CONVERGENCE OF AN ENGQUIST-OSHER SCHEME FOR A MULTI-DIMENSIONAL TRIANGULAR SYSTEM OF CONSERVATION LAWS

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ABSTRACT. We consider a multi-dimensional triangular system of conservation laws. This system arises as a model of three-phase flow in porous media and includes multi-dimensional conservation laws with discontinuous coefficients as a special case. The system is neither strictly hyperbolic nor symmetric. We propose an Engquist-Osher type scheme for this system and show that the approximate solutions generated by the scheme converge to a weak solution. Numerical examples are also presented.

1. INTRODUCTION

In this paper, we consider the 2×2 triangular system of conservation laws,

$$\partial_t u + \operatorname{div}\left(\mathbf{f}(u)\right) = 0, \quad \mathbf{x} \in \mathbb{R}^d,\, t > 0, \tag{1.1}$$

$$\partial_t v + \operatorname{div}\left(\mathbf{g}(u,v)\right) = 0, \quad \mathbf{x} \in \mathbb{R}^d,\, t > 0, \tag{1.2}$$

with the initial condition

$$u(\mathbf{x}, 0) = u_0(\mathbf{x}), \quad v(\mathbf{x}, 0) = v_0(\mathbf{x}),\ \mathbf{x} \in \mathbb{R}^d,$$

where u and v are the unknowns, with the initial values u_0 and v_0, and the flux functions $\mathbf{f} = (f_1, f_2, \ldots, f_d)$ and $\mathbf{g} = (g_1, g_2, \ldots, g_d)$ are prescribed.

Note that the evolution of u is independent of v, but that the evolution of v depends on u. Writing (1.1), (1.2) in quasilinear form results in

$$U_t + \sum_{i=1}^{d} A_i U_{x_i} = 0,$$

where $U = \{u, v\}$ and the directional Jacobians are given by

$$A_i = \begin{pmatrix} \frac{\partial f_i}{\partial u} & 0 \\ \frac{\partial g_i}{\partial u} & \frac{\partial g_i}{\partial v} \end{pmatrix}.$$

Let $\mathbf{A} = \{A_1, A_2, \ldots, A_d\}$ and let $\mathbf{n}$ be a unit vector in $\mathbb{R}^d$. Then the matrix $\mathbf{A} \cdot \mathbf{n}$ is lower triangular. Hence, systems of the form (1.1), (1.2) are called triangular systems. Furthermore, the eigenvalues of the matrix $\mathbf{A} \cdot \mathbf{n}$ are real. Hence, the

system is hyperbolic. Since the eigenvalues can coincide, the system is not strictly hyperbolic. Non-strictly hyperbolic systems are complicated to deal with even in the case of one space dimension.

In general, systems of conservation laws in several space dimensions are difficult to study and few rigorous results have been obtained. See however the book by Serre [2] for detailed references and theoretical results.

A special case of the above system occurs when we take $\mathbf{f} = 0$. In this case, the system reduces to a multi-dimensional scalar conservation law with a spatially varying coefficient u (which can be discontinuous). Scalar conservation laws with discontinuous coefficients arise in a wide variety of contexts including two-phase flows in heterogeneous porous media, as models of clarifier-thickener units and in traffic flow. In one spatial dimension such equations have been studied in several papers. An incomplete list includes [1, 3, 7, 9, 15, 17, 20, 21] and other references therein.

Scalar conservation laws with discontinuous coefficients in several space dimensions have not been correspondingly widely studied and the theory is not as well-developed as in the one-dimensional case. In [14], the authors considered a scalar conservation law in two space dimensions with discontinuous coefficients and obtained existence of weak solutions by showing that vanishing viscosity approximations converge. They used a modification of the compensated compactness approach in order to obtain compactness of approximate solutions. In [19], the author was able to treat a multi-dimensional scalar conservation law with discontinuous coefficients in both space and time. The author showed existence of weak solutions by proving compactness of approximations generated by smoothing the coefficients. The compactness technique in [19] uses the tool of H-measures extensively. We will adapt the compactness framework of [19] in this paper.

In one space dimension, the triangular system (1.1), (1.2) was considered in [13]. Existence of weak solutions was shown by constructing finite volume schemes and showing that the approximate solutions generated by these schemes are compact and converge to a weak solution. The compactness technique involved using discrete entropy inequalities and the compensated compactness framework.

Triangular systems occur in a variety of applications. We are motivated partly by a model of three-phase flow in porous media described briefly below.

A multi-dimensional three-phase flow model. Simulation of oil recovery processes involves models of three-phase flow in porous media. The three phases of interest are mostly oil, gas and water. Examples of three-phase flow include water flooding in the presence of gas, gas flooding and water alternating gas injections into a reservoir.

As a model, consider a homogeneous and isotropic porous medium. The phase saturations are given by S_o, S_w and S_g for the oil, water and gas phases respectively. Mass conservation for each phase gives the following continuity equations:

$$(1.3) \qquad \phi(S_l)_t + \mathrm{div}\,(U_l) = q_l, \quad l = \mathrm{w, o, g,}$$

where U_l is the phase flux of the l-th phase, q_l is a source term and ϕ is the porosity of the medium, henceforth assumed to be equal to one. The phase flux for each phase is given by Darcy's law (see [4]) as

$$U_l = \frac{-Kk_l}{\mu_l}(\nabla p_l - \rho_l \tilde{g}),$$

where K is the absolute permeability, k_l, ρ_l, p_l, and μ_l are the relative permeability, density, pressure and viscosity of the l-th phase. The vector $\tilde{g} = \{0, 0, g\}$ represents the gravitational force. The saturations satisfy the following condition:

$$S_{\mathrm{o}} + S_{\mathrm{g}} + S_{\mathrm{w}} = 1.$$

This implies that the gas and water phase saturations can be used as the unknowns. Furthermore, the phase pressures are also unknowns and have to be determined. We use the phase formulation for pressures (see [4] for details) and assume that the capillary pressures between the phases are zero. This assumption is valid when the flow is convection dominated, i.e., when the total flow rate is very high, and in this case we write $p = p_l$ for $l =$ w, o, g. Then introducing the total flux $U = U_{\mathrm{o}} + U_{\mathrm{g}} + U_{\mathrm{w}}$, phase mobilities $\lambda_l = \frac{k_l}{\mu_l}$ and total mobility $\lambda_{\mathrm{T}} = \lambda_{\mathrm{o}} + \lambda_{\mathrm{g}} + \lambda_{\mathrm{w}}$, we obtain the following equations for the total flux and pressure:

(1.4)
$$U = -K\lambda_{\mathrm{T}} \left(\nabla p - \tilde{g} \left(\frac{\rho_{\mathrm{o}}\lambda_{\mathrm{o}} + \rho_{\mathrm{w}}\lambda_{\mathrm{w}} + \rho_{\mathrm{g}}\lambda_{\mathrm{g}}}{\lambda_{\mathrm{T}}} \right) \right),$$
$$\operatorname{div}(U) = q_{\mathrm{o}} + q_{\mathrm{g}} + q_{\mathrm{w}}.$$

The above equation for the pressure is elliptic and gives the total flux. The resulting equations for the phase saturations are given by

(1.5)
$$(S_{\mathrm{g}})_t + F^{\mathrm{g}}(S_{\mathrm{g}}, S_{\mathrm{w}}, S_{\mathrm{o}})_x + G^{\mathrm{g}}(S_{\mathrm{g}}, S_{\mathrm{w}}, S_{\mathrm{o}})_y + H^{\mathrm{g}}(S_{\mathrm{g}}, S_{\mathrm{w}}, S_{\mathrm{o}})_z = q_{\mathrm{g}},$$
$$(S_{\mathrm{w}})_t + F^{\mathrm{w}}(S_{\mathrm{g}}, S_{\mathrm{w}}, S_{\mathrm{o}})_x + G^{\mathrm{w}}(S_{\mathrm{g}}, S_{\mathrm{w}}, S_{\mathrm{o}})_y + H^{\mathrm{w}}(S_{\mathrm{g}}, S_{\mathrm{w}}, S_{\mathrm{o}})_z = q_{\mathrm{w}}.$$

Writing $U = \{u^1, u^2, u^3\}$ and assuming that the gravity acts in the z-direction, then the fluxes are given by

(1.6)
$$F^{\mathrm{g}} = \frac{\lambda_{\mathrm{g}} u^1}{\lambda_{\mathrm{T}}}, \quad G^{\mathrm{g}} = \frac{\lambda_{\mathrm{g}} u^2}{\lambda_{\mathrm{T}}},$$
$$F^{\mathrm{w}} = \frac{\lambda_{\mathrm{w}} u^1}{\lambda_{\mathrm{T}}}, \quad G^{\mathrm{w}} = \frac{\lambda_{\mathrm{w}} u^2}{\lambda_{\mathrm{T}}},$$
$$H^{\mathrm{g}} = \frac{\lambda_{\mathrm{g}} u^3}{\lambda_{\mathrm{T}}} + K \frac{\lambda_{\mathrm{g}} \lambda_{\mathrm{w}}}{\lambda_{\mathrm{T}}} (\rho_{\mathrm{g}} - \rho_{\mathrm{w}})g + K \frac{\lambda_{\mathrm{g}} \lambda_{\mathrm{o}}}{\lambda_{\mathrm{T}}} (\rho_{\mathrm{g}} - \rho_{\mathrm{o}})g,$$
$$H^{\mathrm{w}} = \frac{\lambda_{\mathrm{w}} u^3}{\lambda_{\mathrm{T}}} + K \frac{\lambda_{\mathrm{g}} \lambda_{\mathrm{w}}}{\lambda_{\mathrm{T}}} (\rho_{\mathrm{w}} - \rho_{\mathrm{g}})g + K \frac{\lambda_{\mathrm{w}} \lambda_{\mathrm{o}}}{\lambda_{\mathrm{T}}} (\rho_{\mathrm{w}} - \rho_{\mathrm{o}})g.$$

Hence, the coupled system of equations (1.4), (1.5) and (1.6) model three-phase flow in a porous medium. This system couples an elliptic pressure equation with a hyperbolic saturation equation. These two equations have very different properties, and it is therefore common to use a spitting approach to obtain numerical solutions. In brief, this consists in fixing the saturation and solving the pressure equation (1.4), then fixing the pressure and solving the saturation equations (1.5) for a short time interval Δt, and repeating this process.

If we consider the saturation equations (1.5) with a given total flux U, the equations are a special case of a system of conservation laws in several space dimensions.

Despite this simplification, this system is quite complicated, both theoretically and computationally. Even in one space dimension, the above system is not necessarily hyperbolic and can contain elliptic regions. Therefore, as a further simplification, the following "reduced" model was proposed in [11] and was analysed in [13] in the one-dimensional case. This simplification is based on the observation that in many situations the mobility of the gas phase is much larger than that of

the other phases. This means that the flux of gas is largely independent of whether the other phase is oil or water. As a consequence, we can make the approximations

$$F^{\mathrm{g}}(S_{\mathrm{g}}, S_{\mathrm{w}}, S_{\mathrm{o}}) = \tilde{F}(S_{\mathrm{g}}, 1 - S_{\mathrm{g}}) = \hat{F}(S_{\mathrm{g}}),$$
$$G^{\mathrm{g}}(S_{\mathrm{g}}, S_{\mathrm{w}}, S_{\mathrm{o}}) = \tilde{G}(S_{\mathrm{g}}, 1 - S_{\mathrm{g}}) = \hat{G}(S_{\mathrm{g}}),$$
$$H^{\mathrm{g}}(S_{\mathrm{g}}, S_{\mathrm{w}}, S_{\mathrm{o}}) = \tilde{H}(S_{\mathrm{g}}, 1 - S_{\mathrm{g}}) = \hat{H}(S_{\mathrm{g}}).$$

Assuming this, the system (1.5) reduces to the following system. The resulting equations for the phase saturations are given by

$$
\begin{aligned}
(S_{\mathrm{g}})_t + \hat{F}(S_{\mathrm{g}})_x + \hat{G}(S_{\mathrm{g}})_y + \hat{H}(S_{\mathrm{g}})_z &= q_{\mathrm{g}}, \\
(S_{\mathrm{w}})_t + F^{\mathrm{w}}(S_{\mathrm{g}}, S_{\mathrm{w}}, S_{\mathrm{o}})_x + G^{\mathrm{w}}(S_{\mathrm{g}}, S_{\mathrm{w}}, S_{\mathrm{o}})_y + H^{\mathrm{w}}(S_{\mathrm{g}}, S_{\mathrm{w}}, S_{\mathrm{o}})_z &= q_{\mathrm{w}}.
\end{aligned}
\tag{1.7}
$$

The above equation is a special case of the multi-dimensional triangular system (1.1), (1.2).

It is to be emphasized that although the assumption of independence of the gas phase is not valid for all fractional flow functions, there exists a large class of fractional flow functions for which this assumption appears to be reasonable. In view of the fact that this assumption makes the model simpler and much more tractable, we can use this "reduced" model in several situations. Nevertheless, we point out that a careful numerical study of this model (1.7) as an approximation to the full three-phase flow model needs to be carried out. An essential ingredient for this program is the development of efficient numerical schemes for (1.5) as well as for (1.7). We emphasize that the full three-phase flow model also includes the elliptic equation for the pressure. In this paper we concentrate on the saturation equations in the presence of a given total flux. In one space dimension, the pressure can be easily eliminated and the full model consists of the saturation equations only.

Our aim in this paper is to design an efficient numerical scheme to approximate solutions of the triangular system (1.1) and (1.2). We propose to use an adaptation of the Engquist-Osher scheme based on a staggered grid. The resulting approximate solutions are shown to converge to a weak solution of the triangular system. The proof of convergence is based on the framework developed in [19] and uses entropy inequalities extensively. We present some numerical experiments in order to demonstrate the robustness of the scheme. This numerical scheme can be used as the hyperbolic solver for three-phase flow models in several space dimensions, and can be coupled with a suitable elliptic solver for the pressure in order to design numerical codes for reservoir simulation.

In [5] we studied the same triangular system as in this paper, but as a limit of viscous regularizations. The basic compactness tools used in [5] were the same as in this paper, but obtaining the bounds necessary to use these compactness tools is much harder for a difference scheme than for the viscous approximations.

The rest of this paper is organized as follows: In Section 2, we present the mathematical framework used in this paper. The Engquist-Osher type scheme is proposed and analysed in Section 3 and numerical experiments are reported in Section 4.

2. Mathematical framework

We start with the assumptions on the data and the flux functions,

(**H.1**) $\mathbf{f} \in C^1([-M, M]; \mathbb{R}^d)$, $\mathbf{g} \in C^2([-M, M]^2; \mathbb{R}^d)$;

(**H.2**) $\partial_v \mathbf{g}(u, \pm M) = 0$ for all u;

(**H.3**) $\partial_{uu}^2 \mathbf{g}(\cdot, v)$ is Lipschitz continuous for $v \in [-M, M]$;

(**H.4**) $\mathbf{g}(u, \cdot)$ is genuinely non-linear for all u; i.e., for any unit vector $\mathbf{n}$, the map $v \in [-M, M] \mapsto \mathbf{g}(u, v) \cdot \mathbf{n}$ is not affine on any non-trivial interval for all u and $\mathbf{n} \in \mathbb{R}^d$, $|\mathbf{n}| = 1$;

(**H.5**) u_0 and v_0 are in $L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d) \cap BV(\mathbb{R}^d)$

for some positive constant M.

As mentioned, we deal with weak solutions of (1.1) and (1.2) defined below.

Definition 2.1. We call the pair (u, v) a weak solution of the Cauchy problem (1.1), (1.2) if

(**D.1**) u and v are in $L^\infty(\mathbb{R}^d \times \mathbb{R}^+)$;

(**D.2**) u and v satisfy (1.1), (1.2) in the sense of distributions on $\mathbb{R}^d \times \mathbb{R}^+$, i.e., the following identities,

$$\int_{\mathbb{R}^d \times \mathbb{R}^+} (u \partial_t \varphi + \mathbf{f}(u) \mathrm{div}\,(\varphi))\, dxdydt + \int_{\mathbb{R}^d} u_0(x, y)\varphi(x, y, 0)dxdy = 0,$$

$$\int_{\mathbb{R}^d \times \mathbb{R}^+} (v \partial_t \varphi + \mathbf{g}(u, v) \mathrm{div}\,(\varphi))\, dxdydt + \int_{\mathbb{R}^d} v_0(x, y)\varphi(x, y, 0)dxdy = 0,$$

hold for each test function φ smooth with compact support in $\mathbb{R}^d \times \mathbb{R}^+$;

(**D.3**) for each constant $k \in \mathbb{R}$ the inequality

$$\partial_t |u - k| + \mathrm{div}\,(\mathrm{sign}\,(u - k)\,(\mathbf{f}(u) - \mathbf{f}(k))) \leq 0$$

holds in the sense of distributions on $\mathbb{R}^d \times \mathbb{R}^+$.

As stated in the introduction, we will propose an Engquist-Osher type scheme in order to approximate the triangular system numerically. The convergence proof for the scheme is based on the following crucial lemma, which is an easy adaptation of a result by Panov [19, Theorem 5]. We call $(\eta, \mathbf{q})$ a convex entropy/entropy flux pair for (1.2) if η is a convex function of v and

$$\partial_v \mathbf{q}(u, v) = \eta'(v) \partial_v \mathbf{g}(u, v).$$

In particular, we have the Kružkov entropy/entropy flux pairs

$$\eta_0(v) = |v - k|\,, \quad \mathbf{q}_0(u, v) = \mathrm{sign}\,(v - k)\,(g(u, v) - g(u, k))\,,$$

where k is an arbitrary constant.

Lemma 2.2. *Let u be the unique entropy solution of the Cauchy problem for the single conservation law (1.1), and let $\{v_\nu\}_{\nu>0}$ be a family of functions defined on $\mathbb{R}^d \times \mathbb{R}^+$. If $\{v_\nu\}$ is bounded in $L^\infty(\mathbb{R}^d \times \mathbb{R}^+)$, and*

$$\{\partial_t \eta_0\,(v_\nu) + \mathrm{div}\,(\mathbf{q}_0\,(u, v_\nu))\}_{\nu>0}$$

lies in a compact set of $H_{\mathrm{loc}}^{-1}(\mathbb{R}^d \times \mathbb{R}^+)$ for all constants k, then there exists a sequence $\{\nu_n\}_{n \in \mathbb{N}}$, $\nu_n \to 0$ and a function $v \in L^\infty(\mathbb{R}^d \times \mathbb{R}^+)$ such that

$$v_{\nu_n} \to v \quad \text{a.e. and in} \quad L_{\mathrm{loc}}^p(\mathbb{R}^d \times \mathbb{R}^+),\ 1 \leq p < \infty.$$

We also need the following useful technical lemma.

Lemma 2.3 (see [18]). *Let Ω be a bounded open subset of $\mathbb{R}^d$, $d \geq 2$. Suppose that the sequence $\{\mathcal{L}_n\}_{n\in\mathbb{N}}$ of distributions is bounded in $W^{-1,\infty}(\Omega)$. Suppose also that*

$$\mathcal{L}_n = \mathcal{L}_{1,n} + \mathcal{L}_{2,n},$$

where $\{\mathcal{L}_{1,n}\}_{n\in\mathbb{N}}$ is in a compact subset of $H_{\mathrm{loc}}^{-1}(\Omega)$ and $\{\mathcal{L}_{2,n}\}_{n\in\mathbb{N}}$ is in a bounded subset of $\mathcal{M}_{\mathrm{loc}}(\Omega)$. Then $\{\mathcal{L}_n\}_{n\in\mathbb{N}}$ is in a compact subset of $H_{\mathrm{loc}}^{-1}(\Omega)$.

Throughout the paper, we use the notation C_X to indicate a constant depending on the quantity X (only).

3. The Engquist-Osher method

Now we shall analyze the Engquist-Osher method for (1.1) and (1.2). In order to simplify the notation, we restrict ourselves to two spatial dimensions, i.e., $d = 2$, but the generalization to arbitrary d is straightforward. In this case,

$$\mathbf{f}(u) = (f_1(u), f_2(u)) \quad \text{and} \quad \mathbf{g}(u, v) = (g_1(u, v), g_2(u, v)).$$

We shall use a staggered version of the Engquist-Osher scheme, in which u and v are discretized on grids which are staggered and rotated with respect to each other. Set

$$\tilde{\mathbf{f}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\mathbf{f} \quad \text{and} \quad \tilde{\mathbf{x}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\mathbf{x}.$$

Then we observe that u is (also) the entropy solution of

$$(3.1) \qquad\qquad \partial_t u + \widetilde{\mathrm{div}}(\tilde{\mathbf{f}}) = 0, \quad u(\tilde{\mathbf{x}}, 0) = u_0(\tilde{\mathbf{x}}),$$

where $\widetilde{\mathrm{div}}\tilde{\mathbf{f}} = (\partial_{\tilde{x}}, \partial_{\tilde{y}}) \cdot \tilde{\mathbf{f}}$. We shall consider discrete versions of (3.1) and (1.2).

For a scalar function $f_\ell(u)$, the Engquist-Osher flux is defined as

$$f_\ell^{\mathrm{EO}}(a, b) = \int_0^a (f_\ell'(s) \vee 0)\,ds + \int_0^b (f_\ell'(s) \wedge 0)\,ds + f_\ell(0),$$

where we use the notation $(a \wedge b) = \min\{a, b\}$ and $(a \vee b) = \max\{a, b\}$.

We use the same grid spacing in both directions $\Delta x = \Delta y = h > 0$ and a uniform time step $\Delta t > 0$. Set $x_i = ih$ and $y_j = jh$ for integers i and j, and $x_{i+1/2} = x_i + h/2$, $y_{j+1/2} = y_j + h/2$ and $t^n = n\Delta t$. Let the squares $\mathcal{I}_{i,j+1/2}$ and $\mathcal{I}_{i+1/2,j}$ be defined by

$$\mathcal{I}_{i,j+1/2} = \left\{(x, y) \mid |x - x_i| + |y - y_{j+1/2}| < h/2\right\},$$
$$\mathcal{I}_{i+1/2,j} = \left\{(x, y) \mid |x - x_{i+1/2}| + |y - y_j| < h/2\right\}.$$

For later use, we also define the cells

$$\mathcal{I}_{i,j} = \left\{(x, y) \mid (|x - x_i| \vee |y - y_j|) < h/2\right\}.$$

With $\lambda = \Delta t / h$, the Engquist-Osher scheme for (3.1) reads

$$
\begin{aligned}
u_{i,j+1/2}^{n+1} = u_{i,j+1/2}^n - \sqrt{2}\lambda\Big(&\tilde{f}_1^{\text{EO}}\left(u_{i,j+1/2}^n, u_{i+1/2,j}^n\right) \\
&- \tilde{f}_1^{\text{EO}}\left(u_{i-1/2,j+1}^n, u_{i,j+1/2}^n\right) \\
&+ \tilde{f}_2^{\text{EO}}\left(u_{i,j+1/2}^n, u_{i+1/2,j+1}^n\right) - \tilde{f}_2^{\text{EO}}\left(u_{i-1/2,j}^n, u_{i,j+1/2}^n\right)\Big),
\end{aligned}
$$

(3.2)

$$
\begin{aligned}
u_{i+1/2,j}^{n+1} = u_{i+1/2,j}^n - \sqrt{2}\lambda\Big(&\tilde{f}_1^{\text{EO}}\left(u_{i+1/2,j}^n, u_{i+1,j-1/2}^n\right) \\
&- \tilde{f}_1^{\text{EO}}\left(u_{i,j+1/2}^n, u_{i+1/2,j}^n\right) \\
&+ \tilde{f}_2^{\text{EO}}\left(u_{i+1/2,j}^n, u_{i+1,j+1/2}^n\right) - \tilde{f}_2^{\text{EO}}\left(u_{i,j-1/2}^n, u_{i+1/2,j}^n\right)\Big),
\end{aligned}
$$

for integers i and j and $n \geq 0$. The scheme is initiated by setting

$$
u_{i,j+1/2}^0 = \frac{2}{h^2} \iint_{\mathcal{I}_{i,j+1/2}} u_0(\mathbf{x})\, d\mathbf{x}, \quad u_{i+1/2,j}^0 = \frac{2}{h^2} \iint_{\mathcal{I}_{i+1/2,j}} u_0(\mathbf{x})\, d\mathbf{x}.
$$

The relation between h and Δt is such that the CFL-condition

$$
(3.3) \qquad\qquad\qquad 4\sqrt{2}\lambda \|\tilde{\mathbf{f}}'\|_{L^\infty} \leq 1
$$

holds. Let u_h be defined as the piecewise constant function

$$
u_h(x,y,t) = \begin{cases} u_{i+1/2,j}^n, & \text{for } (x,y) \in \mathcal{I}_{i+1/2,j} \\ u_{i,j+1/2}^n, & \text{for } (x,y) \in \mathcal{I}_{i,j+1/2}, \end{cases} \quad \text{for } t \in [t^n, t^{n+1}),\ n \geq 0.
$$

We have that $\lim_{h\to 0} u_h$ is the unique entropy solution of (1.1). We list some useful properties of u_h in the next lemma [10].

Lemma 3.1. *Assume that the CFL condition* (3.3) *holds. Then for each $h > 0$ we have that*

(a) $-M \leq u_h(x,y,t) \leq M$, *for all (x,y) and $t \geq 0$.*
(b) *For $n \geq 0$ the functions*

$$
n \mapsto \frac{h^2}{2} \sum_{i,j} \left| u_{i+1/2,j}^n \right| + \left| u_{i,j+1/2}^n \right|,
$$

$$
n \mapsto \sum_{i,j} \left| u_{i,j+1/2}^n - u_{i-1/2,j}^n \right| + \left| u_{i+1/2,j}^n - u_{i,j-1/2}^n \right|,
$$

$$
n \mapsto \sum_{i,j} \left| u_{i+1/2,j}^{n+1} - u_{i+1/2,j}^n \right| + \left| u_{i,j+1/2}^{n+1} - u_{i,j+1/2}^n \right|
$$

are non-increasing. In particular this means that the family $\{u_h\}_{h>0}$ is (uniformly in h) bounded in $L^\infty(\mathbb{R}^+; L^1(\mathbb{R}^2)) \cap BV(\mathbb{R}^2 \times \mathbb{R}^+)$.
(c) *The sequence $\{u_h\}_{h>0}$ converges to the unique entropy solution u of the conservation law* (1.1), *i.e.,*

$$
(3.4) \qquad u_h \to u \quad \text{a.e. and in} \quad L_{\text{loc}}^p(\mathbb{R}^2 \times \mathbb{R}^+),\ 1 \leq p < \infty.
$$

When we formulate the Engquist-Osher scheme for v, we need EO-fluxes for functions of two variables $g_\ell(u, v)$. We define this as

$$g_\ell^{\text{EO}}(u, a, b) = \int_0^a (\partial_v g_\ell(u, s) \vee 0) \, ds + \int_0^b (\partial_v g_\ell(u, s) \wedge 0) \, ds + g_\ell(u, 0);$$

i.e., the first variable is regarded as a parameter. The EO-flux is Lipschitz continuous in all variables and has the useful monotonicity properties,

$$\partial_a g_\ell^{\text{EO}}(u, a, b) \geq 0, \; \partial_b g_\ell^{\text{EO}}(u, a, b) \leq 0,$$

and satisfies the bounds

$$\left| \partial_a g_\ell^{\text{EO}}(u, a, b) \right| \leq \| \partial_v g_\ell \|_{L^\infty}, \; \left| \partial_b g_\ell^{\text{EO}}(u, a, b) \right| \leq \| \partial_v g_\ell \|_{L^\infty}.$$

The Engquist-Osher scheme corresponding to (1.2) reads
(3.5)

$$v_{i,j}^{n+1} = v_{i,j}^n - \lambda \bigg(g_1^{\text{EO}} \left(u_{i+1/2,j}^n, v_{i,j}^n, v_{i+1,j}^n \right) - g_1^{\text{EO}} \left(u_{i-1/2,j}^n, v_{i-1,j}^n, v_{i,j}^n \right)$$
$$+ g_2^{\text{EO}} \left(u_{j+1/2,i}^n, v_{i,j}^n, v_{i,j+1}^n \right) - g_2^{\text{EO}} \left(u_{j-1/2,i}^n, v_{i,j-1}^n, v_{i,j}^n \right) \bigg),$$

with initial values

(3.6)
$$v_{i,j}^0 = \frac{1}{h^2} \iint_{\mathcal{I}_{i,j}} v_0(\mathbf{x}) \, d\mathbf{x}.$$

The reason for using this "staggered" scheme is that we want to have u constant across discontinuities in v. This enables the use of simple scalar numerical flux functions such as the Engquist-Osher flux. Figure 1 shows the location of the discrete variables and the grids used for u and v. We shall prove the following convergence result.
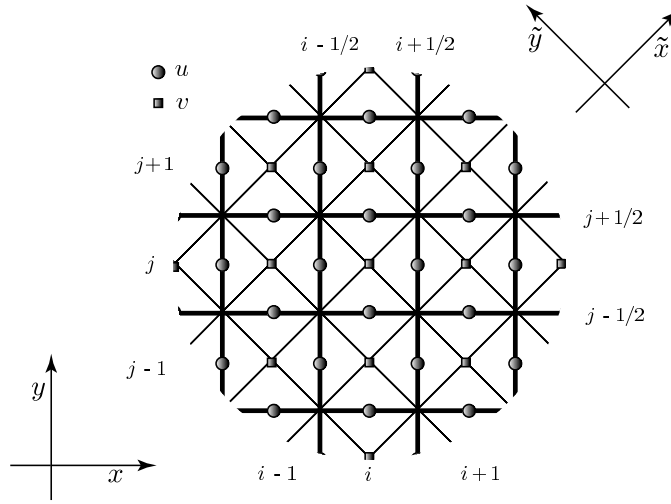

FIGURE 1. The locations of the grid cells for u and v.

Theorem 3.2. *If the CFL-condition*

$$(3.7) \qquad \lambda \leq \min\left\{\frac{1}{4\sqrt{2}\|\tilde{\mathbf{f}}'\|_{L^\infty}}, \frac{1}{4\,\|\partial_v\mathbf{g}\|_{L^\infty}}\right\}$$

holds, then the scheme (3.2)-(3.5) produces a sequence which converges to a weak solution of the Cauchy problem (1.1)-(1.2) in $L^p_{loc}(\mathbb{R}^2 \times \mathbb{R}^+)$, $1 \leq p < \infty$.

Proof. The theorem follows from Lemmas 2.2, 3.4 and 3.5. $\qquad\square$

If the CFL-condition

$$(3.8) \qquad 4\lambda\,\|\partial_v\mathbf{g}\|_{L^\infty} \leq 1$$

holds, the scheme is monotone in the sense described below. We define the piecewise constant functions

$$v^n(\mathbf{x}) = v^n_{i,j} \quad \text{for } \mathbf{x} \in \mathcal{I}_{i,j} \quad \text{and} \quad v_h(\mathbf{x}, t) = v^n(\mathbf{x}) \quad \text{for } t \in [t^n, t^{n+1}).$$

We can write (3.5) as

$$v^{n+1}_{i,j} = F^n_{i,j}\left(v^n_{i,j}, v^n_{i-1,j}, v^n_{i+1,j}, v^n_{i,j-1}, v^n_{i,j+1}\right) \quad \text{or} \quad v^{n+1} = F^n(v^n),$$

and it is easy to see that the CFL-condition (3.8) implies that $F^n_{i,j}$ is non-decreasing in all arguments. Furthermore, by Assumption (**H.2**),

$$F^n_{i,j}(\pm M, \pm M, \pm M, \pm M, \pm M) = \pm M.$$

Thus if $v^n(\mathbf{x}) \in [-M, M]$, then also $v^{n+1} \in [-M, M]$. At this point we recall the useful result by Crandall and Tartar.

Lemma 3.3 (Crandall and Tartar [6])**.** *Let (Ω, μ) be some measure space and let D be a subset of $L^1(\Omega)$ such that if u and v are in D, then so is $(u \vee v)$. Let T be a map $D \mapsto D$ such that*

$$\int_\Omega T(u)\,d\mu = \int_\Omega u\,d\mu \quad \text{for all } u \in D.$$

Then the following statements, valid for all u and v in D, are equivalent:

   (i) *If $u \leq v$, then $T(u) \leq T(v)$.*
  (ii) $\int_\Omega ((T(u) - T(v)) \vee 0)\,d\mu \leq \int_\Omega ((u - v) \vee 0)\,d\mu.$
 (iii) $\int_\Omega |T(u) - T(v)|\,d\mu \leq \int_\Omega |u - v|\,d\mu.$

We can use this lemma for the mapping $v \mapsto F^n(v)$, where D is the subset of $L^1(\mathbb{R}^2)$ consisting of functions that are constant on $\mathcal{I}_{i,j}$. Then the monotonicity of $F^n_{i,j}$ implies that if v and $\tilde{v}$ are in D, then we have

$$\tilde{v} \leq v \Rightarrow F^n(\tilde{v}) \leq F^n(v).$$

By (iii) we then find

$$h^2 \sum_{i,j} \left|F^n_{i,j}(v^n) - F^n_{i,j}(0)\right| \leq h^2 \sum_{i,j} \left|v^n_{i,j}\right|$$

or

$$(3.9) \qquad h^2 \sum_{i,j} \left|v^{n+1}_{i,j}\right| \leq h^2 \sum_{i,j} \left|v^n_{i,j}\right| + C\Delta t\,|u_0|_{BV(\mathbb{R}^2)},$$

where C is a constant depending on $\mathbf{g}$, but not on Δt.

Next, to save space and typing efforts, we introduce the notation $I = (i, j)$, $e_1 = (1, 0)$, $e_2 = (0, 1)$ and

$$D_\ell \alpha_I = \alpha_{I+e_\ell} - \alpha_I.$$

Furthermore, we shall apply D_ℓ to functions of the variables "u" and "v". Let D_i^u be the difference only in the u variable and D_i^v only in the v variable, so that

$$D_\ell^u \alpha \left(u_I, v_I, w_I \right) = \alpha \left(u_{I+e_\ell}, v_I, w_I \right) - \alpha \left(u_I, v_I, w_I \right),$$
$$D_\ell^v \alpha \left(u_I, v_{I-e_\ell}, v_I \right) = \alpha \left(u_I, v_I, v_{I+e_\ell} \right) - \alpha \left(u_I, v_{I-e_\ell}, v_I \right),$$
$$D_\ell \alpha \left(u_I, v_I \right) = D_\ell^u \alpha \left(u_I, v_I \right) + D_\ell^v \alpha \left(u_{I+e_\ell}, v_I \right)$$
$$= D_\ell^v \alpha \left(u_I, v_I \right) + D_\ell^u \alpha \left(u_I, v_{I+e_\ell} \right).$$

We can also use (iii) with $u = v_I^n$ and $v = v_I^{n-1}$ to conclude that

$$h^2 \sum_I \left| v_I^{n+1} - v_I^n \right| \leq h^2 \sum_I \left| v_I^n - v_I^{n-1} \right|$$
$$\leq h^2 \sum_I \left| v_I^1 - v_I^0 \right|$$
$$\leq h^2 \lambda \sum_I \sum_\ell \left| D_\ell g_\ell^{\mathrm{EO}} \left(u_{I-e_\ell/2}^0, v_{I-e_\ell}^0, v_I^0 \right) \right|$$
$$\leq C \Delta t$$

if v_0 and u_0 are of bounded variation.[1] This leads to a bound on the discrete divergence of the numerical flux,

$$(3.10) \qquad h^2 \sum_I \left| \sum_\ell \frac{1}{h} D_\ell g_\ell^{\mathrm{EO}} \left(u_{I-e_\ell/2}^n, v_{I-e_\ell}^n, v_I^n \right) \right| \leq C.$$

Using the "D_ℓ" notation we can write (3.5) as

$$(3.11)$$
$$v_I^{n+1} = v_I^n - \lambda \sum_\ell D_\ell g_\ell^{\mathrm{EO}} \left(u_{I-e_\ell/2}^n, v_{I-e_\ell}^n, v_I^n \right)$$
$$= \underbrace{v_I^n - \lambda \sum_\ell D_\ell^v g_\ell^{\mathrm{EO}} \left(u_{I+e_\ell/2}^n, v_{I-e_\ell/2}^n, v_I^n \right)}_{w_I^n} - \lambda \sum_\ell D_\ell^u g_\ell^{\mathrm{EO}} \left(u_{I-e_\ell}^n, v_{I-e_\ell}^n, v_I^n \right).$$

Let η be a convex entropy, and let $q_\ell^{\mathrm{EO}}(u, a, b)$ denote the associated numerical entropy flux

$$q_\ell^{\mathrm{EO}}(u, a, b) = \int_0^a \eta'(s) \left(\partial_v g_\ell(u, s) \vee 0 \right) ds + \int_0^b \eta'(s) \left(\partial_v g_\ell(u, s) \wedge 0 \right) ds.$$

Since the scheme is monotone we get

$$(3.12) \qquad \eta \left(w_I^n \right) \leq \eta \left(v_I^n \right) - \lambda \sum_\ell D_\ell^v q_\ell^{\mathrm{EO}} \left(u_{I+e_\ell/2}^n, v_{I-e_\ell}^n, v_I^n \right),$$

while by the convexity of η and the definition of w_I^n, see (3.11),

$$(3.13) \qquad \eta \left(w_I^n \right) \geq \eta \left(v_I^{n+1} \right) + \eta' \left(v_I^{n+1} \right) \lambda \sum_\ell D_\ell^u g_\ell^{\mathrm{EO}} \left(u_{I-e_\ell/2}^n, v_{I-e_\ell}^n, v_I^n \right).$$

---

[1] This is only a sufficient condition. We just need to assume that the initial discrete divergence is bounded, i.e., $\sum_I \left| \sum_\ell D_\ell g_{\ell,I}^{\mathrm{EO}}(u_{I-e_\ell/2}^0, v_{I-e_\ell}^0, v_I^0) \right| \leq C$.

Using (3.11), (3.12) and (3.13), we arrive at the following estimate:

$$\eta\left(v_I^{n+1}\right) \le \eta\left(v_I^n\right) - \lambda \sum_\ell D_\ell q_\ell^{\mathrm{EO}}\left(u_{I-e_\ell/2}^n, v_{I-e_\ell}^n, v_I^n\right)$$

$$- \lambda \sum_\ell D_\ell^u\left(\eta'\left(v_I^{n+1}\right) g_\ell^{\mathrm{EO}}\left(u_{I-e_\ell/2}^n, v_{I-e_\ell}^n, v_I^n\right)\right.$$

$$(3.14) \qquad \left. - q_\ell^{\mathrm{EO}}\left(u_{I-e_\ell/2}^n, v_{I-e_\ell}^n, v_I^n\right)\right).$$

Since v_h is bounded, and both g_ℓ^{EO} and q_ℓ^{EO} are Lipschitz continuous, and u_h is of bounded variation we get the bound for non-negative η:

$$(3.15) \qquad \|\eta(v_h(\cdot,t))\|_{L^1(\mathbb{R}^2)} \le \|\eta(v_0)\|_{L^1(\mathbb{R}^2)} + Ct\,|u_0|_{BV(\mathbb{R}^2)},$$

for some constant C which does not depend on h. Choosing $\eta(v) = v^2$ we get the L^2 estimate

$$(3.16) \qquad \|v_h(\cdot,t)\|_{L^2(\mathbb{R}^2)}^2 \le \|v_0\|_{L^2(\mathbb{R}^2)}^2 + Ct\,|u_0|_{BV(\mathbb{R}^2)}.$$

We can actually get a stronger estimate by multiplying the scheme (3.11) by $\eta'(v_I^{n+1})$ and using the Taylor expansion

$$\eta'\left(v_I^{n+1}\right)\left(v_I^{n+1} - v_I^n\right) = \eta\left(v_I^{n+1}\right) - \eta\left(v_I^n\right) + \frac{1}{2}\eta''\left(\xi_I^{n+1/2}\right)\left(v_I^{n+1} - v_I^n\right)^2,$$

where $\xi_I^{n+1/2}$ is an intermediate value. Doing this gives
(3.17)

$$\eta\left(v_I^{n+1}\right) - \eta\left(v_I^n\right) + \frac{1}{2}\eta''\left(\xi_I^{n+1/2}\right)\left(v_I^{n+1} - v_I^n\right)^2$$

$$= -\lambda\eta'\left(v_I^{n+1}\right)\sum_\ell D_\ell g_\ell^{\mathrm{EO}}\left(u_{I-e_\ell/2}^n, v_{I-e_\ell}^n, v_I^n\right)$$

$$= -\lambda\eta'\left(v_I^n\right)\sum_\ell D_\ell g_\ell^{\mathrm{EO}}\left(u_{I-e_\ell/2}^n, v_{I-e_\ell}^n, v_I^n\right)$$

$$- \lambda\left(\eta'\left(v_I^{n+1}\right) - \eta'\left(v_I^n\right)\right)\sum_\ell D_\ell g_\ell^{\mathrm{EO}}\left(u_{I-e_\ell/2}^n, v_{I-e_\ell}^n, v_I^n\right).$$

We follow [16] and define

$$g^-(u,v) = \int_0^v (\partial_v g(u,s) \wedge 0)\,ds, \quad g^+(u,v) = \int_0^v (\partial_v g(u,s) \vee 0)\,ds,$$

and introduce the functions

$$\mathcal{G}_\ell^\pm(u,v) = \int_0^v \eta'(s)\partial_v g_\ell^\pm(u,s)\,ds.$$

Integration by parts gives

$$\mathcal{G}_\ell^\pm(u,b) - \mathcal{G}_\ell^\pm(u,a) = \eta'(b)\left(g_\ell^\pm(u,b) - g_\ell^\pm(u,a)\right)$$

$$(3.18) \qquad - \int_a^b \eta''(s)\left(g_\ell^\pm(u,s) - g_\ell^\pm(u,a)\right)\,ds.$$

Therefore,

$$
\eta'\left(v_I^n\right) D_\ell^v g_\ell^- \left(u_{I-e_\ell/2}^n, v_I^n\right) = D_\ell \mathcal{G}_\ell^- \left(u_{I-e_\ell/2}^n, v_I^n\right) - D_\ell^u \mathcal{G}_\ell^- \left(u_{I-e_\ell/2}^n, v_{I+e_\ell}^n\right)
$$

$$
- \int_{v_{I+e_\ell}^n}^{v_I^n} \eta''(s) \left(g_\ell^-\left(u_{I-e_\ell/2}^n, s\right) - g_\ell^-\left(u_{I-e_\ell/2}^n, v_{I+e_\ell}^n\right)\right) ds,
$$

$$
\eta'\left(v_I^n\right) D_\ell^v g_\ell^+ \left(u_{I-e_\ell/2}^n, v_{I-e_\ell}^n\right) = D_\ell \mathcal{G}_\ell^+ \left(u_{I-e_\ell/2}^n, v_{I-e_\ell}^n\right) - D_\ell^u \mathcal{G}_\ell^+ \left(u_{I-e_\ell/2}^n, v_I^n\right)
$$

$$
+ \int_{v_{I-e_\ell}^n}^{v_I^n} \eta''(s) \left(g_\ell^+\left(u_{I-e_\ell/2}^n, s\right) - g_1^+\left(u_{I-e_\ell/2}^n, v_{I-e_\ell}^n\right)\right) ds.
$$

Hence,

$$
\eta'\left(v_I^n\right) D_\ell g_\ell^{\mathrm{EO}} \left(u_{I-e_\ell/2}^n, v_{I-e_\ell}^n, v_I^n\right) = D_\ell q_\ell^{\mathrm{EO}} \left(u_{I-e_\ell/2}^n, v_{I-e_\ell}^n, v_I^n\right)
$$

$$
- D_\ell^u \left(\eta'\left(v_I^n\right) g_\ell^{\mathrm{EO}} \left(u_{I-e_\ell/2}^n, v_I^n, v_{I+e_\ell}^n\right) - q_\ell^{\mathrm{EO}} \left(u_{I-e_\ell/2}^n, v_I^n, v_{I+e_\ell}^n\right)\right)
$$

$$
\underbrace{- \int_{v_{I+e_\ell}^n}^{v_I^n} \eta''(s) \left(g_\ell^-\left(u_{I-e_\ell/2}^n, s\right) - g_\ell^-\left(u_{I-e_\ell/2}^n, v_{I+e_\ell}^n\right)\right) ds}_{\Theta_{I,\ell}^-}
$$

$$
\underbrace{+ \int_{v_{I-e_\ell}^n}^{v_I^n} \eta''(s) \left(g_\ell^+\left(u_{I-e_\ell/2}^n, s\right) - g_1^+\left(u_{I-e_\ell/2}^n, v_{I-e_\ell}^n\right)\right) ds}_{\Theta_{I,\ell}^+} .
$$

Since g_ℓ^- is decreasing in v, and g_ℓ^+ is increasing in v, $\Theta_{I,\ell}^-, \Theta_{I,\ell}^+ \geq 0$. Thus (3.17) can be written as

$$
\eta\left(v_I^{n+1}\right) - \eta\left(v_I^n\right) + \frac{1}{2}\eta''\left(\xi_I^{n+1/2}\right)\left(v_I^{n+1} - v_I^n\right)^2 + \lambda \sum_\ell \left(\Theta_{I,\ell}^+ + \Theta_{I,\ell}^-\right)
$$

$$
= -\lambda \sum_\ell D_\ell q_\ell^{\mathrm{EO}} \left(u_{I-e_\ell/2}^n, v_{I-e_\ell}^n, v_I^n\right)
$$

(3.19)

$$
- \lambda \left(\eta'\left(v_I^{n+1}\right) - \eta'\left(v_I^n\right)\right) \sum_\ell D_\ell g_\ell^{\mathrm{EO}} \left(u_{I-e_\ell/2}^n, v_{I-e_\ell}^n, v_I^n\right)
$$

$$
- \lambda \sum_\ell D_\ell^u \left(\eta'\left(v_I^n\right) g_\ell^{\mathrm{EO}} \left(u_{I-e_\ell/2}^n, v_I^n, v_{I+e_\ell}^n\right)\right.
$$

$$
\left. - q_\ell^{\mathrm{EO}} \left(u_{I-e_\ell/2}^n, v_I^n, v_{I+e_\ell}^n\right)\right).
$$

Now we can multiply this by h^2, sum over $n \in \{0, \ldots, N-1\}$ and $I \in \mathbb{Z}^2$, and use the divergence bound (3.10) and the fact that u_h has bounded variation to get

(3.20)

$$h^2 \sum_I \eta\left(v_I^N\right) + \frac{1}{2}h^2 \sum_{n,I} \eta''\left(\xi_I^{n+1/2}\right)\left(v_I^{n+1} - v_I^n\right)^2$$

$$+ h^2\lambda \sum_{n,I} \sum_\ell \left(\Theta_{I,\ell}^+ + \Theta_{I,\ell}^-\right)$$

$$\leq h^2 \sum_I \eta\left(v_I^0\right) + 2\|\eta'\|_{L^\infty} h^2\Delta t \sum_{n,I} \frac{1}{h}\left|\sum_\ell D_\ell g_\ell^{\mathrm{EO}}\left(u_{I-e_\ell/2}^n, v_{I-e_\ell}^n, v_I^n\right)\right|$$

$$+ Ch^2\Delta t \sum_{n,I} \sum_\ell \frac{1}{h}\left|u_{I+e_\ell/2}^n - u_{I-e_\ell/2}^n\right|$$

$$\leq h^2 \sum_I \eta\left(v_I^0\right) + C_{\eta'}$$

where the constant $C_{\eta'}$ depends on the initial variation of u_h and the initial divergence bound on $\mathbf{g}$; cf. (3.10).

To get the L^2 bound, choose $\eta(v) = v^2$. Then by an easy lemma in [8],

$$\Theta_{I,\ell}^- \geq \frac{1}{\|\partial_v \mathbf{g}\|_{L^\infty}}\left(D_\ell^v g_\ell^-\left(u_{I-e_\ell/2}^n, v_I^n\right)\right)^2,$$

$$\Theta_{I,\ell}^+ \geq \frac{1}{\|\partial_v \mathbf{g}\|_{L^\infty}}\left(D_\ell^v g_\ell^+\left(u_{I-e_\ell/2}^n, v_{I-e_\ell}^n\right)\right)^2.$$

Therefore,
(3.21)

$$h^2 \sum_I \left(v_I^N\right)^2 + h^2 \sum_{n,I} \left(v_I^{n+1} - v_I^n\right)^2$$

$$+ \frac{\lambda}{\|\partial_v \mathbf{g}\|_{L^\infty}} h^2 \sum_{n,I} \sum_\ell \left(D_\ell^v g_\ell^-\left(u_{I-e_\ell/2}^n, v_I^n\right)\right)^2 + \left(D_\ell^v g_\ell^+\left(u_{I-e_\ell/2}^n, v_{I-e_\ell}^n\right)\right)^2$$

$$\leq h^2 \sum_I \left(v_I^0\right)^2 + Ct^N.$$

This has two immediate consequences. Firstly, since

$$D_\ell g_\ell^{\mathrm{EO}}\left(u_{I-e_\ell/2}^n, v_{I-e_\ell}^n, v_I^n\right) = D_\ell^v g_\ell^-\left(u_{I+e_\ell/2}^n, v_I^n\right) + D_\ell^v g_\ell^+\left(u_{I+e_\ell/2}^n, v_{I-e_\ell}^n\right)$$

$$+ D_\ell^u g_\ell^{\mathrm{EO}}\left(u_{I-e_\ell/2}^n, v_I^n, v_{I+e_\ell}^n\right),$$

we get

$$\left(D_\ell g_\ell^{\mathrm{EO}}\left(u_{I-e_\ell/2}^n, v_{I-e_\ell}^n, v_I^n\right)\right)^2$$

$$\leq 4\left(D_\ell^v g_\ell^-\left(u_{I+e_\ell/2}^n, v_I^n\right)\right)^2 + 4\left(D_\ell^v g_\ell^+\left(u_{I+e_\ell/2}^n, v_{I-e_\ell}^n\right)\right)^2$$

$$+ 2\left(D_\ell^u g_\ell^{\mathrm{EO}}\left(u_{I-e_\ell/2}^n, v_I^n, v_{I+e_\ell}^n\right)\right)^2.$$

Therefore,

(3.22) $$\Delta t h^2 \sum_{n,I} \sum_\ell \left(D_\ell g_\ell^{\mathrm{EO}}\left(u_{I-e_\ell/2}^n, v_{I-e_\ell}^n, v_I^n\right)\right)^2 \leq C_{t^N} h,$$

where C_t is a finite constant depending on t but not on h. Secondly, this gives an estimate for the "time variation" of v_h,

$$(3.23) \qquad \Delta t h^2 \sum_{I,n} \left(v_I^{n+1} - v_I^n\right)^2 \leq C_{t^N} h.$$

We can also bound the variation of the entropy flux. To do this note that

$$q_\ell(u,v) = \mathcal{G}_\ell^+(u,v) + \mathcal{G}_\ell^-(u,v),$$

and by (3.18) and the monotonicity of $g_\ell^\pm$,

$$\left|\mathcal{G}_\ell^\pm(u,b) - \mathcal{G}_\ell^\pm(u,a)\right| \leq |\eta'(b)| \left|g_\ell^\pm(u,b) - g_\ell^\pm(u,a)\right|$$

$$+ \left|\int_a^b \eta''(s)\,ds\right| \left|g_\ell^\pm(u,b) - g_\ell^\pm(u,a)\right|$$

$$\leq 3\,\|\eta'\|_{L^\infty} \left|g_\ell^\pm(u,b) - g_\ell^\pm(u,a)\right|.$$

Thus

$$|q_\ell(u,b) - q_\ell(u,a)| \leq 3\,\|\eta'\|_{L^\infty} \left(\left|g_\ell^+(u,b) - g_\ell^+(u,a)\right| + \left|g_\ell^-(u,b) - g_\ell^-(u,a)\right|\right).$$

Then we arrive at the estimate

$$(3.24) \qquad \Delta t h^2 \sum_{I,n} \sum_\ell \left(D_\ell q_\ell\left(u_{I-e_\ell/2}^n, v_I^n\right)\right)^2 \leq C_{t^N} h.$$

Next, let $\{w_h\}_{h>0}$ be a sequence of piecewise constant functions

$$w_h(\mathbf{x},t) = w_I^n = \frac{1}{4} \sum_\ell \left(u_{I+e_\ell/2}^n + u_{I-e_\ell/2}^n\right) \quad \text{for } \mathbf{x} \in \mathcal{I}_{i,j} \text{ and } t \in [t^n, t^{n+1}).$$

Then w_h is of bounded variation, i.e.,

$$|w_h(\cdot,t)|_{BV(\mathbb{R}^2)} \leq C \quad \text{for all } h > 0 \text{ and for all } t \geq 0.$$

Lemma 3.4. *Let $(\eta_0, \mathbf{q}_0)$ be the Kružkov entropy/entropy flux pair, i.e.,*

$$\eta_0(v) = |v - k|, \qquad q_{0,\ell}(u,v) = \text{sign}\,(v - k)\,(g_\ell(u,v) - g_\ell(u,k)),$$

where k is a constant. Then the sequence

$$\{\partial_t \eta_0(v_h) + \text{div}\,(\mathbf{q}_0(w_h, v_h))\}_h$$

is compact in $H_{\text{loc}}^{-1}(\mathbb{R}^2 \times \mathbb{R}^+)$.

Proof. It will be convenient to work with smooth entropies, rather than η_0. Therefore, we let η_h be a smooth convex approximation to η_0, so that $\eta_h(0) = 0$ and $|\eta_h'| \leq 1$, and we have that

$$\|\eta_h - \eta_0\|_{L^\infty} \leq h.$$

Let $\mathbf{q}_h$ be the entropy flux corresponding to η_h. Then we also have that

$$\|\mathbf{q}_h - \mathbf{q}_0\|_{L^\infty} \to 0 \quad \text{as } h \to 0.$$

Let φ be a function in $C_0^1(\mathbb{R}^2 \times \mathbb{R}^+)$. We use the notation $\hat{\ell} = 3 - \ell$. Set

$$\langle \mathcal{L}, \varphi \rangle = \langle \partial_t \eta_0(v_h) + \text{div}\,(\mathbf{q}_0(w_h, v_h)), \varphi \rangle$$

$$= \langle \mathcal{L}_h, \varphi \rangle + \langle \mathcal{L} - \mathcal{L}_h, \varphi \rangle,$$

where

$$\langle \mathcal{L}_h, \varphi \rangle = \langle \partial_t \eta_h(v_h) + \text{div}\,(\mathbf{q}_0(w_h, v_h)), \varphi \rangle.$$

Now

$$|\langle \mathcal{L} - \mathcal{L}_h, \varphi \rangle| \leq \int_{\mathbb{R}^2 \times \mathbb{R}^+} |\eta_h(v_h) - \eta_0(v_h)| \, |\varphi_t| \, d\mathbf{x}dt$$

$$\leq C \, \|\partial_t \varphi\|_{L^2(\mathbb{R}^2 \times \mathbb{R}^+)} \|\eta_h - \eta_0\|_{L^\infty} \to 0 \text{ as } h \to 0,$$

where the constant C is independent of h but will depend on the support of φ. Thus $\mathcal{L} - \mathcal{L}_h$ is compact in $H_{\text{loc}}^{-1}(\mathbb{R}^2 \times \mathbb{R}^+)$.

Now

$$\langle \mathcal{L}_h, \varphi \rangle = \langle \eta_h(v_h)_t + \text{div}(\mathbf{q}_0(w_h, v_h)), \varphi \rangle$$

$$= \sum_{n \geq 0} \sum_I \left(\eta_h\left(v_I^{n+1}\right) - \eta_h\left(v_I^n\right) \right) \int_{\mathcal{I}_I} \varphi(x_1, x_2, t^{n+1}) \, d\mathbf{x}$$

$$+ \sum_{n \geq 0} \sum_I \sum_\ell D_\ell q_{0,\ell}\left(w_I^n, v_I^n\right) \int_{t^n}^{t^{n+1}} \int_{x_I - e_{\hat{\ell}}/2}^{x_I + e_{\hat{\ell}}/2} \varphi(x_{I+e_\ell/2}, x_{\hat{\ell}}, t) \, dx_{\hat{\ell}} dt.$$

Set

(3.25)
$$\varphi_I^n = \frac{1}{\Delta t h^2} \int_{\mathcal{I}_I^n} \varphi(\mathbf{x}, t) \, d\mathbf{x}dt,$$

where

(3.26)
$$\mathcal{I}_I^n = \mathcal{I}_I \times [t^n, t^{n+1}).$$

We wish to replace the integrals in the definition of $\mathcal{L}_h$ by $h^2 \varphi_I^n$ and $h\Delta t \varphi_I^n$, respectively. The error we make in doing this in the first integral is

$$\left| \sum_{n \geq 0, I} \left(\eta_h\left(v_I^{n+1}\right) - \eta_h\left(v_I^n\right) \right) \left(\int_{\mathcal{I}_I} \varphi(\mathbf{x}, t^{n+1}) \, d\mathbf{x} - h^2 \varphi_I^n \right) \right|$$

$$\leq \|\eta_h'\|_{L^\infty} \sum_{n,I} \left| v_I^{n+1} - v_I^n \right| \frac{1}{\Delta t} \int_{\mathcal{I}_I^n} \left| \varphi(\mathbf{x}, t^{n+1}) - \varphi(\mathbf{x}, t) \right| \, d\mathbf{x}dt$$

$$\leq \sum_{n,I} \left| v_I^{n+1} - v_I^n \right| \frac{1}{\Delta t} \int_{\mathcal{I}_I^n} \int_t^{t^{n+1}} |\partial_t \varphi(\mathbf{x}, s)| \, ds \, d\mathbf{x}dt$$

$$\leq \sum_{n,I} \left| v_I^{n+1} - v_I^n \right| \frac{1}{\Delta t} \int_{\mathcal{I}_I} \int_{t^n}^{t^{n+1}} \sqrt{t^{n+1} - t} \left(\int_{t^n}^{t^{n+1}} |\partial_t \varphi(\mathbf{x}, s)|^2 \, ds \right)^{1/2} d\mathbf{x}dt$$

$$\leq \frac{2}{3} \sum_{n,I} \left| v_I^{n+1} - v_I^n \right| \sqrt{\Delta t} \int_{\mathcal{I}_I} \left(\int_{t^n}^{t^{n+1}} |\partial_t \varphi(\mathbf{x}, s)|^2 \, ds \right)^{1/2} d\mathbf{x}$$

$$\leq \frac{2}{3} \sum_{n,I} \left| v_I^{n+1} - v_I^n \right| h\sqrt{\Delta t} \left(\int_{\mathcal{I}_I^n} (\partial_t \varphi(\mathbf{x}, t))^2 \, d\mathbf{x}dt \right)^{1/2}$$

$$\leq \frac{2}{3} \left(h^2 \Delta t \sum_{n,I} \left(v_I^{n+1} - v_I^n \right)^2 \right)^{1/2} \left(\sum_{n,I} \int_{\mathcal{I}_I^n} (\partial_t \varphi(\mathbf{x}, t))^2 \, d\mathbf{x}dt \right)^{1/2}$$

(3.27)
$$\leq \frac{2}{3} \lambda \sqrt{C_T h} \, \|\varphi\|_{H^1(\mathbb{R}^2 \times R^+)},$$

by (3.23), where T is such that $\varphi(\mathbf{x}, t) = 0$ for $t > T$. Next we observe that

$$
\sum_\ell \left| q_{0,\ell}(u^n_{I-e_\ell/2}, a) - q_{0,\ell}(w^n_I, a) \right|
$$

$$
\leq \frac{\|\partial_u \mathbf{q}_0\|_{L^\infty}}{4} \sum_\ell \left| u^n_{I-e_\ell/2} - u^n_{I+e_\ell/2} \right| + \left| u^n_{I-e_\ell/2} - u^n_{I+e_{\hat\ell}/2} \right| + \left| u^n_{I-e_\ell/2} - u^n_{I-e_{\hat\ell}/2} \right|.
$$

Therefore, the functional $\mathcal{L}_1$, defined by

$$
\langle \mathcal{L}_1, \varphi \rangle = \sum_{n \geq 0} \sum_I \sum_\ell D_\ell \left(q_{0,\ell}(w^n_I, v^n_I) - q_{0,\ell}\left(u^n_{I-e_\ell/2}, v^n_I \right) \right)
$$

$$
\times \int_{t^n}^{t^{n+1}} \int_{x_I - e_{\hat\ell}/2}^{x_I + e_{\hat\ell}/2} \varphi(x_{I-e_\ell/2}, x_{\hat\ell}, t) \, dx_{\hat\ell} dt
$$

$$
= -\sum_{n \geq 0} \sum_I \sum_\ell \left(q_{0,\ell}(w^n_I, v^n_I) - q_{0,\ell}\left(u^n_{I-e_\ell/2}, v^n_I \right) \right) \int_{\mathcal{I}^n_I} \partial_{x_l} \varphi \, d\mathbf{x} dt
$$

$$
\leq \sum_{n,I} \sum_\ell \left| q_{0,\ell}(w^n_I, v^n_I) - q_{0,\ell}\left(u^n_{I-e_\ell/2}, v^n_I \right) \right| h\sqrt{\Delta t}
$$

$$
\times \left(\int_{\mathcal{I}^n_I} (\partial_{x_\ell} \varphi)^2 \, d\mathbf{x} \right)^{1/2}
$$

$$
\leq \left(\sum_{n,I} \sum_\ell \left(q_{0,\ell}(w^n_I, v^n_I) - q_{0,\ell}\left(u^n_{I-e_\ell/2}, v^n_I \right) \right)^2 \right)^{1/2} \|\varphi\|_{H^1(\mathbb{R}^2 \times \mathbb{R}^+)}
$$

$$
\leq \sqrt{CT |u_h(\cdot, 0)|_{BV(\mathbb{R}^2)} h} \, \|\varphi\|_{H^1(\mathbb{R}^2 \times \mathbb{R}^+)},
$$

for some constant C that does not depend on h. Hence, we can replace $D_\ell q_{0,\ell}(w^n_I, v^n_I)$ by $D_\ell q_{0,\ell}(u^n_{I-e_\ell/2}, v^n_I)$ in the second part of $\mathcal{L}_h$, making an error which tends to zero in $H^{-1}_{\text{loc}}(\mathbb{R}^2 \times \mathbb{R}^+)$. Similarly to deriving (3.27), by using (3.24), we get the bound
(3.28)

$$
\left| \sum_{n,I} \sum_\ell D_\ell q_{0,\ell}\left(u^n_{I-e_\ell/2}, v^n_I \right) \left(\int_{t^n}^{t^{n+1}} \int_{x_I - e_{\hat\ell}/2}^{x_I + e_{\hat\ell}/2} \varphi(x_{I+e_\ell/2}, x_{\hat\ell}, t) \, dx_{\hat\ell} dt - \Delta t h \varphi^n_I \right) \right|
$$

$$
\leq \tilde{C}_T \sqrt{h} \, \|\varphi\|_{H^1(\mathbb{R}^2 \times R^+)}.
$$

Summing up the discussion so far, we have established that

$$
\langle \mathcal{L}, \varphi \rangle = h^2 \Delta t \sum_{n,I} \left[\frac{1}{\Delta t} \left(\eta_h \left(v^{n+1}_I \right) - \eta_h \left(v^n_I \right) \right) + \frac{1}{h} \sum_\ell D_\ell q_{0,\ell} \left(u^n_{I-e_\ell/2}, v^n_I \right) \right] \varphi^n_I
$$

$$
+ \text{ terms which are compact in } H^{-1}_{\text{loc}}(\mathbb{R}^2 \times \mathbb{R}^+).
$$

By using the "scheme for η", (3.19), we find that the term in square brackets above can be written as

$$[\cdots] = \underbrace{-\frac{1}{\Delta t}\frac{1}{2}\eta_h''\left(\xi_I^{n+1/2}\right)\left(v_I^{n+1} - v_I^n\right)^2 - \frac{1}{h}\sum_\ell\left(\Theta_{I,\ell}^- + \Theta_{I,\ell}^+\right)}_{A_I^n}$$

$$\underbrace{-\frac{1}{h}\left(\eta'\left(v_I^{n+1}\right) - \eta'\left(v_I^n\right)\right)\sum_\ell D_\ell g_\ell^{\mathrm{EO}}\left(u_{I-e_\ell/2}^n, v_{I-e_\ell}^n, v_I^n\right)}_{B_I^n}$$

$$\underbrace{-\frac{1}{h}\sum_\ell D_\ell^u\left(\eta'\left(v_I^n\right)g_\ell^{\mathrm{EO}}\left(u_{I-e_\ell/2}^n, v_I^n, v_{I+e_\ell}^n\right) - q_\ell^{\mathrm{EO}}\left(u_{I-e_\ell/2}^n, v_I^n, v_{I+e_\ell}^n\right)\right)}_{C_I^n}$$

$$\underbrace{-\frac{1}{h}\sum_\ell D_\ell\left(q_\ell^{\mathrm{EO}}\left(u_{I-e_\ell/2}^n, v_{I-e_\ell}^n, v_I^n\right) - q_{0,\ell}\left(u_{I-e_\ell/2}^n, v_I^n\right)\right)}_{D_I^n}.$$

Thus we can write

$$\langle\mathcal{L},\varphi\rangle = \langle\mathcal{A},\varphi\rangle + \langle\mathcal{B},\varphi\rangle + \langle\mathcal{C},\varphi\rangle + \langle\mathcal{D},\varphi\rangle + \text{compact terms}$$

with

$$\langle\mathcal{A},\varphi\rangle = h^2\Delta t\sum_{n,I} A_I^n\varphi_I^n, \quad \langle\mathcal{B},\varphi\rangle = h^2\Delta t\sum_{n,I} B_I^n\varphi_I^n,$$

$$\langle\mathcal{C},\varphi\rangle = h^2\Delta t\sum_{n,I} C_I^n\varphi_I^n \quad \text{and} \quad \langle\mathcal{D},\varphi\rangle = h^2\Delta t\sum_{n,I} D_I^n\varphi_I^n.$$

By (3.20),

$$\langle\mathcal{A},\varphi\rangle \le \|\varphi\|_{L^\infty(\mathbb{R}^2\times\mathbb{R})}\left(\frac{h^2}{2}\sum_{n,I}\eta_h''\left(\xi_I^{n+1/2}\right)\left(v_I^{n+1} - v_I^n\right)^2 + h\Delta t\sum_{n,I}\sum_\ell\left(\Theta_{I,\ell}^- + \Theta_{I,\ell}^+\right)\right)$$

$$\le C_T\|\varphi\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^+)}.$$

Therefore, $\mathcal{A} \in \mathcal{M}_{\mathrm{loc}}(\mathbb{R}^2\times\mathbb{R}^+)$. By the divergence bound on the numerical flux (3.10) and the BV bound on u_h,

$$(3.29) \qquad\qquad |\langle\mathcal{B} + \mathcal{C},\varphi\rangle| \le C\|\varphi\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^+)};$$

thus also $\mathcal{B} + \mathcal{C} \in \mathcal{M}_{\mathrm{loc}}(\mathbb{R}^2\times\mathbb{R}^+)$.

To estimate $\mathcal{D}$ we start by observing that for $\varepsilon > 0$,

$$q_{\varepsilon,\ell}^{\mathrm{EO}}(u,a,b) - q_{\varepsilon,\ell}(u,b) = q_{\varepsilon,\ell}^{\mathrm{EO}}(u,a,b) - q_{\varepsilon,\ell}^{\mathrm{EO}}(b,b)$$

$$= \int_b^a \eta_\varepsilon'(s)\partial_v g^-(u,s)\,ds$$

$$= \eta_\varepsilon'(a)\left(g^-(u,a) - g^-(u,b)\right)$$

$$- \int_b^a \eta_\varepsilon''(s)\left(g^-(u,s) - g^-(u,a)\right)\,ds.$$

As before, by using the monotonicity of g^- we get

$$\left|q_{\varepsilon,\ell}^{\mathrm{EO}}(u,a,b) - q_{\varepsilon,\ell}(u,b)\right| \le 3\|\eta_\varepsilon'\|_{L^\infty}\left|g^-(u,a) - g^-(u,b)\right| \le 3\left|g^-(u,a) - g^-(u,b)\right|.$$

Sending ε to zero gives

$$(3.30) \qquad \left| q_{0,\ell}^{\mathrm{EO}}(u, a, b) - q_{0,\ell}(u, b) \right| \le 3 \left| g^-(u, a) - g^-(u, b) \right|.$$

Next, by a partial summation,

$$
\left| h^2 \Delta t \sum_{n,I} D_I^n \varphi_I^n \right| = \left| h^2 \Delta t \sum_{n,I} \sum_{\ell} \left(g_\ell^{\mathrm{EO}} \left(u_{I-e_\ell/2}^n, v_{I-e_\ell}^n, v_I^n \right) \right. \right.
$$

$$
\left. \left. - q_{0,\ell} \left(u_{I-e_\ell/2}^n, v_I^n \right) \right) \frac{1}{h} D_\ell \varphi_{I-e_\ell}^n \right|
$$

$$
\le 3 \sum_{\ell} h^2 \Delta t \sum_{n,I} \left| D_\ell^v g^- \left(u_{I-e_\ell/2}^n, v_{I-e_\ell}^n \right) \right| \frac{1}{h} \left| D_\ell \varphi_I^n \right|
$$

$$
\le 3 \sum_{\ell} \sqrt{ h^2 \Delta t \sum_{n,I} \left(D_\ell^v g^- \left(u_{I-e_\ell/2}^n, v_{I-e_\ell}^n \right) \right)^2 } \sqrt{ h^2 \Delta t \sum_{n,I} \left(\frac{1}{h} \left| D_\ell \varphi_I^n \right| \right)^2 }.
$$

Using the bound (3.21) (for g^-) we find that

$$
|\langle \mathcal{C}, \varphi \rangle| \le C_T \sqrt{h} \sum_{\ell} \left\| \partial_{x_\ell} \varphi \right\|_{L^2(\mathbb{R}^2 \times \mathbb{R}^+)}.
$$

Therefore, $\mathcal{D}$ is compact in $H_{\mathrm{loc}}^{-1}(\mathbb{R}^2 \times \mathbb{R}^+)$.

Hence we have established that $\mathcal{L}$ is compact in $H_{\mathrm{loc}}^{-1}(\mathbb{R}^2 \times \mathbb{R}^+)$. $\qquad\square$

Since u_h converges strongly in L^p to u, and u_h is of bounded variation, also w_h will converge strongly to u in L^p for any $p \in [1, \infty)$. In particular this implies

$$
|\langle \mathrm{div} \left(\mathbf{q}_0(w_h, v_h) - \mathbf{q}(u, v_h) \right), \varphi \rangle| \le C \left\| w_h - u \right\|_{L^2(\mathbb{R}^2 \times \mathbb{R}^+)} \left\| \varphi \right\|_{H^1(\mathbb{R}^2 \times \mathbb{R}^+)} \to 0,
$$

as $h \to 0$. Thus the sequence

$$
\left\{ \partial_t \eta_0(v_h) + \mathrm{div} \left(\mathbf{q}_0(u, v_h) \right) \right\}_{h > 0}
$$

is compact in $H_{\mathrm{loc}}^{-1}(\mathbb{R}^2)$. Therefore, by Lemma 2.2 there exists a subsequence $\{v_h\}$ (which we do not relabel) and a function $v \in L^\infty(\mathbb{R}^2 \times \mathbb{R}^+)$ such that

$$(3.31) \qquad v_h \to v \quad \text{a.e. and in } L_{\mathrm{loc}}^p(\mathbb{R}^2 \times \mathbb{R}^+), \text{ for any } p \in [1, \infty).$$

Lemma 3.5. *Let u and v be the maps defined in (3.4) and (3.31), respectively. The pair (u, v) is a weak solution of the Cauchy problem (1.1)-(1.2) in the sense of Definition 2.1.*

Proof. Recalling Lemma 3.1 we only have to verify that v is a weak solution of the conservation law (1.2), i.e., that the second equation in Definition 2.1 holds.

Define

$$
D_t v_I^n = v_I^{n+1} - v_I^n.
$$

With this notation the scheme for v, (3.11), can be written as

$$
\frac{1}{\Delta t} D_t v_I^n + \sum_{\ell} \frac{1}{h} D_\ell g_\ell^{\mathrm{EO}} \left(u_{I-e_\ell/2}^n, v_{I-e_\ell}^n, v_I^n \right) = 0.
$$

Let $\varphi \in C_0^\infty(\mathbb{R}^2 \times \mathbb{R}_0^+)$ be a test function and let φ_I^n be defined by (3.26). Multiplying the above by $h^2 \Delta t \varphi_I^n$ and doing a partial summation, we find that

$$0 = \underbrace{h^2 \sum_I v_I^0 \varphi_I^0}_{I_0} + \underbrace{h^2 \Delta t \sum_{n \geq 1, I} v_I^n \frac{1}{\Delta t} D_t \varphi_I^{n-1}}_{I_1}$$
$$+ \underbrace{h^2 \Delta t \sum_{n,I} \sum_\ell g_\ell^{\mathrm{EO}} \left(u_{I+e_\ell/2}^n, v_I^n, v_{I+e_\ell}^n \right) \frac{1}{h} D_\ell \varphi_I^n}_{I_2}.$$

Now

$$I_0 = \int_{\mathbb{R}^2} v_h(\mathbf{x}, 0) \varphi(\mathbf{x}, 0) \, d\mathbf{x} + \underbrace{\sum_I v_I^0 \int_{\mathcal{I}_I} \int_0^{\Delta t} \frac{\varphi(\mathbf{x}, s) - \varphi(\mathbf{x}, 0)}{\Delta t} \, ds d\mathbf{x}}_{I_{0,1}}.$$

The term $I_{0,1}$ can be estimated by observing that

$$|I_{0,1}| \leq \sum_I |v_I^0| \int_{\mathcal{I}_I} \int_0^{\Delta t} \frac{1}{\Delta t} \int_0^s \partial_t \varphi(\mathbf{x}, \tau) \, d\tau ds d\mathbf{x}$$
$$\leq \sum_I |v_I^0| \int_{\mathcal{I}_I} \frac{\|\partial_t \varphi\|_{L^\infty}}{\Delta t} \int_0^{\Delta t} s \, ds d\mathbf{x}$$
$$\leq \frac{\|\partial_t \varphi\|_{L^\infty} \Delta t}{2} \|v_0\|_{L^1(\mathbb{R}^2)} \to 0 \quad \text{as } \Delta t \to 0.$$

Using this and the bounded convergence theorem,

$$(3.32) \qquad\qquad \lim_{h \to 0} I_0 = \int_{\mathbb{R}^2} v_0(\mathbf{x}) \varphi(\mathbf{x}, 0) \, d\mathbf{x}.$$

Similarly to the estimate on I_0, we can write I_1 as

$$I_1 = \int_{\Delta t}^\infty \int_{\mathbb{R}^2} v_h(\mathbf{x}, t) \partial_t \varphi(\mathbf{x}, t) \, d\mathbf{x} dt + \underbrace{\sum_{n \geq 1, I} v_I^n \int_{\mathcal{I}_I^n} \int_{t-\Delta t}^t \frac{\partial_t \varphi(\mathbf{x}, s) - \partial_t \varphi(\mathbf{x}, t)}{\Delta t} \, ds d\mathbf{x} dt}_{I_{1,1}},$$

where $I_{1,1}$ can be bounded as

$$|I_{1,1}| \leq \Delta t \, \|\partial_{tt} \varphi\|_{L^\infty} \, \|v_h(\cdot, t)\|_{L^1(\mathbb{R})} \to 0 \quad \text{as } \Delta t \to 0.$$

Therefore,

$$\lim_{h \to 0} I_1 = \int_0^\infty \int_{\mathbb{R}^2} v \partial_t \varphi \, d\mathbf{x} dt. \tag{3.33}$$

The term I_2 is slightly more complicated. Writing $I_2 = \sum_\ell I_2^\ell$ we get

$$
\begin{aligned}
I_2^\ell = &\int_0^\infty \int_{\mathbb{R}^2} g\left(w_h, v_h\right) \partial_{x_\ell} \varphi \, d\mathbf{x} dt \\
&\underbrace{- \sum_{n,I} g\left(w_I^n, v_I^n\right) \int_{\mathcal{I}_I^n} \int_0^h \frac{\partial_{x_\ell} \varphi\left(\mathbf{x}, t\right) - \partial_{x_\ell} \varphi\left(\mathbf{x} + x e_\ell\right)}{h} \, dx d\mathbf{x} dt}_{I_{2,1}^\ell} \\
&\underbrace{- h^2 \Delta t \sum_{n,I} \left(g\left(w_I^n, v_I^n\right) - g\left(u_{I+e_\ell/2}^n, v_I^n\right)\right) \frac{1}{h} D_\ell \varphi_I^n}_{I_{2,2}^\ell} \\
&\underbrace{- h^2 \Delta t \sum_{n,I} \left(g_\ell^{\mathrm{EO}}\left(u_{I+e_\ell/2}^n, v_I^n, v_I^n\right) - g_\ell^{\mathrm{EO}}\left(u_{I+e_\ell/2}^n, v_I^n, v_{I+e_\ell}^n\right)\right) \frac{1}{h} D_\ell \varphi_I^n.}_{I_{2,3}^\ell}
\end{aligned}
$$

The term $I_{2,1}^\ell$ can be estimated as $I_{1,1}$,

$$\left|I_{2,1}^\ell\right| \leq h \left\|\partial_{x_\ell x_\ell} \varphi\right\|_{L^\infty} \left\|g(w_h, v_h)\right\|_{L^1([-M,M]^2 \times [0,T])} \to 0 \quad \text{as } h \to 0, \tag{3.34}$$

where M and T are such that $\varphi(x, t) = 0$ for $t > T$ and $|\mathbf{x}| > M$. Since u_h and w_h are of bounded variation,

$$\left|I_{2,2}^\ell\right| \leq C \left|u_0\right|_{BV(\mathbb{R}^2)} \left\|\partial_{x_\ell} \varphi\right\|_{L^\infty} h \to 0 \quad \text{as } h \to 0, \tag{3.35}$$

for some constant C which is independent of h. For the final term we use (3.21) and write

$$
\begin{aligned}
\left|I_{2,3}^\ell\right| &= h^2 \Delta t \sum_{n,I} D_\ell^v g_\ell^+ \left(u_{I+e_\ell/2}^n, v_I^n\right) \frac{1}{h} D_\ell \varphi_I^n \\
&\leq \left(h^2 \Delta t \sum_{n,I} \left(D_\ell^v g_\ell^+ \left(u_{I+e_\ell/2}^n, v_I^n\right)\right)^2\right)^{1/2} \left(h^2 \Delta t \sum_{n,I} \left(\frac{D_\ell \varphi_I^n}{h}\right)^2\right)^{1/2} \\
&\leq \sqrt{C_T h} \left\|\partial_x \varphi\right\|_{L^2(\mathbb{R}^2 \times \mathbb{R}^+)} \to 0 \quad \text{as } h \to 0. \tag{3.36}
\end{aligned}
$$

Collecting the bounds (3.34), (3.35) and (3.36) and using the strong convergence of w_h and v_h we find that

$$\lim_{h \to 0} I_2 = \int_0^\infty \int_{\mathbb{R}^2} \mathbf{g}(u, v) \mathrm{div}\left(\varphi\right) \, d\mathbf{x} dt. \tag{3.37}$$

Having the limits (3.32), (3.33) and (3.37) we observe that the limit v is a weak solution. $\qquad\square$

4. Numerical examples

We close this paper by demonstrating how this scheme works in practice. In the first example we approximate solutions of the equation

$$(4.1) \qquad \partial_t u + \partial_x (\frac{1}{2} \left(u^2 \right)) + \partial_y (\frac{1}{2} u) = 0,$$

$$(4.2) \qquad \partial_t v + \partial_x \left(uv(1-v) \right) + \partial_y \left(v(v+u) \right) = 0,$$

with initial data

$$(4.3) \qquad u(x,y,0) = 2e^{-4(x^2+y^2)} - 1 \quad \text{and} \quad v(x,y,0) = \frac{1}{2} \left(1 - \sin(\pi x) \right),$$

for $(x,y) \in [-1,1]^2$ and periodically extended outside this square. In Figure 2 we show the approximations u_h and v_h for $t = 0.5$ computed on a grid with $h = 1/128$. The v variable is depicted as a colored plot, and u as superposed contours.
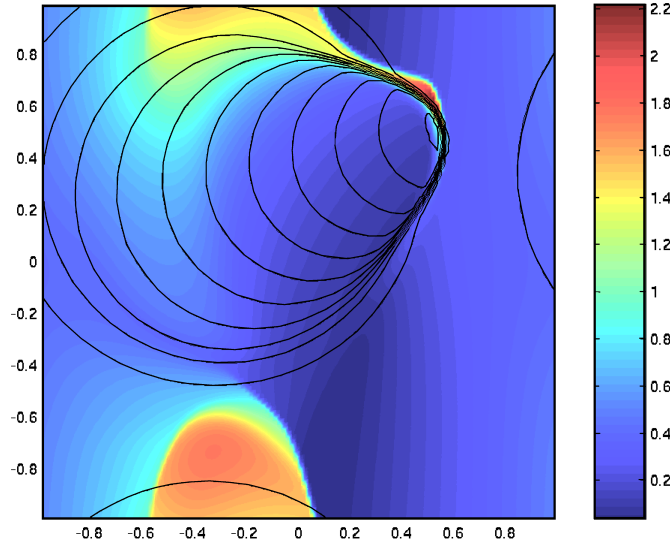

FIGURE 2. The approximate solution of (4.1)-(4.3) at $t = 1/2$.

As a second example, we use a system of equations based on a model of three-phase flow in porous media as mentioned in the introduction. Assume that the porous medium is two-dimensional and that gravity acts in the y-direction. Since we are concerned with the saturation equations, instead of letting the Darcy velocity be defined as the solution of the elliptic equation (1.4), we fix a velocity field $U = (1,0)$. This is supposed to mimic the situation where we are injecting at the left boundary and extracting along the right boundary. Let u and v denote the gas and water saturations, respectively. In this setting, the equations (1.5) take the form

$$(4.4) \qquad \begin{aligned} \partial_t u + \partial_x f_1 \left(u, v \right) + \partial_y f_2 \left(u, v \right) = 0, \\ \partial_t v + \partial_x g_1 \left(u, v \right) + \partial_y g_2(u, v) = 0, \end{aligned}$$

where

$$f_1(u,v) = \frac{\lambda_{\mathrm{g}}(u)}{\lambda_{\mathrm{T}}(u,v)},$$

$$f_2(u,v) = k\frac{\lambda_{\mathrm{g}}(u)}{\lambda_{\mathrm{T}}(u,v)}\left(\lambda_{\mathrm{w}}(v)\left(\rho_{\mathrm{g}} - \rho_{\mathrm{w}}\right) + \lambda_{\mathrm{o}}(1 - u - v)\left(\rho_{\mathrm{g}} - \rho_{\mathrm{o}}\right)\right),$$

$$g_1(u,v) = \frac{\lambda_{\mathrm{w}}(v)}{\lambda_{\mathrm{T}}(u,v)},$$

$$g_2(u,v) = k\frac{\lambda_{\mathrm{w}}(v)}{\lambda_{\mathrm{T}}(u,v)}\left(\lambda_{\mathrm{g}}(u)\left(\rho_{\mathrm{w}} - \rho_{\mathrm{g}}\right) + \lambda_{\mathrm{o}}(1 - u - v)\left(\rho_{\mathrm{w}} - \rho_{\mathrm{o}}\right)\right),$$

where

$$\lambda_{\mathrm{g}}(u) = \frac{u^2}{\nu_{\mathrm{g}}}, \quad \lambda_{\mathrm{w}}(v) = \frac{v^2}{\nu_{\mathrm{w}}}, \quad \lambda_{\mathrm{o}}(u) = \frac{w^2}{\nu_{\mathrm{o}}}, \quad \text{and} \quad \lambda_{\mathrm{T}} = \lambda_{\mathrm{g}} + \lambda_{\mathrm{w}} + \lambda_{\mathrm{o}}.$$

The model (4.4) is not a triangular model, but often the viscosity of the gaseous phase is much smaller than the viscosities of oil and water. Thus, "from the gas' point of view", oil and water are very similar, or $f_i(u,v) \approx f_i(u,(1-u)/2)$. Hence, setting $v = (1-u)/2$ in the first equation in (4.4) should be a reasonable approximation. This yields the triangular model (1.7). In the second example we therefore use (4.4) with $f_i(u,v)$ replaced by $f_i(u,(1-u)/2)$, with the parameter values

$$\nu_{\mathrm{g}} = 1, \quad \nu_{\mathrm{w}} = 80, \quad \nu_{\mathrm{o}} = 100,$$
$$\rho_{\mathrm{g}} = 0.05, \quad \rho_{\mathrm{w}} = 1.00, \quad \rho_{\mathrm{o}} = 0.85, \quad \text{and} \quad k = 500.$$

We used the computational domain $(x,y) \in [0,1]^2$, and the initial values are

$$(4.5) \qquad u(x,0) = \begin{cases} 0 & \text{if } x < 0.1, \\ e^{y-1} & \text{otherwise,} \end{cases} \quad v(x,0) = \begin{cases} 1 & \text{if } x < 0.1, \\ 0 & \text{otherwise.} \end{cases}$$

The system (4.4) with these initial values can be viewed as a simplistic model of water injection in a porous medium filled with oil and gas, where most of the lighter gas is on top of the oil. Recall that the velocity in (4.4) is fixed. Nevertheless, if one uses a sequential method to solve the full system, the hyperbolic part of the model will be of the type (4.4).

In Figure 3 we show the time evolution of the three phases for $t \in [0, 0.4]$. We have computed the approximate solution using $h = 1/128$ and the boundary conditions

$$u(0,y,t) = 0, \quad v(0,y,t) = 1, \quad \mathbf{f} = \mathbf{g} = 0 \quad \text{for } y = 0 \text{ and } y = 1,$$
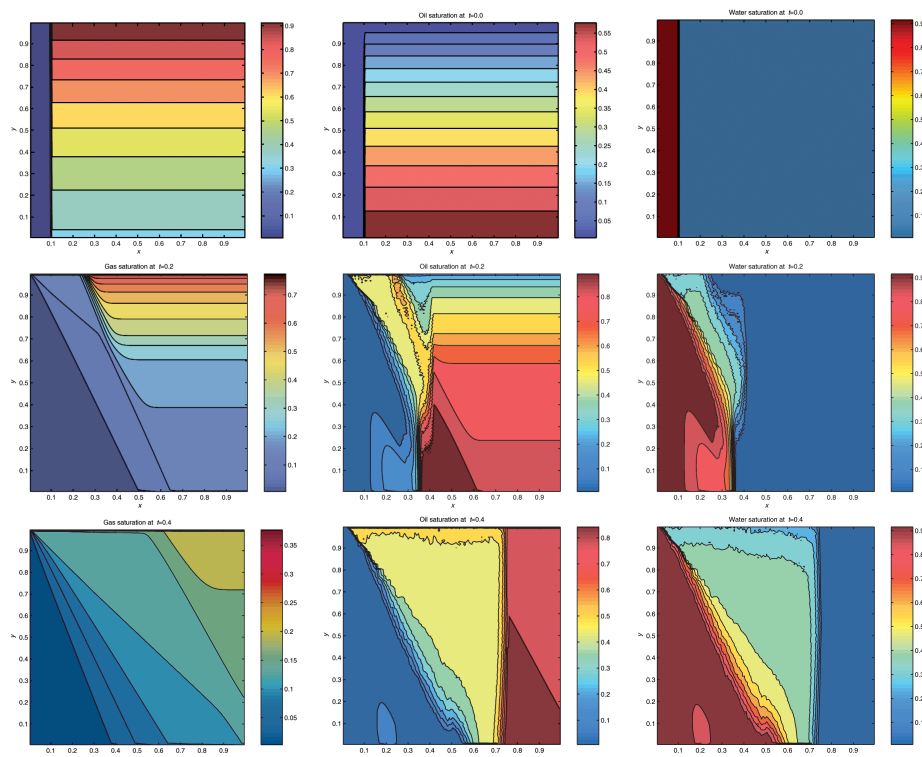$$\partial_x u = \partial_x v = 0 \quad \text{on } x = 1.$$

FIGURE 3. The saturations at $t = 0$ (top), $t = 0.2$ (middle) and $t = 0.4$ (bottom). Left column: gas (u), middle column: oil ($1 - u - v$), right column: water (v).

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