MATHEMATICS OF COMPUTATION
Volume 77, Number 264, October 2008, Pages 1967–1999
S 0025-5718(08)02108-X
Article electronically published on April 29, 2008

POLYNOMIAL EXTENSION OPERATORS FOR H^1, $\boldsymbol{H}(\mathbf{curl})$ AND $\boldsymbol{H}(\mathrm{div})$-SPACES ON A CUBE

M. COSTABEL, M. DAUGE, AND L. DEMKOWICZ

ABSTRACT. This paper is devoted to the construction of continuous trace lifting operators compatible with the de Rham complex on the reference hexahedral element (the unit cube). We consider three trace operators: The standard one from H^1, the tangential trace from $\boldsymbol{H}(\mathbf{curl})$ and the normal trace from $\boldsymbol{H}(\mathrm{div})$. For each of them we construct a continuous right inverse by separation of variables. More importantly, we consider the same trace operators acting from the polynomial spaces forming the exact sequence corresponding to the Nédélec hexahedron of the first type of degree p. The core of the paper is the construction of polynomial trace liftings with operator norms bounded independently of the polynomial degree p. This construction relies on a spectral decomposition of the trace data using discrete Dirichlet and Neumann eigenvectors on the unit interval, in combination with a result on interpolation between Sobolev norms in spaces of polynomials.

1. INTRODUCTION

Many finite element discretizations of Maxwell's equations in three space dimensions rely on the reproduction at the discrete level of the exact sequence

$$H^1(\Omega) \xrightarrow{\boldsymbol{\nabla}} \boldsymbol{H}(\mathbf{curl},\Omega) \xrightarrow{\mathbf{curl}} \boldsymbol{H}(\mathrm{div},\Omega) \xrightarrow{\mathrm{div}} L^2(\Omega). \tag{1.1}$$

General constructions of finite elements related to (1.1) are analyzed in [20], and a survey on recent discoveries of deeper connections between finite element analysis and exact sequences can be found in [2].

In this paper, we address a central issue in connection with the discretization of (1.1) on a hexahedral mesh by means of the polynomial spaces corresponding to Nédélec's hexahedron of the first type [24]. The discrete spaces corresponding to (1.1) are built from reference spaces defined on the master hexahedron $\Omega = I\times I\times I$ with $I = (-1, 1)$, forming the exact sequence,

$$W_p(\Omega) \xrightarrow{\boldsymbol{\nabla}} \boldsymbol{Q}_p(\Omega) \xrightarrow{\mathbf{curl}} \boldsymbol{V}_p(\Omega) \xrightarrow{\mathrm{div}} Y_p(\Omega). \tag{1.2}$$

Here p is any positive integer, $W_p(\Omega) = \mathbb{P}^p(I)\otimes\mathbb{P}^p(I)\otimes\mathbb{P}^p(I)$, $Y_p(\Omega) = W_{p-1}(\Omega)$ and $\boldsymbol{Q}_p(\Omega)$, $\boldsymbol{V}_p(\Omega)$ are suitably defined; see the next section. Combined with suitable inter-element compatibility, the exact polynomial sequence (1.2) mainly serves for

Received by the editor March 21, 2007 and, in revised form, September 6, 2007.
2000 *Mathematics Subject Classification.* Primary 65N35, 65N30; Secondary 78M10, 35J05.
Key words and phrases. Trace lifting, polynomial extension, de Rham complex, separation of variables.
The work of the third author was supported in part by the Air Force under Contract F49620-98-1-0255.

p and hp versions of finite elements. In order to ensure or control the inter-element compatibility, the trace operators and their right inverses play an important role.

The present work is concerned with the construction of trace liftings at the continuous and polynomial levels, satisfying uniform bounds independently of the degree p. At the continuous level, the trace operators naturally associated with $H^1(\Omega)$, $\boldsymbol{H}(\mathbf{curl}, \Omega)$ and $\boldsymbol{H}(\mathrm{div}, \Omega)$ are the standard trace γ_0, the tangential trace $\gamma_{\mathbf{t}}$ and the normal trace $\gamma_{\mathbf{n}}$, respectively. The associated trace spaces are $H^{\frac{1}{2}}(\partial\Omega)$, $\boldsymbol{H}^{-\frac{1}{2}}(\mathrm{curl}, \partial\Omega)$ and $H^{-\frac{1}{2}}(\partial\Omega)$. It is easy to characterize these trace spaces on a cube; see Section 2. For trace spaces defined on general polyhedra, we refer to the work of Buffa and Ciarlet [9], and on general Lipschitz domains to the paper of Buffa, Costabel and Sheen [10].

At the polynomial level, the question is to find extension operators defined on polynomial subspaces of these three trace spaces, which take their values in $W_p(\Omega)$, $\boldsymbol{Q}_p(\Omega)$ and $\boldsymbol{V}_p(\Omega)$, respectively. The construction of such operators becomes a non-trivial task if we request the corresponding norms to be bounded *independently of the polynomial degree* p. A more demanding task would be to construct extension operators on the continuous spaces that are *polynomial preserving*; see the work of Schoeberl, Gopalakrishnan and Demkowicz [26].

Existence of polynomial extension operators with p-independent bounds for the norms is a crucial step in proving convergence for the p and hp finite element methods. First, H^1-extension operators were constructed by Babuška and Suri in two space dimensions in [4], for both triangular and square elements. Their work was further expanded in [3] and applied to the construction of preconditioners for the p-method. The 2D constructions remain an active area of research; see the recent results of Ainsworth and Demkowicz [1], and Heuer and Leydecker [18]. Construction of an $\boldsymbol{H}(\mathbf{curl})$ extension operator in 2D follows immediately from the corresponding H^1 operator. The operator was utilized in deriving p-estimates for the $\boldsymbol{H}(\mathbf{curl})$-conforming problems by Demkowicz and Babuška in [14].

In three space dimensions, there are fewer publications available. For H^1-extension operators on the cube, Ben Belgacem [6] has some results, and the recent work by Bernardi, Dauge, and Maday [7] contains some constructions and many estimates in various Sobolev norms. Muñoz-Sola [23] deals with the construction of an H^1-operator for a tetrahedron. The incoming contribution of Schoeberl, Gopalakrishan and Demkowicz [26] presents an alternative construction for the tetrahedron and all spaces forming the exact sequence.

Our construction of H^1-, $\boldsymbol{H}(\mathbf{curl})$- and $\boldsymbol{H}(\mathrm{div})$-conforming, polynomial extension operators for a cube mimics closely the corresponding definitions on the continuous level obtained by using the method of separation of variables, and has been stimulated by the work of Pavarino and Widlund [25]. Earlier work on this kind of H^1-extension operators was done by Canuto and Funaro [11] and Bernardi and Maday [8]. It has been observed early on that the uniform p-stability of extension operators is equivalent to certain estimates for $H^{1/2}$ norms in terms of Hilbert space interpolation of polynomial spaces [21, 6]. A complete proof of such norm estimates is now available [7].

By inspecting our constructions, it turns out that all the lifting operators can be characterized by orthogonality properties or variational principles. One could have used these variational principles to *define* the liftings. This simple procedure would, however, not have allowed us to obtain the desired norm bounds independent of the polynomial degree p which we obtain from the explicit constructions.

Organization of the paper. In Section 2 we give details of the function spaces which we use, at the continuous as well as at the polynomial level. We define the trace operators. We also recall the Poincaré map. Section 3 deals with questions of continuous and polynomial extension in two dimensions. This serves as a preparatory step for the 3D case. In Section 4, we address the lifting of traces in the continuous spaces in three dimensions. In Sections 5 and 6 we construct polynomial trace liftings into $H^1(\Omega)$ and $\boldsymbol{H}(\mathrm{div},\Omega)$, with the help of explicit formulae based on the separation of variables and expansions in bases of discrete 1D Laplace eigenvectors. The Poincaré map then allows us to derive from the previous two liftings the construction of a lifting for $\boldsymbol{H}(\mathbf{curl},\Omega)$ in Section 7. We emphasize that the direct construction of such a lifting without the help of the Poincaré map would have been a very difficult task. We conclude our paper in Section 8.

2. Sobolev and polynomial spaces

2.1. **Sobolev spaces.** We use Hörmander's definitions for all considered Sobolev spaces; see e.g. [22]. The closure of test functions $C_0^\infty(\Omega)$ in $H^s(\mathbb{R}^n)$ is denoted by $\widetilde{H}^s(\Omega)$, and for $s \geq -\frac{1}{2}$ can be identified with distributions from $H^s(\Omega)$; see [22]. For Lipschitz domains, $\widetilde{H}^s(\Omega)$ and $H^{-s}(\Omega)$ are dual to each other, for any $s \in \mathbb{R}$.

Moreover each scale $\big(\widetilde{H}^s(\Omega)\big)_{s\in\mathbb{R}}$ and $\big(H^s(\Omega)\big)_{s\in\mathbb{R}}$ is an interpolation scale.

For any $s \geq 0$, the constant functions belong to $H^s(\Omega)$ and to $\widetilde{H}^{-s}(\Omega)$. Thus it makes sense to denote by $H^s_{avg}(\Omega)$ and $\widetilde{H}^{-s}_{avg}(\Omega)$ the functions and distributions with zero average.

We will also use the tensor product form of these spaces, in particular on the square $I \times I = (-1,1)^2$. The space $L^2(I, H^s(I))$ denotes the space of all L^2-integrable functions on I with values in the space $H^s(I)$. As a function space on $I \times I$, this space is isomorphic, via an exchange of independent variables, to the space $H^s(I, L^2(I))$ of $L^2(I)$-valued H^s functions on I. We will sometimes indicate by indices the coordinates involved, so that we can write $L^2\big(I_y, H^s(I_x)\big) = H^s\big(I_x, L^2(I_y)\big)$. Analogous definitions are used with $\widetilde{H}^s$ replacing H^s. For any $s \geq 0$, it follows that

$$H^s(I\times I) = L^2\big(I, H^s(I)\big) \cap H^s\big(I, L^2(I)\big), \tag{2.1}$$

$$H^{-s}(I\times I) = L^2\big(I, H^{-s}(I)\big) + H^{-s}\big(I, L^2(I)\big) \tag{2.2}$$

and analogously for $\widetilde{H}^{\pm s}$ spaces.

2.2. **Trace operators.** We denote by γ_0 the standard trace operator

$$\gamma_0 : H^1(\Omega) \longrightarrow H^{\frac{1}{2}}(\partial\Omega), \qquad U \longmapsto u = U|_{\partial\Omega}. \tag{2.3}$$

Here Ω is a Lipschitz 2D or 3D domain.

In 2D, the space $\boldsymbol{H}(\mathrm{curl},\Omega)$ is defined as $\{\boldsymbol{E} \in L^2(\Omega)^2 :\ \mathrm{curl}\,\boldsymbol{E} \in L^2(\Omega)\}$, and, in 3D, as

$$\boldsymbol{H}(\mathbf{curl},\Omega) = \{\boldsymbol{E} \in L^2(\Omega)^3 :\ \mathbf{curl}\,\boldsymbol{E} \in L^2(\Omega)^3\}.$$

The tangential trace $\gamma_{\mathbf{t}}$ is well defined on $\boldsymbol{H}(\mathrm{curl},\Omega)$ in 2D and $\boldsymbol{H}(\mathbf{curl},\Omega)$ in 3D. In 2D, $\gamma_{\mathbf{t}}$ acts continuously from $\boldsymbol{H}(\mathrm{curl},\Omega)$ into $H^{-\frac{1}{2}}(\partial\Omega)$, and in 3D:

$$\gamma_{\mathbf{t}} : \boldsymbol{H}(\mathbf{curl},\Omega) \longrightarrow \boldsymbol{H}^{-\frac{1}{2}}(\mathrm{curl},\partial\Omega), \qquad \boldsymbol{E} \longmapsto \boldsymbol{e}_{\mathbf{t}} = \big(\boldsymbol{E} - (\boldsymbol{E}\cdot\mathbf{n})\mathbf{n}\big)\big|_{\partial\Omega}, \tag{2.4}$$

where $\mathbf{n}$ denotes the outward normal unit vector on the boundary $\partial\Omega$. The trace space $\boldsymbol{H}^{-\frac{1}{2}}(\mathrm{curl},\partial\Omega)$ is the space of tangential fields $\boldsymbol{e} \in \boldsymbol{H}^{-\frac{1}{2}}(\partial\Omega)$ such that their surface curls $\mathrm{curl}\,\boldsymbol{e}$ belong to $H^{-\frac{1}{2}}(\partial\Omega)$.

The space $\boldsymbol{H}(\mathrm{div},\Omega)$ is the space of vector fields $\boldsymbol{H}$ with components in $L^2(\Omega)$ such that $\mathrm{div}\,\boldsymbol{H} \in L^2(\Omega)$. The normal trace $\gamma_{\mathbf{n}}$ is well defined on $\boldsymbol{H}(\mathrm{div},\Omega)$. In 2D and 3D:

$$\gamma_{\mathbf{n}} : \boldsymbol{H}(\mathrm{div},\Omega) \longrightarrow H^{-\frac{1}{2}}(\partial\Omega), \qquad \boldsymbol{H} \longmapsto h = (\boldsymbol{H}\cdot\mathbf{n})|_{\partial\Omega}. \tag{2.5}$$

All three trace operators γ_0, $\gamma_{\mathbf{t}}$ and $\gamma_{\mathbf{n}}$ are surjective [9], and therefore there exist continuous liftings between the spaces in (2.3), (2.4) and (2.5), respectively.

By $H^1_0(\Omega)$, $\boldsymbol{H}_0(\mathrm{curl},\Omega)$, $\boldsymbol{H}_0(\mathbf{curl},\Omega)$, $\boldsymbol{H}_0(\mathrm{div},\Omega)$ we denote the null-spaces of the corresponding trace operators.

2.3. **Tensor product polynomial spaces.** Let $I = (-1, 1)$ be the reference interval. For any integer $p \in \mathbb{N}$ we denote by $\mathbb{P}^p(I)$ the space of polynomials of degree $\le p$ on I. For $p \ge 2$, let $\mathbb{P}^p_0(I)$ be the subspace consisting of those $u \in \mathbb{P}^p(I)$ that are zero at ± 1.

Let p, q, $r \in \mathbb{N}$. We introduce the tensor product spaces

$$\mathbb{P}^{(p,q)}(I^2) = \mathbb{P}^p(I) \otimes \mathbb{P}^q(I) \quad \text{and} \quad \mathbb{P}^{(p,q,r)}(I^3) = \mathbb{P}^p(I) \otimes \mathbb{P}^q(I) \otimes \mathbb{P}^r(I).$$

Let Ω be the reference square $(-1, 1)^2$ in 2D or the reference cube $(-1, 1)^3$ in 3D. The spaces associated with $H^1(\Omega)$ are

$$W_p(I^2) = \mathbb{P}^{(p,p)}(I^2) \quad \text{and} \quad W_p(I^3) = \mathbb{P}^{(p,p,p)}(I^3). \tag{2.6}$$

The spaces associated with $\boldsymbol{H}(\mathrm{curl},\Omega)$ and $\boldsymbol{H}(\mathbf{curl},\Omega)$ are

$$\begin{aligned} \boldsymbol{Q}_p(I^2) &= \mathbb{P}^{(p-1,p)}(I^2) \times \mathbb{P}^{(p,p-1)}(I^2), \\ \boldsymbol{Q}_p(I^3) &= \mathbb{P}^{(p-1,p,p)}(I^3) \times \mathbb{P}^{(p,p-1,p)}(I^3) \times \mathbb{P}^{(p,p,p-1)}(I^3). \end{aligned} \tag{2.7}$$

In 3D, the spaces associated with $\boldsymbol{H}(\mathrm{div},\Omega)$ are

$$\boldsymbol{V}_p(I^3) = \mathbb{P}^{(p,p-1,p-1)}(I^3) \times \mathbb{P}^{(p-1,p,p-1)}(I^3) \times \mathbb{P}^{(p-1,p-1,p)}(I^3). \tag{2.8}$$

Finally the spaces associated with $L^2(\Omega)$ are $Y_p(\Omega) = W_{p-1}(\Omega)$.

Using these spaces, we have for all $p \ge 1$ the following exact sequences:

$$W_p(\Omega) \xrightarrow{\boldsymbol{\nabla}} \boldsymbol{Q}_p(\Omega) \xrightarrow{\mathrm{curl}} Y_p(\Omega) \qquad \text{in 2D}, \tag{2.9}$$

$$W_p(\Omega) \xrightarrow{\boldsymbol{\nabla}} \boldsymbol{Q}_p(\Omega) \xrightarrow{\mathbf{curl}} \boldsymbol{V}_p(\Omega) \xrightarrow{\mathrm{div}} Y_p(\Omega) \qquad \text{in 3D}. \tag{2.10}$$

2.4. **Polynomial traces.** Let $p \in \mathbb{N}$. In 2D, we denote by $\mathsf{e}_1, \ldots, \mathsf{e}_4$ the edges of the square $\Omega = I^2$; see Fig. 1. We introduce the following two trace spaces:

$$\begin{aligned} W_p(\partial\Omega) &= \{u \in H^{\frac{1}{2}}(\partial\Omega) : \; u|_{\mathsf{e}_i} \in \mathbb{P}^p(\mathsf{e}_i), \; i = 1, \ldots, 4\}, \\ Q_p(\partial\Omega) &= \{e \in H^{-\frac{1}{2}}(\partial\Omega) : \; e|_{\mathsf{e}_i} \in \mathbb{P}^{p-1}(\mathsf{e}_i), \; i = 1, \ldots, 4\}. \end{aligned} \tag{2.11}$$

Note that in $W_p(\partial\Omega)$, the edge traces $\underline{u}_i := u|_{\mathsf{e}_i}$ share common values at the corners, whereas in $Q_p(\partial\Omega)$, the edge traces $\underline{e}_i := e|_{\mathsf{e}_i}$ are independent of each other.

In 3D, we denote by $\mathsf{f}_1, \ldots, \mathsf{f}_6$ the faces of the cube $\Omega = I^3$; see Fig. 2. We introduce the following three trace spaces:

$$
\begin{aligned}
W_p(\partial\Omega) &= \{u \in H^{\frac{1}{2}}(\partial\Omega) : \ u|_{\mathsf{f}_i} \in \mathbb{P}^{(p,p)}(\mathsf{f}_i), \ i = 1, \ldots, 6\}, \\
\boldsymbol{Q}_p(\partial\Omega) &= \{\boldsymbol{e} \in \boldsymbol{H}^{-\frac{1}{2}}(\mathrm{curl}, \partial\Omega) : \ \boldsymbol{e}|_{\mathsf{f}_i} \in \boldsymbol{Q}_p(\mathsf{f}_i), \ i = 1, \ldots, 6\}, \\
V_p(\partial\Omega) &= \{h \in H^{-\frac{1}{2}}(\partial\Omega) : \ h|_{\mathsf{f}_i} \in \mathbb{P}^{(p-1,p-1)}(\mathsf{f}_i), \ i = 1, \ldots, 6\}.
\end{aligned}
\tag{2.12}
$$

Note that

- In $W_p(\partial\Omega)$, the face traces $\underline{u}_i := u|_{\mathsf{f}_i}$ share common values along the edges of the cube.
- In $\boldsymbol{Q}_p(\partial\Omega)$, the face traces $\underline{\boldsymbol{e}}_i := \boldsymbol{e}|_{\mathsf{f}_i}$ share common values for their tangential traces along the edges of the cube.
- In $V_p(\partial\Omega)$, the face traces $\underline{h}_i := h|_{\mathsf{f}_i}$ are independent of each other.

We end this subsection by the introduction of the polynomial subspaces with zero trace:

$$
\begin{aligned}
W_{p,0}(\Omega) &= \{U \in W_p(\Omega) : \ \gamma_0 U = 0\} = W_p(\Omega) \cap H_0^1(\Omega), \\
\boldsymbol{Q}_{p,0}(\Omega) &= \{\boldsymbol{E} \in \boldsymbol{Q}_p(\Omega) : \ \gamma_{\mathbf{t}} \boldsymbol{E} = 0\}.
\end{aligned}
\tag{2.13}
$$

2.5. **Poincaré map.** Recall the relevant Poincaré map in three space dimensions,

$$
\boldsymbol{K} : \boldsymbol{H}(\mathrm{div}, \Omega) \to \boldsymbol{H}(\mathbf{curl}, \Omega), \quad (\boldsymbol{K}\boldsymbol{H})(\boldsymbol{x}) = -\boldsymbol{x} \times \int_0^1 t\boldsymbol{H}(t\boldsymbol{x})\, dt.
\tag{2.14}
$$

For a general definition of the Poincaré map in terms of differential forms in any space dimension, we refer, e.g., to [19] or [2]. Direct, elementary computations (see [17]) show that:

- the map is a right-inverse of the curl operator,

$$
\mathrm{div}\, \boldsymbol{H} = 0 \quad \Longrightarrow \mathbf{curl}\, \boldsymbol{K}\boldsymbol{H} = \boldsymbol{H},
\tag{2.15}
$$

- the map is continuous from $\boldsymbol{H}(\mathrm{div}, \Omega)$ into $\boldsymbol{H}(\mathbf{curl}, \Omega)$,
- the map preserves polynomials, i.e. it maps $\boldsymbol{V}_p(\Omega)$ into $\boldsymbol{Q}_p(\Omega)$.

Among other results, the map has been used in [14, 15] to prove that the constant in the discrete Friedrichs' inequality is independent of the polynomial degree p.

3. Polynomial extension operators in 2D

In this section we work with the reference square $\Omega = I^2$.

Construction of the polynomial extension operators follows closely the corresponding construction of extension operators on the continuous level by using separation of variables, and we review the continuous case first.

3.1. Continuous extensions using separation of variables.

We construct extensions on the continuous level for $H^1(\Omega)$ and $\boldsymbol{H}(\mathrm{curl}, \Omega)$.

3.1.1. *H^1 extension operator.* Consider $u \in H^{\frac{1}{2}}(\partial\Omega)$. The most natural, finite-energy lift of u is obtained by considering the extension $U \in H^1(\Omega)$ with minimum H^1-seminorm. This means that U is the solution of the Dirichlet problem for the homogeneous Laplace equation with boundary data u.

We can alternatively construct U through four successive steps involving each time one edge only. We begin by considering restriction $\underline{u}_3 \in H^{\frac{1}{2}}(\mathsf{e}_3)$ of boundary data u to the third edge. The lift U_3 of $\underline{u}_3$ is constructed by solving the mixed boundary-value problem for the Laplacian,

$$
\left\{
\begin{array}{rcll}
U & \in & H^1(\Omega), & \\
-\Delta U & = & 0 & \text{in } \Omega, \\
U & = & \underline{u}_3 & \text{on edge } \mathsf{e}_3, \\
U & = & 0 & \text{on edge } \mathsf{e}_1, \\
\partial_{\mathsf{n}} U & = & 0 & \text{on edges } \mathsf{e}_2, \mathsf{e}_4.
\end{array}
\right.
\tag{3.1}
$$

Here ∂_{n} is the outer normal derivative. Problem (3.1) is well-posed. Its solution can be represented using the expansion of $\underline{u}_3$ in the basis of 1D Neumann eigenvectors: Let (Ψ_n, μ_n), $n = 0, 1, \dots$ denote the Neumann eigenpairs for the 1D Laplacian

$$
-\Psi_n'' = \mu_n \Psi_n, \quad \Psi_n'(-1) = 0,\ \Psi_n'(1) = 0. \tag{3.2}
$$

Eigenvectors are orthogonal in both L^2 and H^1-products, and we assume that Ψ_n has a unit L^2-norm. We write the expansion of $\underline{u}_3$ as

$$
\underline{u}_3 = \sum_{n=0}^{\infty} u_n \Psi_n(x) \quad \text{with} \quad u_n = \int_{\mathsf{e}_3} u(x) \Psi_n(x)\, dx. \tag{3.3}
$$

Then we find that $U = U_3$ is given by

$$
U_3(x, y) = \sum_{n=0}^{\infty} u_n \Psi_n(x) \beta_n^{\mu}(y), \tag{3.4}
$$

where the corresponding functions $\beta = \beta_n^{\mu}$ are found by solving the two-point Dirichlet boundary-value problem,

$$
-\beta'' + \mu_n \beta = 0, \quad \beta(-1) = 0,\ \beta(1) = 1. \tag{3.5}
$$

We can check that for $\underline{u}_3 \in H^{\frac{1}{2}}(\mathsf{e}_3)$, the solution lives in $H^1(\Omega)$.

Lift U_3 of boundary data $\underline{u}_3$ vanishes on the first edge, but it does not vanish on the vertical edges. Notice that the use of Neumann boundary conditions on vertical edges is essential. The solution of a problem with homogeneous Dirichlet conditions replacing the Neumann conditions involves, in general, discontinuous Dirichlet data and, therefore, may not live in $H^1(\Omega)$.

We proceed now in a fully analogous way with the first edge and construct lift U_1 of restriction $\underline{u}_1 = u|_{\mathsf{e}_1}$ of the boundary data to the first edge.

Next, we subtract from function u traces of lifts U_3 and U_1,

$$
v = u - U_1|_{\partial\Omega} - U_3|_{\partial\Omega}. \tag{3.6}
$$

Function $v \in H^{\frac{1}{2}}(\partial\Omega)$ depends continuously upon the original data u and vanishes on the first and on the third edges. Consequently, its restrictions $\underline{v}_2$, $\underline{v}_4$ to the second and the fourth edge live in $\widetilde{H}^{\frac{1}{2}}(I)$ with the norm bounded by the $H^{\frac{1}{2}}$-norm of the original data u.

The fact that the boundary data vanishes at endpoints in a weak sense allows now for considering problems with pure Dirichlet boundary conditions. Lift $U = U_2$

of $\underline{v}_2$ is determined by solving the boundary-value problem

$$\left\{\begin{array}{rcll} U & \in & H^1(\Omega), & \\ -\Delta U & = & 0 & \text{in } \Omega, \\ U & = & \underline{v}_2 & \text{on edge } \mathsf{e}_2, \\ U & = & 0 & \text{on edges } \mathsf{e}_1, \mathsf{e}_3, \mathsf{e}_4. \end{array}\right. \tag{3.7}$$

Its solution U_2 can be represented in terms of 1D Dirichlet eigenpairs (Φ_n, λ_n), $n = 1, 2, \ldots$,

$$-\Phi_n'' = \lambda_n \Phi_n, \quad \Phi_n(-1) = \Phi_n(1) = 0 \tag{3.8}$$

and of the corresponding solutions $\beta = \beta_n^\lambda$ of the two-point boundary value problem,

$$-\beta'' + \lambda_n \beta = 0, \quad \beta(-1) = 0,\ \beta(1) = 1. \tag{3.9}$$

U_2 is given by the formula

$$U_2(x, y) = \sum_{n=1}^{\infty} v_n \beta_n^\lambda(x) \Phi_n(y), \quad u_n = \int_{\mathsf{e}_2} \underline{v}_2 \Phi_n \, dy. \tag{3.10}$$

Lift U_4 is determined in a fully analogous way. The final H^1-extension is computed by summing the four lifts from the individual edges,

$$U = \sum_{j=1}^{4} U_j. \tag{3.11}$$

Finally, we record the formulas for the Neumann and Dirichlet eigenpairs,

$$\begin{aligned} \Psi_0 = \frac{1}{\sqrt{2}},\ \mu_0 = 0, \quad \Psi_n = \cos\left(\frac{n\pi}{2}(x+1)\right),\ \mu_n = \frac{n^2\pi^2}{4},\ n = 1, 2, \ldots, \\ \Phi_n = \sin\left(\frac{n\pi}{2}(x+1)\right),\ \lambda_n = \frac{n^2\pi^2}{4},\ n = 1, 2, \ldots. \end{aligned} \tag{3.12}$$

Notice that (except for the first Neumann eigenpair) the Neumann and Dirichlet eigenvalues are equal and we have a simple relation,

$$\Phi_n = -\mu_n^{-\frac{1}{2}} \Psi_n', \quad \Psi_n = \lambda_n^{-\frac{1}{2}} \Phi_n'. \tag{3.13}$$

Function β_0^μ is simply linear, and the remaining functions $\beta_n^\mu, \beta_n^\lambda$ are expressed in terms of exponentials.

Remark 3.1. The global and elementwise constructions of U result in the same trace lifting, which is the only harmonic $H^1(\Omega)$ function satisfying the trace condition $\gamma_0 U = u$. This defines the lift operator $\mathcal{L}_0$. □

3.1.2. $\boldsymbol{H}$(curl) *extension operator.* In 2D, the construction of the $\boldsymbol{H}$(curl) extension operator can be reduced to the use of the H^1-extension operator. Given a distribution $e_{\mathsf{t}} \in H^{-\frac{1}{2}}(\partial\Omega)$, we compute first its average,

$$e_0 = \langle e_{\mathsf{t}}, 1 \rangle / 8. \tag{3.14}$$

Next, we utilize the isomorphism,

$$\partial_{\mathsf{t}} : H^{\frac{1}{2}}_{avg}(\partial\Omega) \to H^{-\frac{1}{2}}_{avg}(\partial\Omega) \tag{3.15}$$

and introduce the trace potential $u \in H^{\frac{1}{2}}_{avg}(\partial\Omega)$ such that

$$\partial_{\mathsf{t}} u = e_{\mathsf{t}} - e_0. \tag{3.16}$$

Here ∂_{t} is the derivative along the tangential unit vector field t such that (n, t) is direct. The $\boldsymbol{H}$(curl)-extension is now constructed as follows:

$$\boldsymbol{E} = \boldsymbol{\nabla} U + \boldsymbol{E}_0, \tag{3.17}$$

where U is the H^1-extension of the trace potential u, and $\boldsymbol{E}_0 \in \boldsymbol{Q}_1(\Omega)$ is e.g. the extension of the constant trace e_0 expressed in terms of the lowest-order Nédélec shape functions. As a consequence of the construction above we obtain the estimates

$$\begin{aligned} \|\boldsymbol{E}\|_{\boldsymbol{H}(\mathrm{curl},\Omega)} &\leq C\big(\|U\|_{H^1(\Omega)} + \|\boldsymbol{E}_0\|_{\boldsymbol{H}(\mathrm{curl},\Omega)}\big) \\ &\leq C\big(\|u\|_{H^{\frac{1}{2}}(\partial\Omega)} + \|e_0\|_{H^{-\frac{1}{2}}(\partial\Omega)}\big) \leq C\|e_{\mathsf{t}}\|_{H^{-\frac{1}{2}}(\partial\Omega)}. \end{aligned} \tag{3.18}$$

Our alternative edgewise construction relies on the fact that the average value need not be evaluated over the whole boundary. This reveals nicely the difference between spaces $H^{-\frac{1}{2}}(I)$ and $\widetilde{H}^{-\frac{1}{2}}(I)$. We can consider the restriction of e_{t} to the third edge. Utilizing the isomorphism

$$\partial \;:\; H^{\frac{1}{2}}_{avg}(I) \to H^{-\frac{1}{2}}(I), \tag{3.19}$$

we introduce the corresponding potential $\underline{u}_3 \in H^{\frac{1}{2}}_{avg}(\mathsf{e}_3)$ such that $\partial_{\mathsf{t}}\underline{u}_3 = e_{\mathsf{t}}|_{\mathsf{e}_3}$. The corresponding lift from the edge is defined by taking the gradient of the lift of the potential,

$$\boldsymbol{E}_3 = \boldsymbol{\nabla} U_3. \tag{3.20}$$

Notice that we cannot take the average of functionals from $H^{-\frac{1}{2}}(I)$. In the same way, we construct lift $\boldsymbol{E}_1$. Upon subtracting traces of the lifts from the first and third edges,

$$f_{\mathsf{t}} = e_{\mathsf{t}} - \gamma_{\mathsf{t}}(\boldsymbol{E}_1 + \boldsymbol{E}_3), \tag{3.21}$$

we learn that the corresponding restrictions live in smaller spaces $\widetilde{H}^{-\frac{1}{2}}(I)$, for which the computation of the average value is now possible. This can be seen by recalling that space $\widetilde{H}^{-\frac{1}{2}}(I)$ is the dual of space $H^{\frac{1}{2}}(I)$, which includes the unit function. Construction of the lift from the second edge is now done similarly to the global construction. Utilizing the isomorphism

$$\partial \;:\; \widetilde{H}^{\frac{1}{2}}(I) \to \widetilde{H}^{-\frac{1}{2}}_{avg}(I), \tag{3.22}$$

we represent the boundary data as,

$$f_{\mathsf{t}}|_{\mathsf{e}_2} = \partial_{\mathsf{t}}\, \underline{v}_2 + f_{2,0}, \tag{3.23}$$

where $f_{2,0}$ denotes the average value. We then construct the lift by taking

$$\boldsymbol{E}_2 = \boldsymbol{\nabla} U_2 + \boldsymbol{E}_{2,0}, \tag{3.24}$$

where $\boldsymbol{E}_{0,2} \in \boldsymbol{Q}_1(\Omega)$ is obtained with the lowest order Nédélec shape function corresponding to the second edge. In the same way, we lift $f_{\mathsf{t}}|_{\mathsf{e}_4}$ and define the final lift by summing the four edge contributions,

$$\boldsymbol{E} = \sum_{j=1}^{4} \boldsymbol{E}_j. \tag{3.25}$$

Remark 3.2. Here again, the "global" and "local" constructions result in the *same* extension operator $\mathcal{L}_{\mathbf{t}}$, which, moreover, is divergence free. Whatever the construction, we find an extension in the form $\boldsymbol{E} = \boldsymbol{\nabla} U + \boldsymbol{E}_0$ where $U \in H^1(\Omega)$ is harmonic and $\boldsymbol{E}_0 \in \boldsymbol{Q}_1(\Omega)$. Since all elements of $\boldsymbol{Q}_1(\Omega)$ are divergence free, we find $\operatorname{div} \boldsymbol{E} = 0$. Let us prove that such an extension of the zero tangential trace $e_{\mathbf{t}} = 0$ is zero. Since

$$\int_\Omega \operatorname{curl} \boldsymbol{E}\, dx\, dy = \int_{\partial\Omega} \boldsymbol{e}_{\mathbf{t}}\, dt = 0,$$

we find that the average of $\operatorname{curl} \boldsymbol{E}_0$ on Ω is zero. Since $\operatorname{curl} \boldsymbol{E}_0$ is a constant, it is zero. Using the exact sequence (1.1) we obtain $U_0 \in H^1(\Omega)$ such that $\boldsymbol{E}_0 = \boldsymbol{\nabla} U_0$, and we find that $\boldsymbol{E} = \boldsymbol{\nabla}(U + U_0)$. Since $\operatorname{div} \boldsymbol{E} = 0$, we find that $\Delta(U + U_0) = 0$. Since, moreover, $\gamma_{\mathbf{t}} \boldsymbol{\nabla}(U + U_0) = 0$, we finally deduce that $U + U_0$ is constant; hence $\boldsymbol{E}$ is zero. ☐

Conclusions in 2D continuous case. The lift operators $\mathcal{L}_0$ and $\mathcal{L}_{\mathbf{t}}$ satisfy the following exact sequence and commuting diagram properties:

$$\begin{array}{ccccc} H^1(\Omega) & \xrightarrow{\boldsymbol{\nabla}} & \boldsymbol{H}(\operatorname{curl}, \Omega) & \xrightarrow{\operatorname{curl}} & L^2(\Omega) \\ \gamma_0 \big\downarrow \big\uparrow \mathcal{L}_0 & & \gamma_{\mathbf{t}} \big\downarrow \big\uparrow \mathcal{L}_{\mathbf{t}} & & \gamma_{avg} \big\downarrow \big\uparrow \mathcal{L}_{avg} \\ H^{\frac12}(\partial\Omega) & \xrightarrow{\partial} & H^{-\frac12}(\partial\Omega) & \xrightarrow{\gamma_{avg}} & \mathbb{R} \end{array} \tag{3.26}$$

Here, γ_{avg} is the averaging operator, and $\mathcal{L}_{avg}$ is its lifting by a constant function. The operator $\mathcal{L}_0$ is uniquely determined by the condition that $\Delta \circ \mathcal{L}_0 = 0$, and $\mathcal{L}_{\mathbf{t}}$ by the conditions that $\operatorname{div} \circ \mathcal{L}_{\mathbf{t}}$ is zero and that $\operatorname{curl} \circ \mathcal{L}_{\mathbf{t}}$ takes its values in $\mathbb{R}$. ☐

3.2. **Special families of 1D polynomials.** We shall mimic now the continuous construction on the discrete level. We begin by defining a number of polynomial families defined on the master interval $I = (-1, 1)$, with $p \geq 1$ denoting a polynomial degree.

Function $\phi_0^{(p)}$ will denote the minimum L^2-norm extension of 0 and 1 values at the interval endpoints in the space of polynomials of order less than or equal to p.

Lemma 3.3. *Let $\phi_0^{(p)} \in \mathbb{P}^p(I)$ satisfy $\phi_0^{(p)}(-1) = 0$ and $\phi_0^{(p)}(1) = 1$, with minimum norm in $L^2(I)$. It follows that*

$$\|\phi_0^{(p)}\|^2_{L^2(I)} = \frac{2}{p(p+2)}. \tag{3.27}$$

Proof. See [25, Lemma 1]. ☐

Next we introduce the discrete Dirichlet eigenpairs,

$$\left\{ \begin{array}{l} \Phi_i \in \mathbb{P}^p_0(I) \\ \displaystyle\int_I \Phi_i' v' = \lambda_i^{(p)} \int_I \Phi_i v, \quad \forall v \in \mathbb{P}^p_0(I) \\ \hfill i = 2, \ldots, p \end{array} \right. \tag{3.28}$$

and the discrete Neumann eigenpairs,

$$\left\{ \begin{array}{l} \Psi_i \in \mathbb{P}^p(I) \\ \displaystyle\int_I \Psi_i' v' = \mu_i^{(p)} \int_I \Psi_i v, \quad \forall v \in \mathbb{P}^p(I) \\ \hfill i = 0, \ldots, p. \end{array} \right. \tag{3.29}$$

Obviously both eigenvalues and the corresponding eigenvectors depend upon the polynomial degree p, $\lambda_i = \lambda_i^{(p)}, \Phi_i = \Phi_i^{(p)}, \mu_i = \mu_i^{(p)}, \Psi_i = \Psi_i^{(p)}$. For simplicity, we will frequently omit the superscript (p), hoping that this does not lead to confusion with the continuous eigenpairs introduced in the previous section.

We shall assume that all discrete eigenvectors have been normalized to have a unit L^2-norm,

$$\|\Phi_i\|_{L^2(I)} = 1, \quad \|\Psi_i\|_{L^2(I)} = 1. \tag{3.30}$$

Notice that the first Neumann eigenvalue $\mu_0^{(p)} = 0$, independently of p. For each of the discrete Dirichlet and Neumann eigenvalues, we introduce the following solution of the associated discrete 1D boundary-value problem; cf. Section 3.1.

$$\begin{cases} \beta_i^\lambda \in \mathbb{P}^p(I), \quad \beta_i^\lambda(-1) = 0, \beta_i^\lambda(1) = 1 \\ \displaystyle\int_I (\beta_i^\lambda)' v' + \lambda_i^{(p)} \int_I \beta_i^\lambda v = 0 \quad \forall v \in \mathbb{P}_0^p(I) \end{cases} \qquad (i = 2, \dots, p), \tag{3.31}$$

$$\begin{cases} \beta_i^\mu \in \mathbb{P}^p(I), \quad \beta_i^\mu(-1) = 0, \beta_i^\mu(1) = 1 \\ \displaystyle\int_I (\beta_i^\mu)' v' + \mu_i^{(p)} \int_I \beta_i^\mu v = 0 \quad \forall v \in \mathbb{P}_0^p(I) \end{cases} \qquad (i = 0, \dots, p). \tag{3.32}$$

Notice that function β_0^μ, corresponding to the zero Neumann eigenvalue, is linear for all values of p.

Lemma 3.4. *The inverse inequality*

$$|f|^2_{H^1(I)} \le \frac{(p+1)^4}{2} \|f\|^2_{L^2(I)}, \quad \forall f \in \mathbb{P}^p(I) \tag{3.33}$$

holds.

Proof. See [5]. ☐

Lemma 3.5. *There exists a constant $C > 0$, independent of p, such that*

$$\begin{aligned} |\beta_i^\lambda|^2_{H^1(I)} + \lambda_i^{(p)} \|\beta_i^\lambda\|^2_{L^2(I)} &\le C\,(\lambda_i^{(p)})^{\frac{1}{2}}, \quad i = 2, \dots, p, \\ |\beta_i^\mu|^2_{H^1(I)} + \mu_i^{(p)} \|\beta_i^\mu\|^2_{L^2(I)} &\le C\,(\mu_i^{(p)})^{\frac{1}{2}}, \quad i = 1, \dots, p. \end{aligned} \tag{3.34}$$

Proof. For completeness, we shall reproduce the reasoning from [25] and prove the second inequality. The proof of the first inequality is fully analogous. Let $i \ge 1$.

- It follows from the definition (3.32) that

$$\begin{aligned} |\beta_i^\mu|^2_{H^1(I)} + \mu_i^{(p)} \|\beta_i^\mu\|^2_{L^2(I)} \le |\sigma|^2_{H^1(I)} + \mu_i^{(p)} \|\sigma\|^2_{L^2(I)} \\ \forall \sigma \in \mathbb{P}^p(I) \,:\, \sigma(-1) = 0,\ \sigma(1) = 1. \end{aligned} \tag{3.35}$$

- Next, we have

$$\begin{aligned} \mu_i^{(p)} &= \int_I |\Psi_i'|^2 && \text{(set } v = \Psi \text{ in the definition of } \Psi_i) \\ &\le \frac{(p+1)^4}{2} && \text{(Lemma 3.4).} \end{aligned} \tag{3.36}$$

- Since $i \geq 1$, the discrete eigenvalue $\mu_i^{(p)}$ is larger than μ_i, which is in turn larger than or equal to $\frac{\pi^2}{4} > \frac{1}{2}$. Thus we deduce from (3.36) that there exists an integer q, $1 \leq q \leq p$ such that

$$\frac{q^4}{2} \leq \mu_i^{(p)} \leq \frac{(q+1)^4}{2}. \tag{3.37}$$

Now select $\sigma = \phi_0^{(q)}$ with $\phi_0^{(q)}$ defined in Lemma 3.3. We have

$$\begin{aligned} |\phi_0^{(q)}|^2_{H^1(I)} + \mu_i^{(p)} \|\phi_0^{(q)}\|^2_{L^2(I)} &\leq (q+1)^4 \, \|\phi_0^{(q)}\|^2_{L^2(I)} && \text{(Lemma 3.4)} \\ &\leq \; 12q^2 && \text{(Lemma 3.3).} \end{aligned} \tag{3.38}$$

- Substituting into (3.35),

$$|\beta_i^\mu|^2_{H^1(I)} + \mu_i^{(p)} \|\beta_i^\mu\|^2_{L^2(I)} \leq 12q^2 \leq 12\sqrt{2}\,(\mu_i^{(p)})^{\frac{1}{2}}. \tag{3.39}$$

Notice however that the inequality $(3.34)_2$ does not hold for index $i = 0$, as the first Neumann eigenvalue is equal to zero. $\square$

3.3. **Construction of the polynomial H^1 extension operator.** We shall consider the master square Ω shown in Fig. 1. Let u belong to the trace space $W_p(\partial\Omega)$; cf. $(2.11)_1$.

Step 1: Lifting from a horizontal edge. We shall consider first the third edge e_3. Let $\underline{u}_3 = \underline{u}$ be the restriction of u to e_3. Then $\underline{u}(x) \in \mathbb{P}^p(\mathsf{e}_3)$, and we expand it into the discrete Neumann eigenvectors,

$$\underline{u}(x) = \sum_{j=0}^{p} u_j \Psi_j(x), \quad u_j = \int_I \underline{u}\Psi_j \tag{3.40}$$

and define its polynomial extension as

$$U(x,y) := \sum_{j=0}^{p} u_j \Psi_j(x) \beta_j^\mu(y). \tag{3.41}$$

An evaluation of the H^1-norm follows, which is straightforward if we use the definition of Ψ_j and Lemma 3.5.

$$\begin{aligned} \int_\Omega |\boldsymbol{\nabla} U|^2 &= \sum_{i=0}^{p}\sum_{j=0}^{p} u_i \bar{u}_j \left(\int_I \Psi_i'\Psi_j' \int_I \beta_i^\mu \beta_j^\mu + \int_I \Psi_i \Psi_j \int_I (\beta_i^\mu)'(\beta_j^\mu)' \right) \\ &= \sum_{j=0}^{p} |u_j|^2 \left(\mu_j^{(p)} \|\beta_j^\mu\|^2_{L^2(I)} + |\beta_j^\mu|^2_{H^1(I)} \right) \\ &\leq |u_0|^2 + C \sum_{j=1}^{p} |u_j|^2 (\mu_j^{(p)})^{\frac{1}{2}}, \\ \int_\Omega |U|^2 &= \sum_{j=0}^{p} |u_j|^2 \|\beta_j^\mu\|^2_{L^2(I)} \\ &\leq C \sum_{j=0}^{p} |u_j|^2. \end{aligned}$$

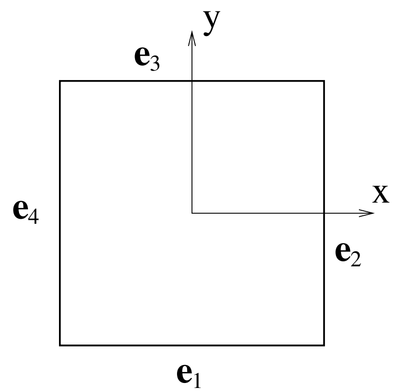

FIGURE 1. Master square element. Enumeration of edges. The first and third edges are parametrized with x, and the second and fourth edges are parametrized with y.

Consequently,

$$\|U\|^2_{H^1(\Omega)} \le C \sum_{j=0}^{p} |u_j|^2 \left(1 + (\mu_j^{(p)})^{\frac{1}{2}}\right). \tag{3.42}$$

Clearly, the right-hand side of (3.42) is the fractional $\frac{1}{2}$- norm for the polynomial space $\mathbb{P}^p(I)$ obtained by the standard interpolation argument applied to the L^2- and the H^1-norms and the space of polynomials. The following theorem of Bernardi, Dauge, and Maday provides the key argument for our result.

Proposition 3.6. *There exists a constant $C > 0$, independent of the polynomial degree p, such that*

$$\|u\|^2_{H^{\frac{1}{2}}(I)} \le \sum_{j=0}^{p} |u_j|^2 \left(1 + (\mu_j^{(p)})^{\frac{1}{2}}\right) \le C\|u\|^2_{H^{\frac{1}{2}}(I)} \tag{3.43}$$

for every polynomial $u \in \mathbb{P}^p(I)$, where $u_j = \int_I u\,\Psi_j^{(p)}$.

Proof. See [7, Ch. II, Thm. 4.2]. □

Consequently, the H^1-norm of the lift $U = U_3$ given by (3.41) is bounded by the $H^{\frac{1}{2}}(\mathsf{e}_3)$-norm of trace $\underline{u} = \underline{u}_3$, which, in turn, is bounded by the global norm of the trace u on the whole boundary. Notice that, by construction, the lift U_3 has a zero trace on the lower edge e_1. Repeating thus exactly the same construction for the first edge, we obtain a lift U_1 of the trace $\underline{u}_1$. The polynomial trace

$$v = u - \gamma_0(U_1 + U_3) \tag{3.44}$$

vanishes along the horizontal edges and, consequently, has zero vertex values.

Step 2: Lifting from a vertical edge. We consider now the function v given by (3.44). Its restrictions $\underline{v}_2$ and $\underline{v}_4$ to either of the two vertical edges are now bounded in the $\widetilde{H}^{\frac{1}{2}}$-norm,

$$\|\underline{v}_i\|_{\widetilde{H}^{\frac{1}{2}}(\mathsf{e}_i)} \le C\|u\|_{H^{\frac{1}{2}}(\partial\Omega)}, \quad i = 2, 4. \tag{3.45}$$

Consider the second edge and expand the function $\underline{v}_2$ now in terms of the discrete Dirichlet eigenvectors,

$$\underline{v}_2(y) = \sum_{j=2}^{p} v_j \Phi_j(y), \quad v_j = \int_I v\Phi_j. \tag{3.46}$$

The extension $U = U_2$ to the square element is now defined as follows:

$$U(y) = \sum_{j=2}^{p} v_j \beta_j^{\lambda}(x) \Phi_j(y). \tag{3.47}$$

By exactly the same arguments as in the previous paragraph, we show that

$$\|U\|^2_{H^1(\Omega)} \leq C \sum_{j=2}^{p} |v_j|^2 \left(1 + (\lambda_j^{(p)})^{\frac{1}{2}}\right). \tag{3.48}$$

The discrete norm on the right-hand side of (3.48) turns out to be equivalent to the continuous $\widetilde{H}^{\frac{1}{2}}$-norm.

Proposition 3.7. *There exists a constant $C > 0$, independent of the polynomial degree p, such that*

$$\|v\|^2_{\widetilde{H}^{\frac{1}{2}}(I)} \leq \sum_{j=2}^{p} |v_j|^2 \left(1 + (\lambda_j^{(p)})^{\frac{1}{2}}\right) \leq C\|v\|^2_{\widetilde{H}^{\frac{1}{2}}(I)} \tag{3.49}$$

for every polynomial $v \in \mathbb{P}_0^p(I)$, where $u_j = \int_I u\, \Phi_j^{(p)}$.

Proof. Such norm equivalences were stated in [6]. For a proof, see [7, Ch. II, Theorem 4.6 and inequality (4.9)]. □

Notice that the lift does not alter the trace on the remaining edges. Let U_4 denote the analogous lift from the vertical edge e_4. We have proved

Theorem 3.8. *The operator*

$$\mathcal{L}_0^{(p)} : W_p(\partial\Omega) \ni u \longmapsto U := \sum_{j=1}^{4} U_j \in W_p(\Omega) \tag{3.50}$$

defines a polynomial lift of traces. Its norm from $H^{\frac{1}{2}}(\partial\Omega)$ into $H^1(\Omega)$ is independent of the degree p.

Remark 3.9. We can see that for any $u \in W_p(\partial\Omega)$, the lift $\mathcal{L}_0^{(p)} u$ coincides with the solution $U \in W_p(\Omega)$ of the discrete Dirichlet problem

$$\gamma_0 U = u \quad \text{and} \quad \int_\Omega \boldsymbol{\nabla} U \cdot \boldsymbol{\nabla} V = 0 \quad \forall V \in W_{p,0}(\Omega). \tag{3.51}$$

Thus, despite its nonsymmetric construction, $\mathcal{L}_0^{(p)}$ is canonical. □

3.4. **Construction of the polynomial $\boldsymbol{H}(\mathrm{curl})$ extension operator.** Let $p \geq 1$ be an integer and let $e_{\mathbf{t}}$ be an element of $Q_p(\partial\Omega)$; cf. $(2.11)_2$. Using the construction in the continuous case first, we obtain an extension in $\boldsymbol{H}(\mathrm{curl},\Omega)$ in the form $\boldsymbol{E} = \boldsymbol{\nabla} U + \boldsymbol{E}_0$ with harmonic $U \in H^1(\Omega)$ and $\boldsymbol{E}_0 \in \boldsymbol{Q}_1(\Omega)$. We define our polynomial extension $\mathcal{L}_{\mathbf{t}}^{(p)} e_{\mathbf{t}}$ by the formula

$$\mathcal{L}_{\mathbf{t}}^{(p)} e_{\mathbf{t}} = \boldsymbol{\nabla}\big(\mathcal{L}_0^{(p)} u\big) + \boldsymbol{E}_0, \quad \text{with} \quad u = \gamma_0 U \in W_p(\partial\Omega). \tag{3.52}$$

Indeed we check that

$$\begin{aligned} \gamma_{\mathbf{t}}\, \mathcal{L}_{\mathbf{t}}^{(p)} e_{\mathbf{t}} &= \gamma_{\mathbf{t}} \boldsymbol{\nabla}(\mathcal{L}_0^{(p)} u) + \gamma_{\mathbf{t}} \boldsymbol{E}_0 \\ &= \partial_{\mathbf{t}} (\mathcal{L}_0^{(p)} u)|_{\partial\Omega} + \gamma_{\mathbf{t}} \boldsymbol{E}_0 \\ &= \partial_{\mathbf{t}} u + \gamma_{\mathbf{t}} \boldsymbol{E}_0 = e_{\mathbf{t}}. \end{aligned}$$

Let us prove furthermore that definition (3.52) is independent of the way that $\boldsymbol{E}$ is split into $\boldsymbol{\nabla} U + \boldsymbol{E}_0$ with harmonic $U \in H^1(\Omega)$ and $\boldsymbol{E}_0 \in \boldsymbol{Q}_1(\Omega)$: Let us consider two such representations of $\boldsymbol{E}$, $\boldsymbol{E} = \boldsymbol{\nabla} U + \boldsymbol{E}_0 = \boldsymbol{\nabla} U' + \boldsymbol{E}_0'$, with harmonic $U, U' \in H^1(\Omega)$ and $\boldsymbol{E}_0$, $\boldsymbol{E}_0' \in \boldsymbol{Q}_1(\Omega)$. Therefore $\boldsymbol{\nabla}(U - U') = \boldsymbol{E}_0' - \boldsymbol{E}_0$. Using the exact sequence (2.9), we find that $\boldsymbol{E}_0' - \boldsymbol{E}_0 = \boldsymbol{\nabla} U_0$ with a (harmonic) $U_0 \in W_1(\Omega)$. We deduce that, after the possible addition of a constant, $U - U' = U_0$. Hence

$$\boldsymbol{\nabla} \mathcal{L}_0^{(p)} \gamma_0 (U - U') = \boldsymbol{\nabla} \mathcal{L}_0^{(p)} \gamma_0 U_0.$$

Since U_0 is a harmonic element of $W_1(\Omega) \subset W_p(\Omega)$, it satisfies

$$\mathcal{L}_0^{(p)} \gamma_0 U_0 = U_0.$$

Finally

$$\boldsymbol{\nabla} \mathcal{L}_0^{(p)} \gamma_0 (U - U') = \boldsymbol{\nabla} U_0 = \boldsymbol{E}_0' - \boldsymbol{E}_0,$$

which proves the independence of $\mathcal{L}_{\mathbf{t}}^{(p)} \boldsymbol{e}_{\mathbf{t}}$ of the representation of $\boldsymbol{E}$.

Theorem 3.10. *The operator*

$$\mathcal{L}_{\mathbf{t}}^{(p)} : Q_p(\partial\Omega) \ni e_{\mathbf{t}} \longmapsto \boldsymbol{E} = \boldsymbol{\nabla}\big(\mathcal{L}_0^{(p)} u\big) + \boldsymbol{E}_0 \in \boldsymbol{Q}_p(\Omega) \tag{3.53}$$

defines a polynomial lift of tangential traces. Its norm from $H^{-\frac{1}{2}}(\partial\Omega)$ to $\boldsymbol{H}(\mathrm{curl},\Omega)$ is bounded independently of the degree p.

Proof. The definition of $\mathcal{L}_{\mathbf{t}}^{(p)}$ and the fact that it lifts tangential traces are clear from the considerations above. The uniform boundedness with respect to p is a consequence of estimates (3.18) for the extension in the continuous case, and of the uniform boundedness of the scalar extensions $\mathcal{L}_0^{(p)}$ (Theorem 3.8). □

Conclusions in 2D polynomial case. With $p \geq 1$ any integer, the lift operators $\mathcal{L}_0^{(p)}$ and $\mathcal{L}_{\mathbf{t}}^{(p)}$ satisfy the following exact sequence and commuting diagram properties, reproducing those of the continuous case:

$$\begin{array}{ccccc} W_p(\Omega) & \xrightarrow{\boldsymbol{\nabla}} & \boldsymbol{Q}_p(\Omega) & \xrightarrow{\mathrm{curl}} & Y_p(\Omega) \\ \gamma_0 \big\downarrow \big\uparrow \mathcal{L}_0^{(p)} & & \gamma_{\mathbf{t}} \big\downarrow \big\uparrow \mathcal{L}_{\mathbf{t}}^{(p)} & & \gamma_{avg} \big\downarrow \big\uparrow \mathcal{L}_{avg} \\ W_p(\partial\Omega) & \xrightarrow{\partial} & Q_p(\partial\Omega) & \xrightarrow{\gamma_{avg}} & \mathbb{R} \end{array} \tag{3.54}$$

Here, γ_{avg} is the averaging operator, and $\mathcal{L}_{avg}$ is its lifting by a constant function.

The operator $\mathcal{L}_0^{(p)}$ is uniquely determined by the extra condition of orthogonality (3.51), and $\mathcal{L}_{\mathsf{t}}^{(p)}$ by

$$\int_\Omega \boldsymbol{E} \cdot \boldsymbol{\nabla} V = 0 \quad \forall V \in W_{p,0}(\Omega) \quad \text{and} \quad \operatorname{curl} \boldsymbol{E} \in \mathbb{R} \tag{3.55}$$

for $\boldsymbol{E} = \mathcal{L}_{\mathsf{t}}^{(p)} e_{\mathsf{t}}$ with e_{t} any element of $Q_p(\partial\Omega)$. □

4. 3D extension operators on the continuous level

As in 2D, we will discuss first the construction of the extension operators on the continuous level. In this section, we will use the 1D Dirichlet and Neumann eigenpairs (Φ_n, λ_n) and (Ψ_n, μ_n) as defined in Section 3.1, equations (3.8) and (3.2).

4.1. H^1 **extension operator.** The procedure is fully analogous to its 2D counterpart. Let Ω denote the master hexahedron shown in Fig. 2. We begin with the top face f_2. Let $\underline{u}_2 \in H^{\frac{1}{2}}(\mathsf{f}_2)$ denote the restriction of the boundary data $u \in H^{\frac{1}{2}}(\partial\Omega)$ to face f_2. The corresponding lift U_2 is constructed by solving the following mixed boundary-value problem for the Laplacian:

$$\begin{cases} U & \in & H^1(\Omega), & \\ -\Delta U & = & 0 & \text{in } \Omega, \\ U & = & \underline{u}_2 & \text{on face } \mathsf{f}_2, \\ U & = & 0 & \text{on face } \mathsf{f}_1, \\ \partial_{\mathbf{n}} U & = & 0 & \text{on faces } \mathsf{f}_3, \mathsf{f}_4, \mathsf{f}_5, \mathsf{f}_6. \end{cases} \tag{4.1}$$

Separation of variables leads to two Neumann eigenvalue problems in terms of x and y coordinates, and a two-point Dirichlet boundary-value problem in terms of the coordinate z,

$$-\beta'' + (\mu_n + \mu_m)\beta = 0, \quad \beta(-1) = 0,\ \beta(1) = 1. \tag{4.2}$$

Functions $\beta = \beta_{nm}^{\mu\mu}, n, m = 0, 1, \ldots$ depend upon pairs of Neumann eigenvalues μ_n, μ_m. The solution $U = U_2$ reads as follows:

$$U_2(x,y,z) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} u_{nm} \Psi_n(x) \Psi_m(y) \beta_{nm}^{\mu\mu}(z), \tag{4.3}$$
$$\text{with} \quad u_{nm} = \int_{\mathsf{f}_2} \underline{u}_2(x,y) \Psi_n(x) \Psi_m(y)\, dxdy.$$

In the same way we lift from the bottom face f_1. We subtract then the traces of the lifts U_1, U_2 from the original boundary data,

$$v = u - (U_1 + U_2)|_{\partial\Omega}. \tag{4.4}$$

Next, we construct the lifts from the pair of vertical faces $\mathsf{f}_3, \mathsf{f}_5$. The key point is that, after the subtraction of the first two lifts, function v vanishes over the horizontal faces. Its restriction $\underline{v}_5$ to face f_5 lives in space $H^{\frac{1}{2}}(I_x, L^2(I_z)) \cap L^2(I_x, \widetilde{H}^{\frac{1}{2}}(I_z))$ with the norm controlled by the $H^{\frac{1}{2}}$ norm of the original data. Consequently, we can impose homogeneous Dirichlet boundary conditions on faces $\mathsf{f}_1, \mathsf{f}_2$. Lift $U = U_5$ is defined as the solution of the problem

$$
\left\{
\begin{array}{rcll}
U & \in & H^1(\Omega), & \\
-\Delta U & = & 0 & \text{in } \Omega, \\
U & = & \underline{v}_5 & \text{on face } \mathsf{f}_5, \\
U & = & 0 & \text{on faces } \mathsf{f}_1, \mathsf{f}_2, \mathsf{f}_3, \\
\partial_{\mathsf{n}} U & = & 0 & \text{on faces } \mathsf{f}_4, \mathsf{f}_6.
\end{array}
\right.
\tag{4.5}
$$

Separation of variables now leads to the Neumann eigenvalue problem in the x coordinate and the Dirichlet eigenvalue problem in the z direction. We need to define a new family of solutions $\beta = \beta_{nm}^{\mu\lambda}$ to the two-point boundary-value problem,

$$
-\beta'' + (\mu_n + \lambda_m)\beta = 0, \quad \beta(-1) = 0,\ \beta(1) = 1. \tag{4.6}
$$

The solution $U = U_5$ reads as follows:

$$
U_5(x, y, z) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} u_{nm} \Psi_n(x) \beta_{nm}^{\mu\lambda}(y) \Phi_m(z) \tag{4.7}
$$

$$
\text{with} \quad u_{nm} = \int_{\mathsf{f}_5} \underline{v}_5(x, y) \Psi_n(x) \Phi_m(z)\, dxdz.
$$

In the same way we construct lift U_3. Upon subtracting lifts U_3, U_5, the modified boundary data vanishes over faces $\mathsf{f}_1, \mathsf{f}_2, \mathsf{f}_3, \mathsf{f}_5$,

$$
w = u - (U_1 + U_2 + U_3 + U_5)|_{\partial\Omega}. \tag{4.8}
$$

Consequently, the traces $\underline{w}_4$ and $\underline{w}_6$ of function w on the remaining vertical faces $\mathsf{f}_4, \mathsf{f}_6$ live in space $\widetilde{H}^{\frac{1}{2}}(I^2)$. We determine lift $U = U_4$ by solving a Dirichlet problem for the Laplacian,

$$
\left\{
\begin{array}{rcll}
U & \in & H^1(\Omega), & \\
-\Delta U & = & 0 & \text{in } \Omega, \\
U & = & \underline{w}_4 & \text{on face } \mathsf{f}_4, \\
U & = & 0 & \text{on faces } \mathsf{f}_1, \mathsf{f}_2, \mathsf{f}_3, \mathsf{f}_5, \mathsf{f}_6.
\end{array}
\right.
\tag{4.9}
$$

The solution depends continuously on $\|\underline{w}_4\|_{\widetilde{H}^{\frac{1}{2}}(\mathsf{f}_4)}$, which, in turn, is bounded by the $H^{\frac{1}{2}}(\partial\Omega)$-norm of the original data. Lift U_4 is expressed in terms of Dirichlet eigenfunctions,

$$
U_4(x, y, z) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} u_{nm} \beta_{nm}^{\lambda\lambda}(x) \Phi_n(y) \Phi_m(z), \tag{4.10}
$$

$$
\text{with} \quad u_{nm} = \int_{\mathsf{f}_4} \underline{w}_4(y, z) \Phi_n(y) \Phi_m(z)\, dydz
$$

and solutions $\beta = \beta_{nm}^{\lambda\lambda}$ to the two-point boundary-value problem,

$$
-\beta'' + (\lambda_n + \lambda_m)\beta = 0, \quad \beta(-1) = 0,\ \beta(1) = 1. \tag{4.11}
$$

In the same way we construct the lift from face f_6. The final extension is constructed by summing the six individual face lifts,

$$U = \sum_{j=1}^{6} U_j. \tag{4.12}$$

Remark 4.1. As in 2D this local construction coincides with the global extension operator $\mathcal{L}_0$ defined as the Dirichlet harmonic extension. □

4.2. $\boldsymbol{H}(\mathrm{div})$ **extension operator.** We shall continue now from the other end of the exact sequence. Given a function $h \in H^{-\frac{1}{2}}(\partial\Omega)$, we will construct an extension $\boldsymbol{H} \in \boldsymbol{H}(\mathrm{div}, \Omega)$ such that $\gamma_{\mathbf{n}}\boldsymbol{H} = h$. As in the previous cases, we will work with one face at a time. We first consider the restriction $\underline{h}_2 = h|_{\mathsf{f}_2} solve \in H^{-\frac{1}{2}}(\mathsf{f}_2)$ and solve an auxiliary mixed problem for the Laplace operator,

$$\left\{\begin{array}{rcll} U & \in & H^1(\Omega), & \\ -\Delta U & = & 0 & \text{in } \Omega, \\ \partial_{\mathbf{n}} U & = & \underline{h}_2 & \text{on face } \mathsf{f}_2, \\ \partial_{\mathbf{n}} U & = & 0 & \text{on face } \mathsf{f}_1, \\ U & = & 0 & \text{on faces } \mathsf{f}_3, \mathsf{f}_4, \mathsf{f}_5, \mathsf{f}_6. \end{array}\right. \tag{4.13}$$

Notice that, due to the presence of Dirichlet boundary conditions, the Neumann data $\underline{h}_2$ need not satisfy any compatibility conditions. The lift from face f_2 is now set to the gradient of function $U = U_2$,

$$\boldsymbol{H}_2 = \boldsymbol{\nabla} U_2. \tag{4.14}$$

As U_2 is harmonic, lift $\boldsymbol{H}_2$ is divergence-free, and its L^2-norm, equal to the H^1-seminorm of U_2, is bounded by the $H^{-\frac{1}{2}}$-norm of data $\underline{h}_2$. Moreover U_2 can be computed using separation of variables,

$$U_2(x, y, z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} h_{nm}\Phi_n(x)\Phi_m(y)\gamma_{nm}^{\lambda\lambda}(z), \tag{4.15}$$

$$\text{with} \quad h_{nm} = \int_{\mathsf{f}_2} \underline{h}_2(x, y)\Phi_n(x)\Phi_m(y)\, dxdy.$$

where $\gamma = \gamma_{nm}^{\lambda\lambda}$ are solutions to the two-point boundary-value problems with Neumann boundary conditions,

$$-\gamma'' + (\lambda_n + \lambda_m)\gamma = 0, \quad \gamma'(-1) = 0,\ \gamma'(1) = 1. \tag{4.16}$$

Extension $\boldsymbol{H}_1$ from face f_1 is constructed in the same way.

Next, we subtract from the original data h, the normal traces of the first two face extensions,

$$g = h - \gamma_{\mathbf{n}}(\boldsymbol{H}_1 + \boldsymbol{H}_2) \tag{4.17}$$

and proceed in the same way as for the H^1-extensions. Extension $\boldsymbol{H}_5$ is constructed by taking the gradient of solution U to the problem

$$\left\{\begin{array}{rcll} U & \in & H^1(\Omega), & \\ -\Delta U & = & 0 & \text{in } \Omega, \\ \partial_{\mathsf{n}} U & = & \underline{g}_5 & \text{on face } \mathsf{f}_5, \\ \partial_{\mathsf{n}} U & = & 0 & \text{on face } \mathsf{f}_1, \mathsf{f}_2, \mathsf{f}_3, \\ U & = & 0 & \text{on faces } \mathsf{f}_4, \mathsf{f}_6. \end{array}\right. \tag{4.18}$$

Here $\underline{g}_5$ is in the space $H^{-\frac{1}{2}}(I_x, L^2(I_z)) + L^2(I_x, \widetilde{H}^{-\frac{1}{2}}(I_z))$, with the corresponding norm controlled by the $H^{-\frac{1}{2}}(\partial\Omega)$-norm of the original data h. Solution $U = U_5$ is obtained using the separation of variables,

$$U_5(x, y, z) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} g_{nm} \Phi_n(x) \gamma_{nm}^{\lambda\mu}(y) \Psi_m(z), \tag{4.19}$$

$$\text{with} \quad g_{nm} = \int_{\mathsf{f}_5} \underline{g}_5(x, y) \Phi_n(x) \Psi_m(z)\, dxdz,$$

where $\gamma = \gamma_{nm}^{\lambda\mu}$ are solutions to the two-point boundary-value problems with Neumann boundary conditions,

$$-\gamma'' + (\lambda_n + \mu_m)\gamma = 0, \quad \gamma'(-1) = 0,\ \gamma'(1) = 1. \tag{4.20}$$

In the same way we construct lift $\boldsymbol{H}_3$. We subtract then from the original data h normal traces of the four face extensions,

$$f = h - \gamma_{\mathsf{n}}(\boldsymbol{H}_1 + \boldsymbol{H}_2 + \boldsymbol{H}_3 + \boldsymbol{H}_5). \tag{4.21}$$

Functional f vanishes over faces $\mathsf{f}_1, \mathsf{f}_2, \mathsf{f}_3, \mathsf{f}_5$. Consequently, its restriction $\underline{f}_4$ to face f_4 lives in space $\widetilde{H}^{-\frac{1}{2}}(I^2)$ with the corresponding norm bounded by the $H^{-\frac{1}{2}}(\partial\Omega)$-norm of the original data h. For functionals in $\widetilde{H}^{-\frac{1}{2}}(I^2)$, we can compute their average values. Let $f_{4,0}$ be the average of $\underline{f}_4$,

$$f_{4,0} = \langle \underline{f}_4, 1\rangle / 4. \tag{4.22}$$

We solve now a pure Neumann problem for $U = U_4$,

$$\left\{\begin{array}{rcll} U & \in & H^1(\Omega), & \\ -\Delta U & = & 0 & \text{in } \Omega, \\ \partial_{\mathsf{n}} U & = & \underline{f}_4 - f_{4,0} & \text{on face } \mathsf{f}_4, \\ \partial_{\mathsf{n}} U & = & 0 & \text{on faces } \mathsf{f}_1, \mathsf{f}_2, \mathsf{f}_3, \mathsf{f}_5, \mathsf{f}_6, \end{array}\right. \tag{4.23}$$

and construct the corresponding face extension as

$$\boldsymbol{H}_4 = \boldsymbol{\nabla} U_4 + \boldsymbol{H}_{4,0}, \tag{4.24}$$

where $\boldsymbol{H}_{4,0} \in \boldsymbol{V}_1(\Omega)$ is obtained with the lowest-order Raviart-Thomas shape function for face f_4. Function $U = U_4$ is computed using separation of variables,

$$U_4(x, y, z) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} f_{nm} \gamma_{nm}^{\mu\mu}(x) \Psi_n(y) \Psi_m(z), \tag{4.25}$$

$$\text{with} \quad f_{nm} = \int_{\mathsf{f}_4} (\underline{f}_4(x, y) - f_{4,0}) \Psi_n(y) \Psi_m(z)\, dxdz,$$

where $\gamma = \gamma_{nm}^{\mu\mu}$ are solutions to the two-point boundary-value problems with Neumann boundary conditions,

$$-\gamma'' + (\mu_n + \mu_m)\gamma = 0, \quad \gamma'(-1) = 0,\ \gamma'(1) = 1. \tag{4.26}$$

In the same way we construct lift $\boldsymbol{H}_6$. The final extension, obtained by summing the six face lifts,

$$\boldsymbol{H} = \sum_{j=1}^{6} \boldsymbol{H}_j \tag{4.27}$$

is bounded in the $\boldsymbol{H}(\mathrm{div}, \Omega)$-norm by the $H^{-\frac{1}{2}}(\partial\Omega)$-norm of data h.

We finish this section by recording relations between functions β and γ:

(4.28)
$$\beta_{nm}^{\lambda\lambda} = \left(\gamma_{nm}^{\lambda\lambda}\right)', \qquad \beta_{mn}^{\mu\lambda} = \left(\gamma_{nm}^{\lambda\mu}\right)', \qquad \beta_{nm}^{\mu\mu} = \left(\gamma_{nm}^{\mu\mu}\right)',$$
$$\gamma_{nm}^{\lambda\lambda} = \frac{1}{\lambda_n + \lambda_m}\left(\beta_{nm}^{\lambda\lambda}\right)', \quad \gamma_{nm}^{\lambda\mu} = \frac{1}{\lambda_n + \mu_m}\left(\beta_{mn}^{\mu\lambda}\right)', \quad \gamma_{nm}^{\mu\mu} = \frac{1}{\mu_n + \mu_m}\left(\beta_{nm}^{\mu\mu}\right)'.$$

Remark 4.2. This local construction results in fact in a global extension operator $\mathcal{L}_{\mathsf{n}}$, which, moreover, is curl free and with constant divergence. Whatever the construction, we find an extension in the form $\boldsymbol{H} = \boldsymbol{\nabla} U + \boldsymbol{H}_0$ where $U \in H^1(\Omega)$ is harmonic and $\boldsymbol{H}_0 \in \boldsymbol{V}_1(\Omega)$. Since all elements of $\boldsymbol{V}_1(\Omega)$ are curl free, we find $\mathbf{curl}\,\boldsymbol{H} = 0$. Let us prove that such an extension of the zero normal trace $h = 0$ is zero. Since

$$\int_\Omega \mathrm{div}\,\boldsymbol{H}\,dx\,dy = \int_{\partial\Omega} h\,dS = 0,$$

we find that the average of $\mathrm{div}\,\boldsymbol{H}_0$ on Ω is zero. Since $\mathrm{div}\,\boldsymbol{H}_0$ is a constant, it is zero; hence $\mathrm{div}\,\boldsymbol{H} = 0$. Using the identity $\mathbf{curl}\,\boldsymbol{H}_0 = 0$ with the exact sequence (1.1) we obtain $U_0 \in H^1(\Omega)$ such that $\boldsymbol{H}_0 = \boldsymbol{\nabla} U_0$, and we find that $\boldsymbol{H} = \boldsymbol{\nabla}(U + U_0)$. Since $\mathrm{div}\,\boldsymbol{H} = 0$, we find that $\Delta(U + U_0) = 0$. Since, moreover, $\gamma_{\mathsf{n}}\boldsymbol{\nabla}(U + U_0) = 0$, we finally deduce that $U + U_0$ is constant; hence $\boldsymbol{H}$ is zero. □

4.3. $\boldsymbol{H}$(curl) **extension operator.** Given a boundary data $\boldsymbol{e}_{\mathsf{t}} \in \boldsymbol{H}^{-\frac{1}{2}}(\mathrm{curl}, \partial\Omega)$, we are set to construct an extension $\boldsymbol{E} = \mathcal{L}_{\mathsf{t}}\boldsymbol{e}_{\mathsf{t}} \in \boldsymbol{H}(\mathbf{curl}, \Omega)$ such that $\gamma_{\mathsf{t}}\boldsymbol{E} = \boldsymbol{e}_{\mathsf{t}}$ and

$$\|\boldsymbol{E}\|_{\boldsymbol{H}(\mathbf{curl},\Omega)} \le C\|\boldsymbol{e}_{\mathsf{t}}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\mathrm{curl},\partial\Omega)} \tag{4.29}$$

with constant C independent of the functional $\boldsymbol{e}_{\mathsf{t}}$.

We will construct the operator so that we have a commuting diagram property such as in 2D. For this we use the Poincaré map $\boldsymbol{K}$; see §2.5. We take the surface curl of the boundary data,

$$h = \mathrm{curl}_{\partial\Omega}\boldsymbol{e}_{\mathsf{t}}, \tag{4.30}$$

and consider the corresponding $\boldsymbol{H}(\mathrm{div})$-extension $\boldsymbol{H} = \mathcal{L}_{\mathsf{n}}h \in \boldsymbol{H}(\mathrm{div}, \Omega)$. Since the average of h on $\partial\Omega$ is zero, the divergence of $\boldsymbol{H} = \mathcal{L}_{\mathsf{n}}h$ is zero.

We then use the Poincaré map $\boldsymbol{K}$ to pull function $\boldsymbol{H}$ back into space $\boldsymbol{H}(\mathbf{curl}, \Omega)$,

$$\boldsymbol{E}_0 = \boldsymbol{K}\boldsymbol{H}. \tag{4.31}$$

It follows from the continuity of map $\boldsymbol{K}$ that $\boldsymbol{E}_0$ is bounded in the $\boldsymbol{H}(\mathbf{curl}, \Omega)$-norm by the $\boldsymbol{H}(\mathrm{div}, \Omega)$-norm of $\boldsymbol{H}$ and, in turn, by the $\boldsymbol{H}^{-\frac{1}{2}}(\mathrm{curl}, \partial\Omega)$-norm of data $\boldsymbol{e}_{\mathsf{t}}$. Since $\operatorname{div}\boldsymbol{H} = 0$, by the fundamental property (2.15) of the Poincaré map we find $\operatorname{\mathbf{curl}}\boldsymbol{E}_0 = \boldsymbol{H}$ and, hence,

$$\operatorname{curl}\gamma_{\mathsf{t}}\boldsymbol{E}_0 = \gamma_{\mathsf{n}}\operatorname{\mathbf{curl}}\boldsymbol{E}_0 = \gamma_{\mathsf{n}}\boldsymbol{H} = h. \tag{4.32}$$

After subtracting from $\boldsymbol{e}_{\mathsf{t}}$ the tangential trace of $\boldsymbol{E}_0$,

$$\boldsymbol{f}_t = \boldsymbol{e}_{\mathsf{t}} - \gamma_{\mathsf{t}}\boldsymbol{E}_0, \tag{4.33}$$

the resulting functional $\boldsymbol{f}_t$ has zero surface curl and it can be identified as the surface gradient of a potential $u \in H^{\frac{1}{2}}_{avg}(\partial\Omega)$,

$$\boldsymbol{f}_t = \boldsymbol{\nabla}_{\partial\Omega}u. \tag{4.34}$$

The potential u can now be extended to $U \in H^1(\Omega)$ using the H^1-extension operator $\mathcal{L}_0$, and the final $\boldsymbol{H}(\mathbf{curl})$-extension is defined by summing up the two contributions,

$$\boldsymbol{E} = \boldsymbol{\nabla}U + \boldsymbol{K}\boldsymbol{H}. \tag{4.35}$$

Remark 4.3. The use of the Poincaré map $\boldsymbol{K}$ introduces a "noncanonical" element into this construction. We can get a canonical construction by replacing $\boldsymbol{E}_0 = \boldsymbol{K}\boldsymbol{H}$ in (4.31) by $\widetilde{\boldsymbol{E}}_0$ defined by

$$\widetilde{\boldsymbol{E}}_0 = \boldsymbol{K}\boldsymbol{H} + \boldsymbol{\nabla}U_0 \quad \text{with} \quad U_0 \in H^1_0(\Omega) \quad \text{such that} \quad \|\widetilde{\boldsymbol{E}}_0\|_{L^2(\Omega)} \quad \text{is minimal.} \tag{4.36}$$

The $\boldsymbol{H}(\operatorname{curl},\Omega)$ norm of $\widetilde{\boldsymbol{E}}_0$ is not larger than the one of $\boldsymbol{E}_0$, and the lift $\widetilde{\boldsymbol{E}}$ constructed in this way satisfies the orthogonality conditions

$$\int_\Omega \operatorname{\mathbf{curl}}\widetilde{\boldsymbol{E}}\cdot\operatorname{\mathbf{curl}}\boldsymbol{F} = 0 \quad \forall\boldsymbol{F} \in \boldsymbol{H}_0(\mathbf{curl},\Omega) \quad \text{and} \quad \int_\Omega \widetilde{\boldsymbol{E}}\cdot\boldsymbol{\nabla}V = 0 \quad \forall V \in H^1_0(\Omega). \tag{4.37}$$

It is easy to see that these two orthogonality relations determine $\widetilde{\boldsymbol{E}}$ uniquely. □

Conclusions in 3D continuous case. The lift operators $\mathcal{L}_0$, $\mathcal{L}_{\mathsf{t}}$ and $\mathcal{L}_{\mathsf{n}}$ satisfy the following exact sequence and commuting diagram properties:

$$\begin{array}{ccccccc}
H^1(\Omega) & \xrightarrow{\boldsymbol{\nabla}} & \boldsymbol{H}(\operatorname{curl},\Omega) & \underset{\boldsymbol{K}}{\overset{\mathbf{curl}}{\rightleftarrows}} & \boldsymbol{H}(\operatorname{div},\Omega) & \xrightarrow{\operatorname{div}} & L^2(\Omega) \\
\gamma_0\Big\downarrow\Big\uparrow\mathcal{L}_0 & & \gamma_{\mathsf{t}}\Big\downarrow\Big\uparrow\mathcal{L}_{\mathsf{t}} & & \gamma_{\mathsf{n}}\Big\downarrow\Big\uparrow\mathcal{L}_{\mathsf{n}} & & \gamma_{avg}\Big\downarrow\Big\uparrow\mathcal{L}_{avg} \\
H^{\frac{1}{2}}(\partial\Omega) & \xrightarrow{\boldsymbol{\nabla}} & H^{-\frac{1}{2}}(\mathbf{curl},\partial\Omega) & \xrightarrow{\operatorname{curl}} & H^{-\frac{1}{2}}(\partial\Omega) & \xrightarrow{\gamma_{avg}} & \mathbb{R}
\end{array} \tag{4.38}$$

The operator $\mathcal{L}_0$ is uniquely determined by the condition that $\Delta\circ\mathcal{L}_0$ is zero, and $\mathcal{L}_{\mathsf{n}}$ by the conditions that $\operatorname{\mathbf{curl}}\circ\mathcal{L}_{\mathsf{n}}$ is zero and that $\operatorname{div}\circ\mathcal{L}_{\mathsf{n}}$ takes its values in $\mathbb{R}$. □

5. H^1 POLYNOMIAL EXTENSION OPERATOR IN 3D

5.1. **Additional 1D polynomials.** In this section, we use the notation introduced in Section 3.2 for the discrete 1D Dirichlet eigenpairs $(\Phi_n, \lambda_n) = (\Phi_n^{(p)}, \lambda_n^{(p)})$ and for the discrete 1D Neumann eigenpairs $(\Psi_n, \mu_n) = (\Psi_n^{(p)}, \mu_n^{(p)})$. On top of the polynomials discussed in Section 3.2, we need now discrete counterparts of functions $\beta_{nm}^{\lambda\lambda}, \beta_{nm}^{\mu\lambda}$ and $\beta_{nm}^{\mu\mu}$. For each pair of discrete Neumann eigenvalues $\mu_i^{(p)}$ and $\mu_j^{(p)}$, we introduce the corresponding polynomial $\beta_{ij}^{\mu\mu}$ that solves the following 1D variational problem, compare with (3.32),

$$\left\{ \begin{array}{l} \beta_{ij}^{\mu\mu} \in \mathbb{P}^p,\ \beta_{ij}^{\mu\mu}(-1) = 0,\ \beta_{ij}^{\mu\mu}(1) = 1 \\ \displaystyle\int_I (\beta_{ij}^{\mu\mu})' v' + (\mu_i^{(p)} + \mu_j^{(p)}) \int_I \beta_{ij}^{\mu\mu}\, v = 0, \quad \forall v \in \mathbb{P}_0^p \end{array} \right. \qquad (i, j, = 0, \dots, p). \tag{5.1}$$

In an analogous way, for each couple of Dirichlet eigenvalues, we introduce the corresponding functions $\beta_{ij}^{\lambda\lambda}, i, j = 2, \dots, p$, and then functions $\beta_{ij}^{\mu\lambda}, i = 0, \dots, p,\ j = 2, \dots, p$, corresponding to pairs $(\mu_i^{(p)}, \lambda_j^{(p)})$.

Lemma 5.1. *There exists a constant $C > 0$, independent of p, such that*

$$\begin{aligned} |\beta_{ij}^{\lambda\lambda}|^2_{H^1(I)} + (\lambda_i^{(p)} + \lambda_j^{(p)}) \|\beta_{ij}^{\lambda\lambda}\|^2_{L^2(I)} &\le C \left(\lambda_i^{(p)} + \lambda_j^{(p)}\right)^{\frac{1}{2}}, \\ &\qquad i = 2, \dots, p, \quad j = 2, \dots, p, \\ |\beta_{ij}^{\mu\mu}|^2_{H^1(I)} + (\mu_i^{(p)} + \mu_j^{(p)}) \|\beta_{ij}^{\mu\mu}\|^2_{L^2(I)} &\le C \left(\mu_i^{(p)} + \mu_j^{(p)}\right)^{\frac{1}{2}}, \\ &\qquad i, j = 0, \dots, p, \quad (i, j) \ne (0, 0), \\ |\beta_{ij}^{\mu\lambda}|^2_{H^1(I)} + (\mu_i^{(p)} + \lambda_j^{(p)}) \|\beta_{ij}^{\mu\lambda}\|^2_{L^2(I)} &\le C \left(\mu_i^{(p)} + \lambda_j^{(p)}\right)^{\frac{1}{2}}, \\ &\qquad i = 0, \dots, p, \quad j = 2, \dots, p. \end{aligned}$$

Proof. The proof is fully analogous to the proof of (3.34). □

5.2. **Step 1: Lifting from a horizontal face.** Let $u \in W_p(\partial\Omega)$ be a polynomial trace; cf. (2.12). We first consider the horizontal face $z = 1$ of the master cube $\Omega = I^3$; see Fig. 2. Let $\underline{u}_2(x, y)$ be the restriction of u on this face. We expand it into the discrete 1D Neumann eigenvectors,

$$\underline{u}_2(x, y) = \sum_{i=0}^{p} \sum_{j=0}^{p} u_{ij} \Psi_i(x) \Psi_j(y)\,, \tag{5.2}$$

and define the extension $U = U_2$ as follows:

$$U_2(x, y, z) = \sum_{i=0}^{p} \sum_{j=0}^{p} u_{ij} \Psi_i(x) \Psi_j(y) \beta_{ij}^{\mu\mu}(z). \tag{5.3}$$

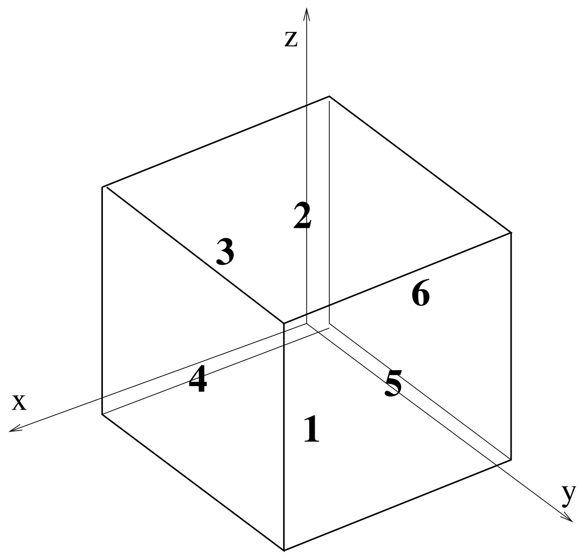

FIGURE 2. Master cube element. Enumeration of faces. Face numbers are located in the middle of each face. The faces are parametrized with the corresponding master element coordinates in the lexicographic order, i.e., faces $\mathsf{f}_1, \mathsf{f}_2$ are parametrized with x, y, faces $\mathsf{f}_3, \mathsf{f}_5$ are parametrized with x, z, and faces $\mathsf{f}_4, \mathsf{f}_6$ are parametrized with y, z.

A straightforward evaluation of the H^1-norm gives (we write μ_i, μ_j for $\mu_i^{(p)}$, $\mu_j^{(p)}$)
(5.4)

$$\begin{aligned}
\int_\Omega |\boldsymbol{\nabla} U|^2 &= \sum_{i=0}^{p}\sum_{j=0}^{p} |u_{ij}|^2 \Big((\mu_i+\mu_j)\|\beta_{ij}^{\mu\mu}\|^2_{L^2(\Omega)} + |\beta_{ij}^{\mu\mu}|^2_{H^1(I)} \Big) \\
&= |u_{00}|^2 |\beta_{00}^{\mu\mu}|^2_{H^1(I)} + \sum_{(i,j)\neq(0,0)} |u_{ij}|^2 \Big((\mu_i+\mu_j)\|\beta_{ij}^{\mu\mu}\|^2_{L^2(\Omega)} + |\beta_{ij}^{\mu\mu}|^2_{H^1(I)} \Big) \\
&\le 2|u_{00}|^2 + C \sum_{(i,j)\neq(0,0)} |u_{ij}|^2 (\mu_i+\mu_j)^{\frac12}, \\
\int_\Omega |U|^2 &= \sum_{i=0}^{p}\sum_{j=0}^{p} |u_{ij}|^2 \|\beta_{ij}^{\mu\mu}\|^2_{L^2(I)} \\
&\le C \sum_{i=0}^{p}\sum_{j=0}^{p} |u_{ij}|^2 .
\end{aligned}$$

In the last line we have used the fact that all discrete eigenvalues $\mu_i, i > 0$ are greater than the exact eigenvalues and, therefore, are uniformly bounded away from zero, which, by Lemma 5.1 implies that the corresponding L^2-norms $\|\beta_{ij}^{\mu\mu}\|^2_{L^2(I)}$ are uniformly bounded. In summary, we have obtained the estimate

$$\|U\|^2_{H^1(\Omega)} \le C \sum_{i=0}^{p}\sum_{j=0}^{p} |u_{ij}|^2 \Big(1 + (\mu_i+\mu_j)^{\frac12} \Big) . \tag{5.5}$$

Lemma 5.2. *The discrete norm on the right-hand side of* (5.5) *is equivalent to the standard* $H^{\frac{1}{2}}$*-norm on the face.*

Proof. The standard L^2- and H^1-norms are equal to the corresponding discrete norms. Thus, by the standard interpolation argument, the continuous $H^{\frac{1}{2}}$-norm must be bounded by the discrete $H^{\frac{1}{2}}$-norm. On the other side, by Proposition 3.6, we have

$$\sum_{i=0}^{p}\sum_{j=0}^{p}|u_{ij}|^2\left(1+\mu_i^{\frac{1}{2}}\right)\le C\|u\|^2_{L^2(I_y,H^{\frac{1}{2}}(I_x))}, \tag{5.6}$$

and, by the same argument,

$$\sum_{i=0}^{p}\sum_{j=0}^{p}|u_{ij}|^2\left(1+\mu_j^{\frac{1}{2}}\right)\le C\|u\|^2_{L^2(I_x,H^{\frac{1}{2}}(I_y))}. \tag{5.7}$$

Summing the last two inequalities we get

$$\begin{aligned}&\sum_{i=0}^{p}\sum_{j=0}^{p}|u_{ij}|^2\left(1+(\mu_i+\mu_j)^{\frac{1}{2}}\right)\\ &\quad\le C(\|u\|^2_{L^2(I_x,H^{\frac{1}{2}}(I_y))}+\|u\|^2_{L^2(I_y,H^{\frac{1}{2}}(I_x))})\approx\|u\|^2_{H^{\frac{1}{2}}(I^2)}.\end{aligned} \tag{5.8}$$ $\square$

In conclusion, the H^1-norm of extension $U=U_2$ is bounded by the $H^{\frac{1}{2}}$-norm of the data u. In exactly the same way, we construct an extension U_1 from the lower face $z=-1$. Notice that both extensions are zero on the opposite face and, therefore, do not alter the original values there.

5.3. **Lifting from the vertical faces.** We proceed now along the lines of the construction on the continuous level discussed in Section 4.1. Let U_1 and U_2 denote the extensions from the lower and the upper horizontal faces, respectively. We consider the trace

$$v=v(x,z)=(u-U_1-U_2)(x,1,z). \tag{5.9}$$

Function $u-U_1-U_2$ vanishes over the horizontal faces and, therefore, the trace v is bounded in the norm of the $H^{\frac{1}{2}}(I_x,L^2(I_z))\cap L^2(I_x,\widetilde{H}^{\frac{1}{2}}(I_z))$ space. We expand function v now in the products of discrete Neumann and Dirichlet eigenvectors,

$$v(x,z)=\sum_{i=0}^{p}\sum_{j=2}^{p}v_{ij}\Psi_i(x)\Phi_j(z)\ , \tag{5.10}$$

and define extension $U=U_5$ as

$$U_5(x,y,z)=\sum_{i=0}^{p}\sum_{j=2}^{p}v_{ij}\Psi_i(x)\beta_{ij}^{\mu\lambda}(y)\Phi_j(z). \tag{5.11}$$

Following the same steps as in the previous section, we demonstrate that the H^1-norm of extension U_5 is bounded by the $H^{\frac{1}{2}}(I_x,L^2(I_z))\cap L^2(I_x,\widetilde{H}^{\frac{1}{2}}(I_z))$-norm of trace v and, consequently, by the $H^{\frac{1}{2}}$-norm of the trace u on the whole boundary. Then, in the same way, we construct lift U_3 from face $y=-1$.

Having constructed the lifts U_3 and U_5 from the third and the fifth face, we subtract them from $u - U_1 - U_2$ and consider the trace of the remaining function on face $x = 1$,

$$w(y, z) = (u - U_1 - U_2 - U_3 - U_5)(1, y, z). \tag{5.12}$$

As function $u - U_1 - U_2 - U_3 - U_5$ vanishes over faces 1, 2, 3 and 5, the trace v is bounded in the $\widetilde{H}^{\frac{1}{2}}$-norm. We expand it into the discrete Dirichlet eigenvectors,

$$w(x, z) = \sum_{i=2}^{p} \sum_{j=2}^{p} w_{ij} \Phi_i(y) \Phi_j(z) \,, \tag{5.13}$$

and define the extension as

$$U_4(x, y, z) = \sum_{i=2}^{p} \sum_{j=2}^{p} w_{ij} \beta_{ij}^{\lambda\lambda}(x) \Phi_i(y) \Phi_j(z) \,. \tag{5.14}$$

Again, following the same steps as in the previous section, we demonstrate that the H^1-norm of extension U_4 is bounded by the $\widetilde{H}^{\frac{1}{2}}(I^2)$-norm of trace v and, consequently, by the $H^{\frac{1}{2}}$-norm of the trace u on the whole boundary. Then, in the same way, we construct the lift U_6 from face $x = -1$. The final extension is defined as the sum of the contributions from the six faces, and we have proved

Theorem 5.3. *On the cube $\Omega = I^3$, the operator*

$$\mathcal{L}_0^{(p)} : W_p(\partial\Omega) \ni u \longmapsto U := \sum_{j=1}^{6} U_j \in W_p(\Omega) \tag{5.15}$$

defines a polynomial lift of traces. Its norm from $H^{\frac{1}{2}}(\partial\Omega)$ into $H^1(\Omega)$ is bounded independently of the degree p.

6. $\boldsymbol{H}(\mathrm{div})$ polynomial extension operator in 3D

Before we proceed along the lines outlined in Section 4.2, we make one important modification. Motivated with relations (4.28) between functions γ and β on the continuous level, we shall replace all functions γ (that have not been defined on the discrete level at all) with derivatives of the corresponding functions β and, similarly, express 1D Neumann eigenvectors with the derivatives of Dirichlet eigenvectors, and 1D Dirichlet eigenvectors with the derivatives of Neumann eigenvectors. Throughout this section, we use the condensed notation (Φ_n, λ_n) and (Ψ_n, μ_n) for the discrete 1D Dirichlet and Neumann eigenpairs introduced in Section 3.2.

6.1. **Step 1: Lifting from a horizontal face.** Let h be a polynomial trace in the trace space $V_p(\partial\Omega)$; see (2.12). We begin with the restriction $\underline{h}_2$ of function h to the second face. This restriction belongs to $\mathbb{P}^{p-1} \otimes \mathbb{P}^{p-1}$. We expand it in terms of derivatives of the discrete 1D Neumann eigenvectors,

$$\underline{h}_2(x, y) = \sum_{n=1}^{p} \sum_{m=1}^{p} h_{nm} \frac{\Psi_n'(x)}{\mu_n^{\frac{1}{2}}} \frac{\Psi_m'(y)}{\mu_m^{\frac{1}{2}}}. \tag{6.1}$$

Notice that the functions $\mu_n^{-\frac{1}{2}}\,\Psi'_n$ are L^2-orthonormal. The vector-valued extension $\boldsymbol{H} = \boldsymbol{H}_2$ is now defined as follows:

$$\boldsymbol{H}_2 = \Bigg(-\sum_{n=1}^{p}\sum_{m=1}^{p} h_{nm}\,\mu_n^{\frac{1}{2}}\Psi_n(x)\,\frac{1}{\mu_m^{\frac{1}{2}}}\Psi'_m(y)\,\frac{1}{\mu_n+\mu_m}\big(\beta_{nm}^{\mu\mu}\big)'(z), \quad -\sum_{n=1}^{p}\sum_{m=1}^{p} h_{nm}\,\frac{1}{\mu_n^{\frac{1}{2}}}\Psi'_n(x)\,\mu_m^{\frac{1}{2}}\Psi_m(y)\,\frac{1}{\mu_n+\mu_m}\big(\beta_{nm}^{\mu\mu}\big)'(z), \quad \sum_{n=1}^{p}\sum_{m=1}^{p} h_{nm}\,\frac{1}{\mu_n^{\frac{1}{2}}}\Psi'_n(x)\,\frac{1}{\mu_m^{\frac{1}{2}}}\Psi'_m(y)\,\beta_{nm}^{\mu\mu}(z)\Bigg). \tag{6.2}$$

Notice a direct correspondence of Formula (6.2) with the gradient of function (4.15) with the replacements outlined in the beginning of this section.

A direct calculation reveals that polynomial $\boldsymbol{H}_2$ is divergence-free. The $\boldsymbol{H}(\mathrm{div})$-norm of function $\boldsymbol{H}_2$ reduces then to its L^2-norm, which we now calculate. Utilizing the L^2-orthogonality of eigenvectors Ψ_n and their derivatives, we obtain

$$\|\boldsymbol{H}_2\|^2_{L^2(\Omega)} = \sum_{n=1}^{p}\sum_{m=1}^{p} |h_{nm}|^2 \left(\frac{1}{\mu_n+\mu_m}\|\big(\beta_{nm}^{\mu\mu}\big)'\|^2_{L^2(I)} + \|\beta_{nm}^{\mu\mu}\|^2_{L^2(I)}\right). \tag{6.3}$$

Making use of Lemma 5.1, we obtain the bound

$$\|\boldsymbol{H}_2\|^2_{L^2(\Omega)} \le C\sum_{n=1}^{p}\sum_{m=1}^{p} |h_{nm}|^2\,(\mu_n+\mu_m)^{-\frac{1}{2}}. \tag{6.4}$$

It remains to show now that the weighted sum on the right-hand side is equivalent to the $H^{-\frac{1}{2}}$-norm of data h_2. We begin with a simple result concerning the $H^{-\frac{1}{2}}$-norm in one space dimension.

Lemma 6.1. *Let $w \in \mathbb{P}^{p-1}$ be expanded in terms of the derivatives of discrete Neumann eigenvectors,*

$$w = \sum_{n=1}^{p} w_n \frac{\Psi'_n}{\mu_n^{\frac{1}{2}}}. \tag{6.5}$$

The following discrete norm is equivalent to the $H^{-\frac{1}{2}}$-norm of w with equivalence constants independent of the degree p,

$$\|w\|^2_{H^{-\frac{1}{2}}(I)} \approx \sum_{n=1}^{p} |w_n|^2 \mu_n^{-\frac{1}{2}}. \tag{6.6}$$

Proof. The proof follows from the isomorphism,

$$\partial \;:\; H^{\frac{1}{2}}_{avg}(I) \to H^{-\frac{1}{2}}(I) \tag{6.7}$$

and Proposition 3.6. □

We can prove now a similar result for two space dimensions.

Lemma 6.2. *Let polynomial $h \in \mathbb{P}^{p-1}\otimes\mathbb{P}^{p-1}$ be expanded in terms of the derivatives of the discrete Neumann eigenvectors as in Formula* (6.1). *The discrete norm* (6.4) *is equivalent to the $H^{-\frac{1}{2}}$-norm of h with equivalence constants independent of the degree p.*

Proof. We begin by recalling the standard tensorization result for the space $H^{-\frac{1}{2}}(I^2)$,

$$H^{-\frac{1}{2}}(I^2) = L^2(I_x, H^{-\frac{1}{2}}(I_y)) + L^2(I_y, H^{-\frac{1}{2}}(I_x)), \tag{6.8}$$

where

$$L^2(I_x, H^{-\frac{1}{2}}(I_y)) := \{h(x,y) \ : \ \int_I \|h(x,\cdot)\|^2_{H^{-\frac{1}{2}}(I)}\, dx < \infty\} \tag{6.9}$$

with an analogous definition for the second space. The norm in the sum of two normed spaces can be defined as follows:

$$V = X + Y, \quad \|v\|_{X+Y} = \inf_{x+y=v} \left(\|x\|^2_X + \|y\|^2_Y\right)^{\frac{1}{2}}. \tag{6.10}$$

Lemma 6.1 implies that

$$\begin{aligned} \|h\|^2_{L^2(I_x,H^{-\frac{1}{2}}(I_y))} &\approx \sum_{n=1}^{p}\sum_{m=1}^{p} |h_{nm}|^2 \mu_m^{-\frac{1}{2}}, \\ \|h\|^2_{L^2(I_y,H^{-\frac{1}{2}}(I_x))} &\approx \sum_{n=1}^{p}\sum_{m=1}^{p} |h_{nm}|^2 \mu_n^{-\frac{1}{2}}. \end{aligned} \tag{6.11}$$

Finally, definition (6.10) and elementary algebraic arguments lead to

$$\begin{aligned} \|h\|^2_{H^{-\frac{1}{2}}(I^2)} &= \inf_{u_{nm}+v_{nm}=h_{nm}} \left(\sum_{n=1}^{p}\sum_{m=1}^{p} |u_{nm}|^2 \mu_m^{-\frac{1}{2}} + \sum_{n=1}^{p}\sum_{m=1}^{p} |v_{nm}|^2 \mu_n^{-\frac{1}{2}} \right) \\ &\approx \sum_{n=1}^{p}\sum_{m=1}^{p} |h_{nm}|^2 \min\{\mu_m^{-\frac{1}{2}},\ \mu_n^{-\frac{1}{2}}\} \\ &\approx \sum_{n=1}^{p}\sum_{m=1}^{p} |h_{nm}|^2 (\mu_n + \mu_m)^{-\frac{1}{2}}. \end{aligned} \tag{6.12}$$

Here sign $A \approx B$ indicates the existence of constants C_1, C_2 independent of the function h and the polynomial degree p, such that $A \le C_1 B$ and $B \le C_2 A$. $\square$

We have demonstrated therefore the continuity of the lift operator,

$$\|\boldsymbol{H}\|_{\boldsymbol{H}(\mathrm{div},\Omega)} \le C \|h_2\|_{H^{-\frac{1}{2}}(\mathsf{f}_2)}. \tag{6.13}$$

In the same way we construct then the lift from the bottom face f_1.

6.2. **Step 2: Lifting from faces** f_3, f_5**.** We mimic the construction on the continuous level. First normal traces of lifts $\boldsymbol{H}_1, \boldsymbol{H}_2$ are subtracted from the original data,

$$g := h - \gamma_{\mathsf{n}}(\boldsymbol{H}_1 + \boldsymbol{H}_2). \tag{6.14}$$

Restriction $\underline{g}_5$ of polynomial g to face f_5 is expanded in terms of derivatives of the discrete 1D Neumann and Dirichlet eigenvectors,

$$\underline{g}_5(x,z) = \sum_{n=1}^{p} g_{n1} \frac{\Psi_n'(x)}{\mu_n^{\frac{1}{2}}} \frac{1}{\sqrt{2}} + \sum_{n=1}^{p}\sum_{m=2}^{p} g_{nm} \frac{\Psi_n'(x)}{\mu_n^{\frac{1}{2}}} \frac{\Phi_m'(z)}{\lambda_m^{\frac{1}{2}}}. \tag{6.15}$$

Notice that the functions $\lambda_m^{-\frac{1}{2}}\,\Phi'_m,\ m=2,\dots,p$, complemented with constant $1/\sqrt{2}$, are L^2-orthonormal. The vector-valued extension $\boldsymbol{H}=\boldsymbol{H}_5$ is now defined as follows:

(6.16)

$$\begin{aligned}\boldsymbol{H}_5=\Bigg(&-\sum_{n=1}^{p} g_{n1}\,\mu_n^{\frac{1}{2}}\Psi_n(x)\frac{\left(\beta_n^{\mu}\right)'(y)}{\mu_n}\frac{1}{\sqrt{2}}-\sum_{n=1}^{p}\sum_{m=2}^{p} g_{nm}\,\mu_n^{\frac{1}{2}}\Psi_n(x)\frac{\left(\beta_{nm}^{\mu\lambda}\right)'(y)}{\mu_n+\lambda_m}\,\frac{\Phi'_m(z)}{\lambda_m^{\frac{1}{2}}},\\&\sum_{n=1}^{p} g_{n1}\,\frac{1}{\mu_n^{\frac{1}{2}}}\Psi'_n(x)\,\beta_n^{\mu}(y)\,\frac{1}{\sqrt{2}}\,+\,\sum_{n=1}^{p}\sum_{m=2}^{p} g_{nm}\,\frac{1}{\mu_n^{\frac{1}{2}}}\Psi'_n(x)\,\beta_{nm}^{\mu\lambda}(y)\,\frac{\Phi'_m(z)}{\lambda_m^{\frac{1}{2}}},\\&-\sum_{n=1}^{p}\sum_{m=2}^{p} g_{nm}\,\frac{1}{\mu_n^{\frac{1}{2}}}\Psi'_n(x)\,\frac{\left(\beta_{nm}^{\mu\lambda}\right)'(y)}{\mu_n+\lambda_m}\,\lambda_m^{\frac{1}{2}}\Phi_m(z).\Bigg)\end{aligned}$$

A direct computation reveals that $\boldsymbol{H}_5$ is divergence-free. We proceed with the evaluation of the L^2-norm,

$$\begin{aligned}\|\boldsymbol{H}_5\|^2_{L^2(\Omega)}\quad&=\sum_{n=1}^{p}|g_{n1}|^2\left(\frac{1}{\mu_n}\|(\beta_n^{\mu})'\|^2_{L^2(I)}+\|\beta_n^{\mu}\|^2_{L^2(I)}\right)\\&+\sum_{n=1}^{p}\sum_{m=2}^{p}|g_{nm}|^2\left(\frac{1}{\mu_n+\lambda_m}\|(\beta_{nm}^{\mu\lambda})'\|^2_{L^2(I)}+\|\beta_{nm}^{\mu\lambda}\|^2_{L^2(I)}\right)\\&\le C\left(\sum_{n=1}^{p}|g_{n1}|^2\mu_n^{-\frac{1}{2}}+\sum_{n=1}^{p}\sum_{m=2}^{p}|g_{nm}|^2(\mu_n+\lambda_m)^{-\frac{1}{2}}\right)\end{aligned}\tag{6.17}$$

where, in the last inequality, we have used Lemma 3.5 and Lemma 5.1. We continue now with a result concerning the $\widetilde{H}^{-\frac{1}{2}}$-norm in one space dimension.

Lemma 6.3. *Let $w\in\mathbb{P}^{p-1}$ be expanded in terms of the derivatives of discrete Dirichlet eigenvectors and a constant function,*

$$w=w_1\frac{1}{\sqrt{2}}+\sum_{m=2}^{p}w_m\frac{\Phi'_m}{\lambda_m^{\frac{1}{2}}}.\tag{6.18}$$

The following discrete norm is equivalent to the $\widetilde{H}^{-\frac{1}{2}}$-norm of w with equivalence constants independent of the degree p,

$$\|w\|^2_{\widetilde{H}^{-\frac{1}{2}}(I)}\approx|w_1|^2+\sum_{m=2}^{p}|w_m|^2\lambda_m^{-\frac{1}{2}}.\tag{6.19}$$

Proof. The proof follows from the stable decomposition

$$\widetilde{H}^{-\frac{1}{2}}(I)=\mathbb{C}\oplus\widetilde{H}^{-\frac{1}{2}}_{avg}(I),\quad w=w_0+(w-w_0),\ w_0=\langle w,1\rangle/2,\tag{6.20}$$

the isomorphism

$$\partial\ :\ \widetilde{H}^{\frac{1}{2}}(I)\to\widetilde{H}^{-\frac{1}{2}}_{avg}(I),\tag{6.21}$$

and Proposition 3.7. □

We have now a discrete version of the norm in $H^{-\frac{1}{2}}(I_x,L^2(I_z))+L^2(I_x,\widetilde{H}^{-\frac{1}{2}}(I_z))$.

Lemma 6.4. *Let $h\in\mathbb{P}^{p-1}\otimes\mathbb{P}^{p-1}$ be a polynomial expanded in terms of derivatives of discrete Neumann and Dirichlet eigenvectors as in Formula* (6.15). *The* *following discrete norm is equivalent to the norm in the space* $H^{-\frac{1}{2}}(I_x, L^2(I_z)) + L^2(I_x, \widetilde{H}^{-\frac{1}{2}}(I_z))$ *with equivalence constants independent of the polynomial degree* p*:*

$$\|h\|^2 = \sum_{n=1}^{p} |g_{n1}|^2 \mu_n^{-\frac{1}{2}} + \sum_{n=1}^{p}\sum_{m=1}^{p} |g_{nm}|^2 (\mu_n + \lambda_m)^{-\frac{1}{2}}. \tag{6.22}$$

Proof. Lemma 6.3 implies that

$$\|h\|^2_{L^2(I_x, \widetilde{H}^{-\frac{1}{2}}(I_z))} \approx \sum_{n=1}^{p} |g_{n1}|^2 + \sum_{n=1}^{p}\sum_{m=2}^{p} |g_{nm}|^2 \lambda_m^{-\frac{1}{2}} \tag{6.23}$$

and from Lemma 6.1 it follows that

$$\|h\|^2_{L^2(I_z, H^{-\frac{1}{2}}(I_x))} \approx \sum_{n=1}^{p} |g_{n1}|^2 \mu_n^{-\frac{1}{2}} + \sum_{n=1}^{p}\sum_{m=2}^{p} |g_{nm}|^2 \mu_n^{-\frac{1}{2}}. \tag{6.24}$$

As discrete eigenvalues, the μ_n are always greater than their exact counterparts, and only a finite number of Neumann eigenvalues is less than one. We have

$$\begin{aligned} \|h\|^2 &\approx \sum_{n=1}^{p} |g_{n1}|^2 \min\{1, \mu_n^{-\frac{1}{2}}\} + \sum_{n=1}^{p}\sum_{m=2}^{p} |g_{nm}|^2 \min\{\mu_n^{-\frac{1}{2}}, \lambda_m^{-\frac{1}{2}}\} \\ &\approx \sum_{n=1}^{p} |g_{n1}|^2 \mu_n^{-\frac{1}{2}} + \sum_{n=1}^{p}\sum_{m=2}^{p} |g_{nm}|^2 (\mu_n + \lambda_m)^{-\frac{1}{2}}, \end{aligned} \tag{6.25}$$

which ends the proof. □

Thus, we have shown that the $\boldsymbol{H}(\mathrm{div}, \Omega)$-norm of extension $\boldsymbol{H}_5$ is bounded by the norm of the face data $\underline{g}_5$ in the space $H^{-\frac{1}{2}}(I_x, L^2(I_z)) + L^2(I_x, \widetilde{H}^{-\frac{1}{2}}(I_z))$, and in turn, by the $H^{-\frac{1}{2}}(\partial\Omega)$-norm of the original boundary data h. The lift from face f_3 is constructed in the same way.

6.3. **Step 3: Lifting from faces f_4, f_6.** We again mimic the construction of the continuous level. Normal traces of lifts $\boldsymbol{H}_1, \boldsymbol{H}_2, \boldsymbol{H}_3, \boldsymbol{H}_5$ are subtracted from the original data,

$$f := h - \gamma_{\mathsf{n}}(\boldsymbol{H}_1 + \boldsymbol{H}_2 + \boldsymbol{H}_3 + \boldsymbol{H}_5). \tag{6.26}$$

The restriction $\underline{f}_4$ of polynomial f to face f_4 is expanded in terms of derivatives of the discrete 1D Dirichlet eigenvectors and constant functions,

$$\begin{aligned} \underline{f}_4(y,z) = f_{11} \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} &+ \sum_{n=2}^{p} f_{n1} \frac{\Phi_n'(y)}{\lambda_n^{\frac{1}{2}}} \frac{1}{\sqrt{2}} \\ &+ \sum_{m=2}^{p} f_{1m} \frac{1}{\sqrt{2}} \frac{\Phi_m'(z)}{\lambda_m^{\frac{1}{2}}} + \sum_{n=2}^{p}\sum_{m=2}^{p} f_{nm} \frac{\Phi_n'(y)}{\lambda_n^{\frac{1}{2}}} \frac{\Phi_m'(z)}{\lambda_m^{\frac{1}{2}}}. \end{aligned} \tag{6.27}$$

As $\underline{f}_4$ is now controlled in the $\widetilde{H}^{-\frac{1}{2}}(I^2)$-norm, the average value f_{11} of $\underline{f}_4$ depends continuously upon the norm, and the corresponding constant function is extended into the element using the lowest-order Raviart-Thomas shape function $\boldsymbol{H}_{\mathsf{RT},4}$. We proceed by defining extension $\boldsymbol{H} = \boldsymbol{H}_4$ of function (6.27) with the constant f_{11}

removed:

$$
\begin{aligned}
H_x = \quad & \sum_{n=2}^{p} f_{n1}\, \beta_n^{\lambda}(x)\, \frac{1}{\lambda_n^{\frac12}} \Phi_n'(y)\, \frac{1}{\sqrt{2}} + \sum_{m=2}^{p} f_{1m}\, \beta_m^{\lambda}(x)\, \frac{1}{\sqrt{2}}\, \frac{1}{\lambda_m^{\frac12}} \Phi_m'(z) \\
& + \sum_{n=2}^{p} \sum_{m=2}^{p} f_{nm}\, \beta_{nm}^{\lambda\lambda}(x)\, \frac{1}{\lambda_n^{\frac12}} \Phi_n'(y)\, \frac{1}{\lambda_m^{\frac12}} \Phi_m'(z), \\
H_y = \quad & -\sum_{n=2}^{p} f_{n1}\, \frac{1}{\lambda_n} \big(\beta_n^{\lambda}\big)'(x)\, \lambda_n^{\frac12} \Phi_n(y)\, \frac{1}{\sqrt{2}} \\
& - \sum_{n=2}^{p} \sum_{m=2}^{p} f_{nm}\, \frac{1}{\lambda_n + \lambda_m} \big(\beta_{nm}^{\lambda\lambda}\big)'(x)\, \lambda_n^{\frac12} \Phi_n(y)\, \frac{1}{\lambda_m^{\frac12}} \Phi_m'(z), \\
H_z = \quad & -\sum_{m=2}^{p} f_{1m}\, \frac{1}{\lambda_m} \big(\beta_m^{\lambda}\big)'(x)\, \frac{1}{\sqrt{2}}\, \lambda_m^{\frac12} \Phi_m(z) \\
& - \sum_{n=2}^{p} \sum_{m=2}^{p} f_{nm}\, \frac{1}{\lambda_n + \lambda_m} \big(\beta_{nm}^{\lambda\lambda}\big)'(x)\, \frac{1}{\lambda_n^{\frac12}} \Phi_n'(y)\, \lambda_m^{\frac12} \Phi_m(z).
\end{aligned}
$$

A direct computation shows again that $\operatorname{div}\boldsymbol{H} = 0$. We proceed with the evaluation of the L^2-norm,

(6.28)

$$
\begin{aligned}
\|\boldsymbol{H}\|^2_{L^2(\Omega)} &= \sum_{n=2}^{p} |f_{n1}|^2 \left(\frac{1}{\lambda_n} \| \left(\beta_n^{\mu}\right)' \|_{L^2(I)} + \|\beta_n^{\mu}\|_{L^2(I)} \right) \\
&\quad + \sum_{n=2}^{p} |f_{1m}|^2 \left(\frac{1}{\lambda_m} \| \left(\beta_m^{\mu}\right)' \|_{L^2(I)} + \|\beta_m^{\mu}\|_{L^2(I)} \right) \\
&\quad + \sum_{n=2}^{p} \sum_{m=2}^{p} |f_{nm}|^2 \left(\frac{1}{\lambda_n + \lambda_m} \| \left(\beta_{nm}^{\lambda\lambda}\right)' \|^2_{L^2(I)} + \|\beta_{nm}^{\lambda\lambda}\|^2_{L^2(I)} \right) \\
&\le C \left(\sum_{n=2}^{p} |f_{n1}|^2 \lambda_n^{-\frac12} + \sum_{m=2}^{p} |f_{1m}|^2 \lambda_m^{-\frac12} + \sum_{n=2}^{p} \sum_{m=2}^{p} |f_{nm}|^2 (\lambda_n + \lambda_m)^{-\frac12} \right),
\end{aligned}
$$

where, in the last inequality, we have used Lemma 3.5 and Lemma 5.1. We have

Lemma 6.5. *Let $h \in \mathbb{P}^{p-1} \otimes \mathbb{P}^{p-1}$ be a polynomial expanded in terms of derivatives of discrete Dirichlet eigenvectors:*

(6.29)

$$
h(y,z) = \sum_{n=2}^{p} f_{n1} \frac{\Phi_n'(y)}{\lambda_n^{\frac12}} \frac{1}{\sqrt{2}} + \sum_{m=2}^{p} f_{1m} \frac{1}{\sqrt{2}} \frac{\Phi_m'(z)}{\lambda_m^{\frac12}} + \sum_{n=2}^{p} \sum_{m=2}^{p} f_{nm} \frac{\Phi_n'(y)}{\lambda_n^{\frac12}} \frac{\Phi_m'(z)}{\lambda_m^{\frac12}}.
$$

The following discrete norm is equivalent to the norm in the space $\widetilde{H}^{-\frac12}(I^2)$ with equivalence constants independent of the polynomial degree p:

$$
\|h\|^2 = \sum_{n=2}^{p} |f_{n1}|^2 \lambda_n^{-\frac12} + \sum_{m=2}^{p} |f_{1m}|^2 \lambda_m^{-\frac12} + \sum_{n=2}^{p} \sum_{m=2}^{p} |f_{nm}|^2 (\lambda_n + \lambda_m)^{-\frac12}. \tag{6.30}
$$

Proof. The proof is based on the tensorization result,

$$
\widetilde{H}^{-\frac12}(I_y \times I_z) = L^2(I_y, \widetilde{H}^{-\frac12}(I_z)) + L^2(I_z, \widetilde{H}^{-\frac12}(I_y)). \tag{6.31}
$$

Lemma 6.3 implies that

$$
\|h\|^2_{L^2(I_y, \widetilde{H}^{-\frac12}(I_z))} \approx \sum_{n=2}^{p} |f_{n1}|^2 + \sum_{m=2}^{p} |f_{1m}|^2 \lambda_m^{-\frac12} + \sum_{n=2}^{p} \sum_{m=2}^{p} |f_{nm}|^2 \lambda_m^{-\frac12} \tag{6.32}
$$

and

$$\|h\|^2_{L^2(I_z,\widetilde{H}^{-\frac{1}{2}}(I_y))} \approx \sum_{n=2}^{p} |f_{n1}|^2 \lambda_n^{-\frac{1}{2}} + \sum_{m=2}^{p} |f_{1m}|^2 + \sum_{n=2}^{p}\sum_{m=2}^{p} |f_{nm}|^2 \lambda_n^{-\frac{1}{2}}. \tag{6.33}$$

We have

$$\begin{aligned}
\|h\|^2 &\approx \sum_{n=2}^{p} |f_{n1}|^2 \min\{1,\lambda_n^{-\frac{1}{2}}\} + \sum_{m=2}^{p} |f_{1m}|^2 \min\{1,\lambda_m^{-\frac{1}{2}}\} \\
&\qquad + \sum_{n=2}^{p}\sum_{m=2}^{p} |f_{nm}|^2 \min\{\lambda_n^{-\frac{1}{2}},\lambda_m^{-\frac{1}{2}}\} \\
&\approx \sum_{n=2}^{p} |f_{n1}|^2 \lambda_n^{-\frac{1}{2}} + \sum_{m=2}^{p} |f_{1m}|^2 \lambda_m^{-\frac{1}{2}} + \sum_{n=2}^{p}\sum_{m=2}^{p} |f_{nm}|^2 (\lambda_n+\lambda_m)^{-\frac{1}{2}},
\end{aligned} \tag{6.34}$$

which ends the proof. □

In the same way we define the lift from face f_6. Our final $\boldsymbol{H}(\mathrm{div},\Omega)$-extension is defined by summing all face contributions,

$$\boldsymbol{H} = \boldsymbol{H}_1 + \boldsymbol{H}_2 + \boldsymbol{H}_3 + f_{4,11}\boldsymbol{H}_{\mathsf{RT},4} + \boldsymbol{H}_4 + \boldsymbol{H}_5 + f_{6,11}\boldsymbol{H}_{\mathsf{RT},6} + \boldsymbol{H}_6, \tag{6.35}$$

where $f_{4,11}$ and $f_{6,11}$ are the average values of function (6.26) over faces f_4, f_6, and $\boldsymbol{H}_{\mathsf{RT},4}$, $\boldsymbol{H}_{\mathsf{RT},6} \in \boldsymbol{V}_1(\Omega)$ are the corresponding lowest-order Raviart-Thomas shape functions. We conclude with

Theorem 6.6. *Operator* (6.35)*,*

$$\mathcal{L}_{\mathsf{n}}^{(p)} : \; V_p(\partial\Omega) \ni h \longmapsto \boldsymbol{H} \in \boldsymbol{V}_p(\Omega), \tag{6.36}$$

defines a polynomial lift of normal traces. Its norm from $H^{-\frac{1}{2}}(\partial\Omega)$ into $\boldsymbol{H}(\mathrm{div},\Omega)$ is bounded independently of the polynomial degree p.

Remark 6.7. We can check that the lifting $\boldsymbol{H}$ is orthogonal to the curls of all fields with zero tangential trace $\boldsymbol{E} \in \boldsymbol{Q}_{p,0}(\Omega)$. Moreover the divergence of $\boldsymbol{H}$ is constant. Conversely, if $\boldsymbol{H} \in \boldsymbol{V}_p(\Omega)$ has these properties and a zero normal trace, it is zero. The proof uses similar arguments as previously. We deduce first that $\operatorname{div}\boldsymbol{H} = 0$. Therefore, there exists $\boldsymbol{E} \in \boldsymbol{Q}_p(\Omega)$ such that $\mathbf{curl}\,\boldsymbol{E} = \boldsymbol{H}$. Since $\gamma_{\mathsf{n}}\boldsymbol{H} = 0$, we deduce that $\mathbf{curl}\,\gamma_{\mathsf{t}}\boldsymbol{E} = 0$. Therefore there exists a surface potential $u \in W_p(\partial\Omega)$ such that $\boldsymbol{\nabla}_{\partial\Omega} u = \gamma_{\mathsf{t}}\boldsymbol{E}$. Setting $U = \mathcal{L}_0^{(p)} u$, we obtain that $\gamma_{\mathsf{t}}\boldsymbol{\nabla} U = \gamma_{\mathsf{t}}\boldsymbol{E}$. Finally

$$\boldsymbol{H} = \mathbf{curl}(\boldsymbol{E} - \boldsymbol{\nabla} U) \quad \text{with} \quad \boldsymbol{E} - \boldsymbol{\nabla} U \in \boldsymbol{Q}_{p,0}(\Omega).$$

The orthogonality condition against the curls of $\boldsymbol{Q}_{p,0}(\Omega)$ gives $\boldsymbol{H} = 0$. □

7. $\boldsymbol{H}$(curl) polynomial extension operator in 3D

Having constructed the polynomial extension operators for the spaces $H^1(\Omega)$ and $\boldsymbol{H}(\mathrm{div},\Omega)$, we proceed along exactly the same lines as for the continuous case discussed in Section 4.3. We consider the exact polynomial sequence (2.10). Given a polynomial trace $\boldsymbol{e}_{\mathsf{t}} \in \boldsymbol{Q}_p(\partial\Omega)$, we compute its surface curl,

$$h = \mathrm{curl}_{\partial\Omega}\, \boldsymbol{e}_{\mathsf{t}} \tag{7.1}$$

and use the $\boldsymbol{H}(\mathrm{div})$-extension operator $\mathcal{L}_{\mathsf{n}}^{(p)}$ to construct an $\boldsymbol{H}(\mathrm{div},\Omega)$-extension $\boldsymbol{H}$ of polynomial h. We then use the Poincaré map to "take out the curl" out of the data $\boldsymbol{e}_{\mathbf{t}}$ and conclude that the resulting function must be a surface gradient of a potential $u \in W_p(\partial\Omega)$ (with a zero average),

$$\boldsymbol{e}_{\mathbf{t}} - \gamma_{\mathbf{t}} \boldsymbol{K}\boldsymbol{H} = \boldsymbol{\nabla}_{\partial\Omega} u. \tag{7.2}$$

The $\boldsymbol{H}(\mathbf{curl}, \Omega)$-extension is now obtained by summing the gradient of the $H^1(\Omega)$-extension of the potential u with $\boldsymbol{K}\boldsymbol{H}$,

$$\boldsymbol{E} = \boldsymbol{\nabla} U + \boldsymbol{K}\boldsymbol{H}. \tag{7.3}$$

Replicating the procedure from the continuous level is possible because the Poincaré map is polynomial preserving. We conclude our construction with

Theorem 7.1. *Operator* (7.3),

$$\mathcal{L}_{\mathbf{t}}^{(p)} : \boldsymbol{Q}_p(\partial\Omega) \ni \boldsymbol{e}_{\mathbf{t}} \longmapsto \boldsymbol{E} \in \boldsymbol{Q}_p, \tag{7.4}$$

defines a polynomial lift of the tangential trace. Its norm from $H^{-\frac{1}{2}}(\mathrm{curl}, \partial\Omega)$ *into* $\boldsymbol{H}(\mathbf{curl}, \Omega)$ *is bounded independently of the polynomial degree* p.

Remark 7.2. As in the continuous case, see Remark 4.3, we can get a canonical construction by replacing the Poincaré map $\boldsymbol{K}$ by a p-depending operator $\widetilde{\boldsymbol{K}}$ defined as follows:

$$\widetilde{\boldsymbol{K}}\boldsymbol{H} = \boldsymbol{K}\boldsymbol{H} + \boldsymbol{\nabla} U_0 \quad \text{with} \quad U_0 \in W_{p,0}(\Omega) \quad \text{such that} \quad \|\widetilde{\boldsymbol{K}}\boldsymbol{H}\|_{L^2(\Omega)} \quad \text{is minimal.} \tag{7.5}$$

The operator norm of $\widetilde{\boldsymbol{K}}$ is not larger than the one of $\boldsymbol{K}$, and the lift $\widetilde{\boldsymbol{E}}$ constructed in this way satisfies the orthogonality conditions

$$\int_\Omega \mathbf{curl}\,\widetilde{\boldsymbol{E}} \cdot \mathbf{curl}\,\boldsymbol{F} = 0 \quad \forall \boldsymbol{F} \in \boldsymbol{Q}_{p,0}(\Omega) \quad \text{and} \quad \int_\Omega \widetilde{\boldsymbol{E}} \cdot \boldsymbol{\nabla} V = 0 \quad \forall V \in W_{p,0}(\Omega). \tag{7.6}$$

It is easy to see that these two orthogonality relations determine $\widetilde{\boldsymbol{E}}$ uniquely. □

Conclusions in 3D polynomial case. With $p \geq 1$ any integer, the lift operators $\mathcal{L}_0^{(p)}$, $\mathcal{L}_{\mathbf{t}}^{(p)}$ and $\mathcal{L}_{\mathbf{t}}^{(p)}$ satisfy the following exact sequence and commuting diagram properties, reproducing those of the continuous case:

$$\begin{array}{ccccccc}
W_p(\Omega) & \xrightarrow{\boldsymbol{\nabla}} & \boldsymbol{Q}_p(\Omega) & \underset{\boldsymbol{K}}{\overset{\mathbf{curl}}{\rightleftarrows}} & \boldsymbol{V}_p & \xrightarrow{\mathrm{div}} & Y_p(\Omega) \\
\gamma_0 \big\downarrow \big\uparrow \mathcal{L}_0^{(p)} & & \gamma_{\mathbf{t}} \big\downarrow \big\uparrow \mathcal{L}_{\mathbf{t}}^{(p)} & & \gamma_{\mathbf{n}} \big\downarrow \big\uparrow \mathcal{L}_{\mathbf{n}}^{(p)} & & \gamma_{avg} \big\downarrow \big\uparrow \mathcal{L}_{avg} \\
W_p(\partial\Omega) & \xrightarrow{\boldsymbol{\nabla}} & \boldsymbol{Q}_p(\partial\Omega) & \xrightarrow{\mathrm{curl}} & V_p(\partial\Omega) & \xrightarrow{\gamma_{avg}} & \mathbb{R}
\end{array} \tag{7.7}$$

The operator $\mathcal{L}_0^{(p)} : u \mapsto U$ is uniquely determined by the extra condition of orthogonality,

$$\int_\Omega \boldsymbol{\nabla} U \cdot \boldsymbol{\nabla} V = 0 \quad \forall V \in W_{p,0}(\Omega) \tag{7.8}$$

and $\mathcal{L}_{\mathbf{n}}^{(p)} : h \mapsto \boldsymbol{H}$ by

$$\int_\Omega \boldsymbol{H} \cdot \mathbf{curl}\,\boldsymbol{E} = 0 \quad \forall \boldsymbol{E} \in \boldsymbol{Q}_{p,0}(\Omega) \quad \text{and} \quad \mathrm{div}\,\boldsymbol{H} \in \mathbb{R}. \tag{7.9}$$

The "canonical" version $\widetilde{\mathcal{L}}_{\mathbf{t}}^{(p)}$ of the lift $\boldsymbol{e}_{\mathbf{t}} \mapsto \widetilde{\boldsymbol{E}}$ as defined in Remark 7.2 is uniquely determined by the orthogonality relations (7.6). □

8. Conclusions

In the paper, we have constructed polynomial extension operators for a master hexahedron and the polynomial spaces forming the exact sequence corresponding to Nédélec's hexahedron of the first type. The polynomial extension operators mimic closely the corresponding constructions on the continuous level based on separation of variables. In the presentation, we have restricted ourselves to the "isotropic" spaces (same polynomial order in each direction), but the whole procedure generalizes easily to the case of anisotropic spaces as well. Given a scalar space $\mathbb{P}^{(p,q,r)}$, we arrange the element system of coordinates in such a way that $p \leq q \leq r$, and the construction presented in this paper goes through without any changes.

The existence of polynomial extension operators completes the theory of *Projection-Based Interpolation*; see [15, 13] for the hexahedral element. The obtained discrete $H^{\frac{1}{2}}(I^2)$ and $H^{-\frac{1}{2}}(I^2)$-norms may be used in the automatic *hp*-adaptivity algorithm presented in [12, 16].

Finally, we finish with a didactic comment on teaching the separation of variables. When presenting the solution of the Dirichlet problem for the Laplace equation on a square or cube, virtually all textbooks recommend splitting the data into edge or face contributions, and the solution of the corresponding single edge or face problems with *pure Dirichlet conditions* using the separation of variables. The superposition principle is then used to obtain the final solution. With a regular boundary data that guarantees the existence of a finite energy solution, the procedure breaks the solution into the corresponding edge or face solutions that, in general, have infinite energy. This can be avoided by using the mixed boundary conditions employed in this paper, which guarantee that all solutions corresponding to a nonhomogeneous single edge or face data remain of finite energy. The same comments apply to the Neumann problem.

Acknowledgment

We want to thank the anonymous referee for the very careful reading and constructive remarks.

References

[1] M. Ainsworth and L. Demkowicz. Explicit polynomial preserving trace liftings on a triangle. *London Mathematical Society*, submitted. See also ICES Report 03-47.

[2] D. N. Arnold, R. S. Falk, and R. Winther. Finite element exterior calculus, homological techniques, and applications. *Acta Numer.*, 15:1–155, 2006. MR2269741 (2007j:58002)

[3] I. Babuška, A. Craig, J. Mandel, and J. Pitkaränta. Efficient preconditioning for the *p*-version finite element method in two dimensions. *SIAM J. Numer. Anal*, 28(3):624–661, 1991. MR1098410 (92a:65282)

[4] I. Babuška and M. Suri. The *h-p* version of the finite element method with quasi-uniform meshes. *RAIRO Modél. Math. Anal. Numér.*, 21(2):199–238, 1987. MR896241 (88d:65154)

[5] R. Bellman. A note on an inequality of E. Schmidt. *Bull. Amer. Math. Soc.*, 50:734–736, 1944. MR0010732 (6:61g)

[6] F. Ben Belgacem. Polynomial extensions of compatible polynomial traces in three dimensions. *Comput. Methods Appl. Mech. Engrg.*, 116(1–4):235–241, 1994. MR1286532 (95c:65019)

[7] C. Bernardi, M. Dauge, and Y. Maday. Polynomials in the Sobolev World. hal-00153795, version 2, `http://hal.archives-ouvertes.fr/hal-00153795/en/`. CNRS and INRIA 2007.

[8] C. Bernardi and Y. Maday. Relèvement polynomial de traces et applications. *RAIRO Modél. Math. Anal. Numér.*, 24(5):557–611, 1990. MR1076961 (91k:65040)

[9] A. Buffa and P. Ciarlet. On traces for functional spaces related to Maxwell's equations. Part I: An integration by parts formula in Lipschitz polyhedra. *Mathematical Methods in the Applied Sciences*, 24:9–30, 2001. MR1809491 (2002b:78024)

[10] A. Buffa, M. Costabel, and D. Sheen. On traces for $\boldsymbol{H}(\mathbf{curl}, \Omega)$ in Lipschitz domains. *J. Math. Anal. Appl.*, 276:845–876, 2002. MR1944792 (2004i:35045)

[11] C. Canuto and D. Funaro. The Schwarz algorithm for spectral methods. *SIAM J. Numer. Anal.*, 25(1):24–40, 1988. MR923923 (89b:65286)

[12] L. Demkowicz. *Computing with hp-adaptive Finite Elements. Vol. I. One and Two Dimensional Elliptic and Maxwell Problems*. Chapman & Hall/CRC Press, Taylor and Francis, 2006. MR2267112 (2007k:65003)

[13] L. Demkowicz. Polynomial exact sequences and projection-based interpolation with applications to Maxwell equations. In D. Boffi and L. Gastaldi, editors, *Mixed Finite Elements, Compatibility Conditions and Applications*, Lecture Notes in Mathematics. Springer-Verlag, 2007. See also ICES Report 06-12.

[14] L. Demkowicz and I. Babuška. p interpolation error estimates for edge finite elements of variable order in two dimensions. *SIAM J. Numer. Anal.*, 41(4):1195–1208 (electronic), 2003. MR2034876 (2004m:65191)

[15] L. Demkowicz and A. Buffa. $H^1, H(\mathrm{curl})$ and $H(\mathrm{div})$-conforming projection-based interpolation in three dimensions. Quasi-optimal p-interpolation estimates. *Comput. Methods Appl. Mech. Engrg*, 194:267–296, 2005. MR2105164 (2005j:65139)

[16] L. Demkowicz, J. Kurtz, D. Pardo, M. Paszyński, W. Rachowicz, and A. Zdunek. *Computing with hp-adaptive Finite Elements. Vol. II. Frontiers: Three-Dimensional Elliptic and Maxwell Problems with Applications*. CRC Press, Taylor and Francis, 2007. In press.

[17] J. Gopalakrishnan and L. Demkowicz. Quasioptimality of some spectral mixed methods. *J. Comput. Appl. Math.*, 167(1):163–182, May 2004. MR2059719 (2005d:65218)

[18] N. Heuer and F. Leydecker. An extension theorem for polynomials on triangles. BICOM, Dept. of Mathematical Sciences, Brunel University, 2006.

[19] R. Hiptmair. Canonical construction of finite elements. *Mathematics of Computation*, 68, 1999. 1325-1346. MR1665954 (2000b:65214)

[20] R. Hiptmair. Finite elements in computational electromagnetism. *Acta Numer.*, 11:237–339, 2002. MR2009375 (2004k:78028)

[21] Y. Maday. Relèvements de traces polynomiales et interpolations hilbertiennes entre espaces de polynômes. *C. R. Acad. Sci. Paris Sér. I Math.*, 309(7):463–468, 1989. MR1055459 (91f:46046)

[22] W. McLean. *Strongly Elliptic Systems and Boundary Integral Equations*. Cambridge University Press, 2000. MR1742312 (2001a:35051)

[23] R. Muñoz-Sola. Polynomial lifting on a tetrahedron and application to the *hp*-version of the finite element method. *SIAM J. Numer. Anal.*, 34:282–314, 1997. MR1445738 (98k:65069)

[24] J. C. Nédélec. Mixed finite elements in $\mathbb{R}^3$. *Numer. Math.*, 35:315–341, 1980. MR592160 (81k:65125)

[25] L. F. Pavarino and O. B. Widlund. A polylogarithmic bound for an iterative substructuring method for spectral elements in three dimensions. *SIAM J. Numer. Anal.*, 33(4):1303–1335, 1996. MR1403547 (97h:65151)

[26] J. Schoeberl, J. Gopalakrishnan, and L. Demkowicz. Polynomial preserving H^1, $\boldsymbol{H}(\mathbf{curl})$ and $\boldsymbol{H}(\mathrm{div})$ extension operators on a tetrahedron. In preparation, 2007.

IRMAR, Université de Rennes 1, 35042 Rennes, France
E-mail address: martin.costabel@univ-rennes1.fr

IRMAR, Université de Rennes 1, 35042 Rennes, France
E-mail address: monique.dauge@univ-rennes1.fr

Institute for Computational Engineering and Sciences, The University of Texas at Austin, Austin, Texas 78712
E-mail address: leszek@ices.utexas.edu