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HIGH PRECISION COMPUTATION OF A CONSTANT IN THE THEORY OF TRIGONOMETRIC SERIES

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ABSTRACT. Using the bisection as well as the Newton-Raphson method, we compute to high precision the Littlewood-Salem-Izumi constant frequently occurring in the theory of trigonometric sums.

1. INTRODUCTION

In Zygmund [15, p. 192] we read that there exists a number $\alpha_0 \in (0,1)$ such that for each $\alpha \geq \alpha_0$ the partial sums of the series $\sum_{n=1}^{\infty} n^{-\alpha}\cos(nx)$ are uniformly bounded below, whereas for $\alpha < \alpha_0$ they are not. It is also shown there that α_0 is the unique solution of the equation

$$\int_0^{3\pi/2} u^{-\alpha}\cos u\,du = 0 \qquad (0 < \alpha < 1). \tag{1.1}$$

(The uniqueness of α_0 will also follow from our analysis in Section 4.)

In this journal ([5], [8] and [14]) we find three short papers dealing with the numerical computation of this critical constant. In the first-mentioned paper the method of computation was not revealed. The result $0.30483 < \alpha_0 < 0.30484$ appears to be incorrect in the third decimal (which was also observed in [8] and [14]). In the second paper, by *conventional numerical quadrature*, it was (correctly) found that $0.308443 < \alpha_0 < 0.308444$. In the third paper, using *differencing and making use of ordinary interpolation techniques*, it was announced that (to 15 D) $\alpha_0 = 0.308443779561985$, which, as we will see, comes quite close to the true solution of (1.1).

The main object of this note is to present some simple elementary procedures for a *high precision* computation of α_0.

Although we will not tackle (1.1) by any integral approximating procedure, anyone persisting to do so might consider first removing the singularity of the integrand in (1.1) at $u = 0$ by integrating by parts, yielding the equivalent equation

$$F(\alpha) := \int_0^{3\pi/2} u^{1-\alpha}\sin u\,du = 0 \qquad (0 < \alpha < 1). \tag{1.2}$$

We might solve (1.1) by directly substituting the power series for $\cos u$. However, instead, we will tackle (1.2) by directly substituting the power series for $\sin u$,

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yielding the equivalent equation

$$\int_0^{3\pi/2} u^{1-\alpha}\sum_{k=0}^{\infty}(-1)^k\frac{u^{2k+1}}{(2k+1)!}\,du=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\int_0^{3\pi/2}u^{2k+2-\alpha}\,du=0$$

(interchanging $\sum$ and $\int$ being permitted here by uniform convergence) or

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\frac{\left(\frac{3\pi}{2}\right)^{2k+3-\alpha}}{2k+3-\alpha}=0\qquad(0<\alpha<1),\tag{1.3}$$

which, in its turn, is clearly equivalent to

$$G(\alpha):=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\frac{\left(\frac{3\pi}{2}\right)^{2k}}{2k+3-\alpha}=0\qquad(0<\alpha<1).\tag{1.4}$$

Note that $F(\alpha)$ and $G(\alpha)$ differ only by a positive factor:

$$G(\alpha)=\left(\frac{3\pi}{2}\right)^{\alpha-3}F(\alpha)=\frac{2}{3\pi}\int_0^1 v^{1-\alpha}\sin\left(\tfrac{3\pi}{2}v\right)dv.\tag{1.5}$$

2. Error analysis for $G(\alpha)$ and $G'(\alpha)$

In order to compute $G(\alpha)$ sufficiently accurately we make the following error analysis. We will make use of the following simple and well-known

Lemma 2.1. *If $a_{M+1}>a_{M+2}>a_{M+3}>\cdots>0$ and $\lim_{k\to\infty}a_k=0$, then the alternating series $\sum_{k=M+1}^{\infty}(-1)^k a_k$ converges and its sum S satisfies $|S|<a_{M+1}$.*

We can now easily show that when truncating (1.4) after M terms we commit an (absolute) error $<\frac{(3\pi/2)^{2M+2}}{(2M+4)!}$. Writing

$$a_k=\frac{\left(\frac{3\pi}{2}\right)^{2k}}{(2k+1)!\,(2k+3-\alpha)}$$

we clearly have $a_k>0$, $\lim_{k\to\infty}a_k=0$ and

$$\frac{a_{k+1}}{a_k}=\frac{\left(\frac{3\pi}{2}\right)^2}{(2k+2)(2k+3)}\,\frac{2k+3-\alpha}{2k+5-\alpha}<\frac{23}{42}<1\quad\text{for } k\geq 2$$

so that the lemma applies. Hence

$$\Bigl|\sum_{k=M+1}^{\infty}(-1)^k\frac{\left(\frac{3\pi}{2}\right)^{2k}}{(2k+1)!\,(2k+3-\alpha)}\Bigr|<a_{M+1}<\frac{\left(\frac{3\pi}{2}\right)^{2M+2}}{(2M+4)!}\quad\text{for } M\geq 1,$$

proving our claim. (Note that we used that $0<\alpha<1$.)

In a similar way it is easily seen that the same M yields an even smaller error when

$$G'(\alpha)=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\frac{\left(\frac{3\pi}{2}\right)^{2k}}{(2k+3-\alpha)^2}$$

is truncated after M terms.

3. The bisection program

We first present a program (for Mathematica Version 4.2) using *bisection* of the α-interval $[0,1]$. This very robust procedure needs no further justification. The result is (accurate to 130 D):

$$\alpha_0 \approx 0.30844\,37795\,61986\,00303\,41969\,50985\,95615\,94093\,74881\,47222\,19050\,10818\,91891$$
$$75633\,33646\,83898\,81583\,89154\,74111\,81428\,85243\,33044\,87005\,90567\,92056\,38627$$

4. Justification of the application of the Newton-Raphson method

In order to solve the equation $G(\alpha) = 0$ we will now apply the much faster Newton-Raphson method. We will show that (1.4) can also be used for this purpose.

Our justification for applying this method here is based on the following three observations:

Observation 1. $G(0) < 0 < G(1)$.

Proof. From (1.5) it easily follows that

$$G(0) = -\Big(\frac{2}{3\pi}\Big)^3 \qquad \text{and} \qquad G(1) = \Big(\frac{2}{3\pi}\Big)^2. \qquad \square$$

Observation 2. $G'(\alpha) > (2/3\pi)^3$ for $0 \le \alpha \le 1$.

Proof. Since (with $\mathrm{Si}(x) = \int_0^x \sin t/t\,dt$)

$$G'(0) = -\frac{2}{3\pi}\int_0^1 v\log v \sin\big(\tfrac{3\pi}{2}v\big)\,dv = \frac{8\big(1+\mathrm{Si}\left(\frac{3\pi}{2}\right)\big)}{27\pi^3} > \Big(\frac{2}{3\pi}\Big)^3,$$

our claim is an easy consequence of the following $\square$

Observation 3. $G''(\alpha) > 0$ for $0 \le \alpha \le 1$.

Proof. Writing $c = \frac{3\pi}{2}$ and $a_k = \frac{2\,c^{2k}}{(2k+1)!\,(2k+3-\alpha)^3}$ (> 0) it follows from (1.4) that

$$G''(\alpha) = \sum_{k=0}^{\infty}(-1)^k a_k = (a_0 - a_1) + \sum_{k=1}^{\infty}(a_{2k} - a_{2k+1}).$$

Since

$$\frac{a_{2k+1}}{a_{2k}} = \frac{c^2}{(4k+2)(4k+3)}\Big(\frac{4k+3-\alpha}{4k+5-\alpha}\Big)^3 < \frac{23}{42}\cdot 1 < 1 \quad \text{for} \quad k \ge 1,$$

we already find that $G''(\alpha) > a_0 - a_1$.

So, it suffices to show that $a_0 \ge a_1$, or, equivalently, that

$$\frac{a_1}{a_0} = \frac{c^2}{3!}\Big(\frac{3-\alpha}{5-\alpha}\Big)^3 \le 1.$$

Since $(0 <)\ \frac{3-\alpha}{5-\alpha} = 1 - \frac{2}{5-\alpha}$ is decreasing (for $0 \le \alpha \le 1$) we need only check whether $\frac{c^2}{6}\left(\frac{3}{5}\right)^3 \le 1$. Since $c^2 < 23$ it suffices to observe that $\frac{23}{6}\frac{27}{125} = \frac{207}{250} < 1$, and the proof is complete. $\square$

From the previous proof it is clear that $\sum_{k=0}^{\infty}(-1)^k a_k$ is an alternating series in the sense of Lemma 2.1. Hence, we might use a similar error estimate as given before for $G(\alpha)$ and $G'(\alpha)$.

The upshot of all this is that (for $0 \le \alpha \le 1$)

(a) $G(\alpha)$ changes sign, (b) $G(\alpha)$ is strictly increasing, (c) $G(\alpha)$ is strictly convex.

It is well known that these conditions are sufficient for a rigorous application of the Newton-Raphson method to solve our equation $G(\alpha) = 0$.

Starting on the large side with $\alpha = \frac{31}{100}$, the program in Section 5 yields the solution α_0 presented there (accurate to 1120 D).

5. The Newton-Raphson program

Considerably more efficient than the bisection procedure is the Newton-Raphson method. The theoretical justification was given in the previous section.

Starting with $\alpha = \frac{31}{100}$ we find after 10 iterations that

$\alpha_0 \approx$ 0.30844 37795 61986 00303 41969 50985 95615 94093 74881 47222 19050 10818 91891 75633
33646 83898 81583 89154 74111 81428 85243 33044 87005 90567 92056 38627 42762 94638
64125 05998 23831 85455 61091 48448 43732 92955 91210 06167 94033 89040 23200 75937
35551 96458 18912 23711 85977 40758 47712 23681 52071 27309 30648 54142 11222 21328
67475 18367 96776 32777 82178 57135 03785 59451 85855 37153 73748 62329 34834 93383
10271 03316 23977 99085 75171 17825 15277 15233 91368 31623 10073 85968 71360 45377
29958 88150 04792 46476 18905 99174 27695 38591 86825 04300 04568 91962 61785 51160
73434 48711 02464 44624 46899 43950 49454 94157 36865 88771 28074 35765 04551 57356
60342 47934 73045 97313 77001 84093 75724 64014 49041 17091 09020 62994 10947 38484
57301 91655 68731 08265 96219 74870 97674 02739 49480 30079 45799 29657 72476 57829
65635 42318 57887 31812 19547 50611 89593 78195 94367 39765 43677 52291 77108 50149
82852 32724 82004 44482 96275 62690 66209 63438 91019 91785 43433 58580 60118 22865
76976 69697 93848 92079 44235 48266 73922 11031 64893 05399 22498 60586 15316 41872
84522 28247 13023 41908 83558 50673 14676 06111 89516 31984 20955 54304 27710 93862
29062 11870 49536 74477 31495 59765 32238 45223 68791 92455 15013 42105 66452 23369
44558 91848 15109 15528 59670 62037 73042 74959 95260 50539 70933 62562 21140 36998

(α_0 accurate to 1120 D, obtained within 18 seconds)

Without any economization of our Newton-Raphson program, the computation of α_0 to 5000 D (requiring 12 iterations) took less than 20 minutes on a Toshiba laptop - 2 GB RAM - 3.2 MHz.

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Note. Zygmund [15, p. 379] writes that the origin of the defining property for α_0 is to be found in an unpublished result of Littlewood and Salem, and that the equation defining α_0 is due to S. Izumi. This justifies calling it the Littlewood-Salem-Izumi constant.

However, the earliest paper where we detected this constant is [10].

For additional information on α_0 we recommend [2] and [9]. α_0 also plays a role in some theorems about positive trigonometric series with general coefficients: [4], [6] and [7], and in theorems about the positivity of some sums of orthogonal polynomials: [12].

Our method can be extended to apply to similar constants. For example, some open problems are mentioned in [1], [3] and [13].

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