}{3}\iint_{\widehat{e}} u_{xxy}v_y - \frac{4}{45}\iint_{\widehat{e}} u_{xxyy}v_{yy} + O(1)|u|_{5,\widehat{e}}|v|_{2,\widehat{e}}.$$

The desired result (3.6) in Lemma 3.4 for EQ_1^{rot} follows from the affine transformation T in (3.32).

Next for Q_1^{rot}, we have from Table 3,

$$|B(u,v)| \le C|u|_{3,\widehat{e}}|v|_{1,\widehat{e}}.$$

The rest of the proof is exactly the same as that for EQ_1^{rot}. This completes the proof of Lemma 3.4. □

4. Numerical experiments

In this section, we provide two numerical experiments of the four elements, $Q_1, Q_1^{rot}, EQ_1^{rot}$ and Wilson's element for solving (1.1) and (1.2).

4.1. **Function $\rho = 1$.** Consider the eigenvalue problem of Laplace's operator with $\rho = 1$,

$$-\triangle u = -\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = \lambda u \text{ in } S,$$
$$u = 0 \text{ on } \Gamma = \partial S,$$

where $S = \{(x, y), 0 \le x, y \le 1\}$. Then we have the exact eigenfunctions and eigenvalues,[5]

$$u_{k,\ell} = 2\sin(k\pi x)\sin(\ell\pi y),\ \lambda_{k,\ell} = (k^2 + \ell^2)\pi^2,\ 1 \le k, \ell \le N-1. \tag{4.1}$$

Since the minimal and the next minimal eigenvalues, denoted by λ_1 and λ_2, are the most interesting, we only provide their computed results. In Tables 4 and 5, we list the numerical eigenvalues, their errors and the ratios $= |\frac{\varepsilon_{2h}}{\varepsilon_h}|$ for all four elements, where $\varepsilon_h = \lambda_h - \lambda$, and λ_h and λ are the approximate and the true eigenvalues, respectively. Denote $h = 1/(2N)$ from Figure 2, and $N = 2^m$, $m = 1, 2, \ldots$ When $|\frac{\varepsilon_{2h}}{\varepsilon_h}| \approx 2^p$, we may conclude the empirical convergence rates $O(h^p)$.

For the Q_1, the EQ_1^{rot} and the Wilson's element, we can see from Tables 4 and 5 that

$$\lambda_{1,h} - \lambda = O(h^2), \tag{4.2}$$

where $\lambda_{\ell,h}$ denotes the computed λ_ℓ ($\ell = 1, 2$) at the mesh size h. However, for the Q_1^{rot},

$$\lambda_{1,h} - \lambda_1 = O(h^4), \tag{4.3}$$

$$\lambda_{2,h} - \lambda_2 = O(h^2). \tag{4.4}$$

Equations (4.2)–(4.4) agree with those in (1.15) perfectly. The high convergence rate $O(h^4)$ in (4.3) results from the symmetry of $u_{xx} = u_{yy}$ for the eigenfunction $u(x, y)$ corresponding to λ_1.

From Table 4, we can find the following relative errors of λ_1 at $N = 32$:

$$\frac{\lambda_{1,h} - \lambda_1}{\lambda_1} = 0.803(-3),\ -0.387(-6),\ -0.802(-3),\ -0.240(-2), \tag{4.5}$$

for Q_1, Q_1^{rot}, EQ_1^{rot} and Wilson's elements, respectively. From (4.5) we can see that Q_1 provides an upper bound due to a positive relative error, and EQ_1^{rot} and Wilson's elements provide lower bounds due to negative relative errors. From Tables 4 and 5, the Q_1^{rot} provide the lower and the upper bounds for λ_1 and λ_2, respectively.

To verify (1.16) we have computed $\widehat{E}_h = E_h\Big|_{Q_1^{rot}} - \frac{1}{2}(E_h\Big|_{Q_1} + E_h\Big|_{EQ_1^{rot}})$, where $E_h = \lambda_{1,h} - \lambda_1$. Table 6 lists the results to display the $O(h^4)$ convergence rate perfectly.

More importantly, the expansions of eigenvalues can be applied to raise the accuracy by the extrapolation techniques. Based on the computed eigenvalues in

[5] The constant 2 of the eigenfunctions in (4.1) is used for $(u, u) = 1$.

Tables 4 and 5, we may use the following extrapolation formulas for $\lambda_{1,h}$:

$$\lambda_h^{(k)} = \frac{2^{2k}\lambda_h^{(k-1)} - \lambda_{2h}^{(k-1)}}{2^{2k}-1}, \; k = 1,2,3,4, \tag{4.6}$$

for Q_1, EQ_1^{rot} and Wilson's elements, where $\lambda_h^0 = \lambda_h$. Eq. (4.6) is also used for $\lambda_{2,h}$ by the Q_1^{rot}. Since the $\lambda_{1,h}$ by the Q_1^{rot} has the higher convergence rate, the following extrapolation formulas should be used:

$$\lambda_h^{(k)} = \frac{2^{2k+2}\lambda_h^{(k-1)} - \lambda_{2h}^{(k-1)}}{2^{2k+2}-1}, \; k = 1,2,3,4. \tag{4.7}$$

Note that in (4.6) and (4.7), $\lambda_h^{(1)}$ denotes the first level of extrapolation. In computation, we have computed from the first to the fourth levels of extrapolation. Such a procedure is like that in the Romberg integration. All the extrapolation results are listed in Tables 7 and 8 for $\lambda_{1,h}$ by Q_1^{rot} and EQ_1^{rot}. From Tables 7 and 8 we can see

$$\lambda_{1,h}^{(1)} - \lambda = O(h^4) \text{ for } EQ_1^{rot}, \tag{4.8}$$

$$\lambda_{1,h}^{(1)} - \lambda = O(h^6) \text{ for } Q_1^{rot}, \tag{4.9}$$

where $\lambda_{1,h}^{(1)}$ is the better approximation of $\lambda_{1,h}$ at the first level of extrapolation. Below, we list the following eigenvalues at the first and fourth levels of extrapolation:

$$\frac{\lambda_{1,h}^{(1)} - \lambda_1}{\lambda_1} = 0.472(-9), \; -0.512(-5), \tag{4.10}$$

$$\frac{\lambda_{1,h}^{(4)} - \lambda_1}{\lambda_1} = -0.135(-13), \; -0.454(-8), \tag{4.11}$$

for Q_1^{rot} and EQ_1^{rot} at $N = 32$ respectively. Evidently, the errors in (4.10) and (4.11) are much smaller than those in (4.5). Interestingly, the $\lambda_{1,h}^{(4)} = 19.73920880217845$ by the Q_1^{rot} has 14 significant digits, which is the most accurate value in our computation.

Suppose that we only carry out the computation for $N = 2, 4, 8$, but not for $N = 16$ and $N = 36$ due to some reasons (e.g., the limitation of computer memory or the CPU time). Based on those results, we may use (4.6) and (4.7) until the second level of extrapolation only. The corresponding results are found from Tables 7 and 8 at $N = 8$:

$$\left|\frac{\lambda_{1,h}^{(2)} - \lambda_1}{\lambda_1}\right| = 0.422(-6), \; 0.347(-3), \tag{4.12}$$

for $Q_1^{rot}$and EQ_1^{rot} respectively. The relative errors in (4.12) are close to those in (4.5), but their signs may be changed. This fact displays a significance of the extrapolation, based on the expansions of eigenvalue solutions given in this paper.

The above examination is for the convergence rate; it is crucial to scrutinize numerically the principal terms of the error expansions in (1.15). First, take EQ_1^{rot} for λ_1 for example. Since the corresponding eigenfunction $u_{1,1} = 2\sin(\pi x)\sin(\pi y)$ from (4.1), we have the principal term from (1.15),

$$E_1 = -\frac{2h^2}{3}\iint_S u_{xy}^2 = -\frac{2h^2\pi^4}{3} = -\frac{\pi^4}{6N^2}, \tag{4.13}$$

where we have used $h = \frac{1}{2N}$. Then the relative value is given by

$$\bar{\epsilon}_1 = \frac{E_1}{\lambda_1} = -\frac{\pi^4}{6N^2(2\pi^2)} = -\frac{\pi^2}{12N^2}. \tag{4.14}$$

Based on (4.14), for $N = 2, 4, 8, 16, 32$, we obtain respectively

$$\bar{\epsilon}_1 = -0.206,\ -0.514(-1),\ -0.129(-1),\ -0.321(-2), -0.803(-3). \tag{4.15}$$

Eq. (4.15) coincides with the numerical data in Table 4 for EQ_1^{rot} very well, which verifies the principal term in (4.13).

Next, consider Q_1^{rot} for λ_2. Since the corresponding eigenfunction $u_{2,1} = 2\sin(2\pi x)\sin(\pi y)$ from (4.1) with $u_{xx} \neq u_{yy}$, we have the principal term from (1.15)

$$E_1 = \frac{h^2}{6}\iint_S (u_{xx} - u_{yy})^2 = \frac{3h^2\pi^4}{2} = \frac{3\pi^4}{8N^2}, \tag{4.16}$$

which gives

$$\bar{\epsilon}_2 = \frac{E_2}{\lambda_2} = \frac{3\pi^4}{8N^2(5\pi^2)} = \frac{3\pi^2}{40N^2}. \tag{4.17}$$

Based on (4.17), for $N = 2, 4, 8, 16, 32$, we obtain respectively

$$\bar{\epsilon}_2 = 0.185,\ 0.463(-1),\ 0.116(-1),\ 0.289(-2), 0.723(-3). \tag{4.18}$$

Eq. (4.18) also coincides with the numerical data in Table 5 for Q_1^{rot}, which verifies the principal term in (4.16).

4.2. **Function $\rho \neq 1$.** Since the error analysis is valid for the function $\rho = \rho(x, y) \geq \rho_0 > 0$, to verify the analysis made, we also carry out the numerical experiments for $\rho \neq 1$. Choose

$$\rho = \rho(x, y) = 1 + (x - \frac{1}{2})(y - \frac{1}{2}), \tag{4.19}$$

which is symmetric with respect to x and y. We have

$$\begin{aligned} &-\triangle u = -\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = \lambda\rho u \text{ in } S,\\ &u = 0 \text{ on } \Gamma = \partial S, \end{aligned}$$

where S is also the unit square. For the ρ in (4.19), we may evaluate $\iint_S \rho uv$ in (1.5) exactly. The FEM as (1.3) can be easily performed. We provide the results for λ_1 by Q_1^{rot} and EQ_1^{rot} only, and list them in Tables 10 and 11. Since for ρ in (4.19), the true solution of λ_1 is unknown, we may compute the ratios of sequential errors to display the empirical convergence rates.[6] The numerical solutions, the sequential errors and their ratios are listed in Tables 10 and 11 for Q_1^{rot} and EQ_1^{rot}. Since only the sign of $\varepsilon^{(0)}$ is significant, it is listed in Tables 10 and 11. From Table 11, we can see the sequential errors

[6] An a posteriori error may be evaluated as follows. Since Q_1^{rot} may provide the most accurate solution, we may choose $\lambda_{1,h}^{(4)} = 19.7322552487$ in Table 10 as the true solution. Then the errors such as those in Tables 10 and 11 can also be computed.

$$\frac{\lambda_{1,2h} - \lambda_{1,4h}}{\lambda_{1,h} - \lambda_{1,2h}} = O(h^2), \tag{4.20}$$

for EQ_1^{rot} elements. However, from Table 10,

$$\frac{\lambda_{1,2h} - \lambda_{1,4h}}{\lambda_{1,h} - \lambda_{1,2h}} = O(h^4), \tag{4.21}$$

for the Q_1^{rot} element. The empirical convergence rates of λ_1 are exactly the same as those in Section 4.1 for $\rho = 1$.

4.3. **Numerical conclusions.** Based on the numerical results, we may draw a few important conclusions:

(1) The Q_1 and the EQ_1^{rot} provide the upper and the lower bounds respectively. The Q_1^{rot} provides the lower bound for λ_2 and other λ whose corresponding function u satisfies $u_{xx} \neq u_{yy}$.[7]

(2) For the minimal eigenvalue $\lambda_{\min} = \lambda_1$, the corresponding eigenfunctions satisfy $u_{xx} = u_{yy}$, and the Q_1^{rot} element yields the high $O(h^4)$ convergence rates. Such an ultraconvergence of Q_1^{rot} holds for any eigenvalues whose eigenfunctions are symmetric with respect to x and y.

(3) We list in Table 6 the computed results, to show the validation of (1.16).

(4) By the first level of extrapolation, the superconvergence $O(h^4)$ can be obtained by all four FEMs.

(5) For Q_1^{rot}, the ultraconvergence for λ_1 as

$$\lambda_{1,h}^{(i)} - \lambda_1 = O(h^{2i+4}),\ i = 0, 1, 2, 3,$$

can be achieved numerically by multiple levels of extrapolation; see Table 8.

(6) The principal terms of the eigenvalue errors for Q_1^{rot} and EQ_1^{rot} have been verified by our numerical experiments.

Concluding remarks. The new expansions of numerical eigenvalues by four FEMs are summarized in (1.15), whose proof for the two nonconforming elements Q_1^{rot} and EQ_1^{rot} is provided in this paper. Not only can (1.15) display an upper or a lower bound of the FEM solution of leading eigenvalues, but it can also lead to higher superconvergence rates by the extrapolation techniques. All the theoretical analyses have been verified by the numerical experiments in Section 4. Moreover, the best convergence rates have been obtained numerically by multiple levels of extrapolation for both Q_1^{rot} and EQ_1^{rot} elements.

[7]Numerically, the Q_1^{rot} also provides the lower bound of λ_1, based on Table 4 for $\rho = 1$, and on Table 10 for $\rho \neq 1$.

TABLE 4. The first eigenvalue solutions $\lambda_{1,h}$ for $-\Delta u = \lambda u$ by the four FEMs, where the true $\lambda_1 = 2\pi^2 \doteq 19.73920880217872$, $\varepsilon_h = \frac{\lambda_{1,h}-\lambda_1}{\lambda_1}$, Ratio $= |\frac{\varepsilon_{2h}}{\varepsilon_h}|$ and $h = \frac{1}{2N}$.

N	2	4	8	16	32
$\lambda_{1,h}$ by Q_1	24.000000	20.773284	19.994161	19.802707	19.755068
$\lambda_{1,h}$ by Q_1^{rot}	19.200000	19.707183	19.737241	19.739086	19.739201
$\lambda_{1,h}$ by EQ_1^{rot}	16.848019	18.818638	19.491886	19.676196	19.723380
$\lambda_{1,h}$ by Wilson's	13.321013	17.296011	19.023223	19.551919	19.691833
ε_h by Q_1	0.216	0.524(-1)	0.129(-1)	0.322(-2)	0.803(-3)
ε_h by Q_1^{rot}	-0.273(-1)	-0.162(-2)	-0.997(-4)	-0.620(-5)	-0.387(-6)
ε_h by EQ_1^{rot}	-0.146	-0.466(-1)	-0.125(-1)	-0.319(-2)	-0.802(-3)
ε_h by Wilson's	-0.325	-0.124	-0.363(-1)	-0.949(-2)	-0.240(-2)
Ratio by Q_1	/	4.12	4.06	4.02	4.00
Ratio by Q_1^{rot}	/	16.8	16.3	16.1	16.0
Ratio by EQ_1^{rot}	/	3.14	3.72	3.92	3.98
Ratio by Wilson's	/	2.63	3.41	3.82	3.95

TABLE 5. The second eigenvalue solutions $\lambda_{2,h}$ for $-\Delta u = \lambda u$ by the four FEMs, where the true $\lambda_2 = 5\pi^2 \doteq 49.3480220054$, $\varepsilon_h = \frac{\lambda_{2,h}-\lambda_2}{\lambda_2}$, Ratio $= |\frac{\varepsilon_{2h}}{\varepsilon_h}|$ and $h = \frac{1}{2N}$.

N	2	4	8	16	32
$\lambda_{2,h}$ by Q_1	/	58.3866	51.5436	49.8897	49.4829
$\lambda_{2,h}$ by Q_1^{rot}	58.5366	51.3290	49.9022	49.4897	49.3836
$\lambda_{2,h}$ by EQ_1^{rot}	41.6696	46.3304	48.4088	49.0993	
$\lambda_{2,h}$ by Wilson's	21.8182	38.6181	45.7010	48.3394	
ε_h by Q_1	/	0.183	0.445(-1)	0.110(-1)	0.273(-2)
ε_h by Q_1^{rot}	0.186	0.401(-1)	0.112(-1)	0.287(-2)	0.722(-3)
ε_h by EQ_1^{rot}	-0.156	-0.611(-1)	-0.190(-1)	-0.504(-2)	
ε_h by Wilson's	-0.558	-0.217	-0.739(-1)	-0.204(-1)	
Ratio by Q_1	/	/	4.12	4.05	4.01
Ratio by Q_1^{rot}	/	4.64	3.57	3.91	3.98
Ratio by EQ_1^{rot}	/	2.55	3.21	3.78	
Ratio by Wilson's	/	2.57	2.94	3.62	

TABLE 6. The errors $\lambda_{1,h} - \lambda_1$ for Q_1, Q_1^{rot} and EQ_1^{rot}, where $E = \lambda_{1,h} - \lambda_1$, $\widehat{E}_h = E_h|_{Q_1^{rot}} - \frac{1}{2}(E_h|_{Q_1} + E_h|_{EQ_1^{rot}})$, Ratio $= \widehat{E}_{2h}/\widehat{E}_h$ and $h = \frac{1}{2N}$.

N	2	4	8	16	32
$\lambda_{1,h}$ by Q_1	24.0000000000	20.7732840104	19.9941613125	19.8027073568	19.7550682351
$\lambda_{1,h}$ by Q_1^{rot}	19.2000000000	19.7071826998	19.7372410126	19.7390863687	19.7392011588
$\lambda_{1,h}$ by EQ_1^{rot}	16.8480192154	18.8186378768	19.4918862529	19.6761961558	19.7233798827
E_h by Q_1	4.26079	1.03408	0.254953	0.634986(-1)	0.158594(-1)
E_h by Q_1^{rot}	-0.539209	-0.320261(-1)	-0.196779(-2)	-0.122433(-3)	-0.764336(-5)
E_h by EQ_1^{rot}	-2.89119	-0.920571	-0.247323	-0.630126(-1)	-0.158289(-1)
$\widehat{E}_h$	-1.22401	-0.887782(-1)	-0.578277(-2)	-0.365388(-3)	-0.229001(-4)
Ratio	/	13.79	15.35	15.83	15.96

TABLE 7. The first $\lambda_{1,h}$ by extrapolation from the Q_1^{rot} solutions, where the true $\lambda_1 = 2\pi^2 \doteq 19.73920880217872$, $\lambda_{1,h}^{(k)} = \frac{2^{2k+2}\lambda_{1,h}^{(k-1)} - \lambda_{1,2h}^{(k-1)}}{2^{2k+2}-1}$, $\varepsilon_h^{(k)} = \frac{\lambda_{1,h}^{(k)} - \lambda_1}{\lambda_1}$, Ratio$(k) = |\varepsilon_{2h}^{(k)}/\varepsilon_h^{(k)}|$, $\lambda_{1,h}^{(0)} = \lambda_{1,h}$ and $h = \frac{1}{2N}$.

N	2	4	8	16	32
$\lambda_{1,h}$	19.20000000000000	19.70718269984467	19.73724101257975	19.73908636870386	19.73920115882359
$\lambda_{1,h}^{(1)}$	/	19.74099487983432	19.73924490009542	19.73920939244547	19.73920881149824
$\lambda_{1,h}^{(2)}$	/	/	19.73921712263925	19.73920882883198	19.73920880227686
$\lambda_{1,h}^{(3)}$	/	/	/	19.73920879630725	19.73920880217272
$\lambda_{1,h}^{(4)}$	/	/	/	/	19.73920880217845
$\varepsilon_h^{(0)}$	-0.273(-1)	-0.162(-2)	-0.997(-4)	-0.620(-5)	-0.387(-6)
$\varepsilon_h^{(1)}$	/	0.905(-4)	0.183(-5)	0.299(-7)	0.472(-9)
$\varepsilon_h^{(2)}$	/	/	0.422(-6)	0.135(-8)	0.497(-11)
$\varepsilon_h^{(3)}$	/	/	/	-0.297(-9)	-0.304(-12)
$\varepsilon_h^{(4)}$	/	/	/	/	-0.135(-13)
Ratio(0)	/	16.8	16.3	16.1	16.0
Ratio(1)	/	/	49.5	61.2	63.3
Ratio(2)	/	/	/	312	272
Ratio(3)	/	/	/	/	979

TABLE 8. The first $\lambda_{1,h}$ by extrapolation from the EQ_1^{rot} solutions, where the true $\lambda_1 = 2\pi^2 \doteq 19.73920880217872$, where $\lambda_{1,h}^{(k)} = \frac{2^{2k}\lambda_{1,h}^{(k-1)} - \lambda_{1,2h}^{(k-1)}}{2^{2k}-1}$, $\varepsilon_h^{(k)} = \frac{\lambda_{1,h}^{(k)} - \lambda_1}{\lambda_1}$, Ratio$(k) = |\varepsilon_{2h}^{(k)}/\varepsilon_h^{(k)}|$, $\lambda_{1,h}^{(0)} = \lambda_{1,h}$ and $h = \frac{1}{2N}$.

N	2	4	8	16	32
$\lambda_{1,h}$	16.8480192154	18.8186378768	19.4918862529	19.6761961558	19.7233798827
$\lambda_{1,h}^{(1)}$	/	19.4755107640	19.7163023782	19.7376327901	19.7391077917
$\lambda_{1,h}^{(2)}$	/	/	19.7323551525	19.7390548176	19.7392061251
$\lambda_{1,h}^{(3)}$	/	/	/	19.7391611615	19.7392085268
$\lambda_{1,h}^{(4)}$	/	/	/	/	19.7392087126
$\varepsilon_h^{(0)}$	-0.146	-0.466(-1)	-0.125(-1)	-0.319(-2)	-0.802(-3)
$\varepsilon_h^{(1)}$	/	-0.134(-1)	-0.116(-2)	-0.798(-4)	-0.512(-5)
$\varepsilon_h^{(2)}$	/	/	-0.347(-3)	-0.780(-5)	-0.136(-6)
$\varepsilon_h^{(3)}$	/	/	/	-0.241(-5)	-0.140(-7)
$\varepsilon_h^{(4)}$	/	/	/	/	-0.454(-8)
Ratio(0)	/	3.14	3.72	3.92	3.98
Ratio(1)	/	/	11.5	14.5	15.6
Ratio(2)	/	/	/	44.5	57.5
Ratio(3)	/	/	/	/	173

TABLE 9. The second $\lambda_{2,h}$ by extrapolation from the Q_1^{rot} solutions, where the true $\lambda_2 = 5\pi^2 \doteq 49.34802200544679$, $\lambda_{2,h}^{(k)} = \frac{2^{2k}\lambda_{2,h}^{(k-1)} - \lambda_{2,2h}^{(k-1)}}{2^{2k}-1}$, $\varepsilon_h^{(k)} = \frac{\lambda_{2,h}^{(k)} - \lambda_2}{\lambda_2}$, $\text{Ratio}(k) = |\varepsilon_{2h}^{(k)}/\varepsilon_h^{(k)}|$, $\lambda_{2,h}^{(0)} = \lambda_{2,h}$ and $h = \frac{1}{2N}$.

N	2	4	8	16	32
$\lambda_{2,h}$	58.5365853659	51.3289965050	49.9021574073	49.4897065300	49.3836320249
$\lambda_{2,h}^{(1)}$	/	48.9264668847	49.4265443748	49.3522229042	49.3482738566
$\lambda_{2,h}^{(2)}$	/	/	49.4598828741	49.3472681395	49.3480105867
$\lambda_{2,h}^{(3)}$	/	/	/	49.3454806040	49.3480223716
$\lambda_{2,h}^{(4)}$	/	/	/	/	49.3480323393
$\varepsilon_h^{(0)}$	0.186	0.401(-1)	0.112(-1)	0.287(-2)	0.722(-3)
$\varepsilon_h^{(1)}$	/	-0.854(-2)	0.159(-2)	0.851(-4)	0.510(-5)
$\varepsilon_h^{(2)}$	/	/	0.227(-2)	-0.153(-4)	-0.231(-6)
$\varepsilon_h^{(3)}$	/	/	/	-0.515(-4)	0.742(-8)
$\varepsilon_h^{(4)}$	/	/	/	/	0.209(-6)
Ratio(0)	/	4.64	3.57	3.91	3.98
Ratio(1)	/	/	5.37	18.7	16.7
Ratio(2)	/	/	/	148	66.0
Ratio(3)	/	/	/	/	6940

TABLE 10. The first $\lambda_{1,h}$ for $-\Delta u = \lambda\rho u$ by Q_1^{rot} , where $\lambda_{1,h}^{(k)} = \frac{2^{2k+2}\lambda_{1,h}^{(k-1)} - \lambda_{1,2h}^{(k-1)}}{2^{2k+2}-1}$, $\varepsilon_h^{(k)} = \lambda_{1,h}^{(k)} - \lambda_{1,2h}^{(k)}$, $\text{Ratio}(k) = |\varepsilon_{2h}^{(k)}/\varepsilon_h^{(k)}|$, $\lambda_{1,h}^{(0)} = \lambda_{1,h}$ and $h = \frac{1}{2N}$.

N	2	4	8	16	32
$\lambda_{1,h}$	19.2000000000	19.7011813280	19.7303925515	19.7321409854	19.7322481452
$\lambda_{1,h}^{(1)}$	/	19.7345933897	19.7323399681	19.7322575477	19.7322552892
$\lambda_{1,h}^{(2)}$	/	/	19.7323041995	19.7322562394	19.7322552534
$\lambda_{1,h}^{(3)}$	/	/	/	19.7322560513	19.7322552495
$\lambda_{1,h}^{(4)}$	/	/	/	/	19.7322552487
$\varepsilon_h^{(0)}$	/	-0.501	-0.292(-1)	-0.175(-2)	-0.107(-3)
$\lvert\varepsilon_h^{(1)}\rvert$	/	/	0.225(-2)	0.824(-4)	0.226(-5)
$\lvert\varepsilon_h^{(2)}\rvert$	/	/	/	0.480(-4)	0.986(-6)
$\lvert\varepsilon_h^{(3)}\rvert$	/	/	/	/	0.802(-6)
Ratio(0)	/	/	17.16	16.71	16.32
Ratio(1)	/	/	/	27.34	36.49
Ratio(2)	/	/	/	/	48.64

TABLE 11. The first $\lambda_{1,h}$ for $-\Delta u = \lambda\rho u$ by EQ_1^{rot} , where $\lambda_{1,h}^{(k)} = \frac{2^{2k}\lambda_{1,h}^{(k-1)}-\lambda_{1,2h}^{(k-1)}}{2^{2k}-1}$, $\varepsilon_h^{(k)} = \lambda_{1,h}^{(k)} - \lambda_{1,2h}^{(k)}$, Ratio$(k) = |\varepsilon_{2h}^{(k)}/\varepsilon_h^{(k)}|$, $\lambda_{1,h}^{(0)} = \lambda_{1,h}$ and $h = \frac{1}{2N}$.

N	2	4	8	16	32
$\lambda_{1,h}$	17.7264222820	19.1414166129	19.5800759724	19.6984491727	19.7286510415
$\lambda_{1,h}^{(1)}$	/	19.6130813898	19.7262957589	19.7379069061	19.7387183311
$\lambda_{1,h}^{(2)}$	/	/	19.7338433835	19.7386809826	19.7387724261
$\lambda_{1,h}^{(3)}$	/	/	/	19.7387577699	19.7387738776
$\lambda_{1,h}^{(4)}$	/	/	/	/	19.7387739407
$\varepsilon_h^{(0)}$	/	-1.415	-0.439	-0.118	-0.302(-1)
$\|\varepsilon_h^{(1)}\|$	/	/	0.113	0.116(-1)	0.811(-3)
$\|\varepsilon_h^{(2)}\|$	/	/	/	0.484(-2)	0.914(-4)
$\|\varepsilon_h^{(3)}\|$	/	/	/	/	0.161(-4)
Ratio(0)	/	/	3.23	3.71	3.92
Ratio(1)	/	/	/	9.75	14.31
Ratio(2)	/	/	/	/	52.90

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Institute of Computational Mathematics and Scientific/Engineering Computing, Academy of Mathematics and System Sciences, Chinese Academy of Sciences, P.O. Box 2719, Beijing 1000080, China
E-mail address: qlin@lsec.cc.ac.cn

Department of Applied Mathematics, I-Shou University, Taiwan 840
E-mail address: huanght@isu.edu.tw

(Corresponding author) Department of Applied Mathematics, and Department of Computer Science and Engineering, National Sun Yat-sen University, Kaohsiung, Taiwan 80424
E-mail address: zcli@math.nsysu.edu.tw