

ON EQUAL SUMS OF NINTH POWERS

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ABSTRACT. In this paper, we develop an elementary method to obtain infinitely many solutions of the Diophantine equation

$$x_1^9 + x_2^9 + x_3^9 + x_4^9 + x_5^9 + x_6^9 = y_1^9 + y_2^9 + y_3^9 + y_4^9 + y_5^9 + y_6^9$$

and we give some numerical results.

1. INTRODUCTION

Weisstein [7] gives a comprehensive survey of known results concerning equal sums of ninth powers. Moessner [4] in considering the Prouhet-Tarry-Escott problem gives a parametrization that *inter alia* gives infinitely many solutions to the Diophantine equation $\sum_{i=1}^{i=10} x_i^9 = \sum_{i=1}^{i=10} y_i^9$. Only a few numerical solutions of the equation $\sum_{i=1}^{i=6} x_i^9 = \sum_{i=1}^{i=6} y_i^9$ are known. Lander, Parkin, and Selfridge [2] give the single solution

$$(1) \quad (23, 18, 14, 13, 13, 1, 22, 21, 15, 10, 9, 5);$$

Ekl [1] lists eight other solutions, and Weisstein references nine more. A few additional solutions are listed in Piezas [5].

The solution in (1) satisfies the set of equalities:

$$\begin{cases} 23 + 18 + 14 + 13 + 13 + 1 = 22 + 21 + 15 + 10 + 9 + 5, \\ 23^3 + 18^3 + 14^3 + 13^3 + 13^3 + 1^3 = 22^3 + 21^3 + 15^3 + 10^3 + 9^3 + 5^3, \\ 23^9 + 18^9 + 14^9 + 13^9 + 13^9 + 1^9 = 22^9 + 21^9 + 15^9 + 10^9 + 9^9 + 5^9. \end{cases}$$

Therefore this solution, as well as two other solutions on Ekl's list, satisfies the system $\{(\mathcal{T}_1), (\mathcal{T}_3), (\mathcal{T}_9)\}$, where (\mathcal{T}_p) is the equality:

$$\sum_{k=1}^6 x_k^p = \sum_{k=1}^6 y_k^p.$$

In this paper we actually prove that the system $\{(\mathcal{T}_1), (\mathcal{T}_2), (\mathcal{T}_3), (\mathcal{T}_9)\}$ has infinitely many rational solutions.

2. A SIMPLER SYSTEM

For any rational numbers $a_1, a_2, a_3, a'_1, a'_2, a'_3$, we put

$$\begin{cases} \sigma_1 = a_1 + a_2 + a_3 \\ \sigma_2 = a_2 a_3 + a_3 a_1 + a_1 a_2 \\ \sigma_3 = a_1 a_2 a_3, \end{cases} \quad \begin{cases} \sigma'_1 = a'_1 + a'_2 + a'_3 \\ \sigma'_2 = a'_2 a'_3 + a'_3 a'_1 + a'_1 a'_2 \\ \sigma'_3 = a'_1 a'_2 a'_3. \end{cases}$$

Received by the editor March 3, 2009.

2000 *Mathematics Subject Classification*. Primary 11D41, 11G05.

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Proposition 1. *If the rational numbers $a_1, a_2, a_3, a'_1, a'_2, a'_3$ satisfy the following system:*

$$(2) \quad (\mathcal{S}) : \begin{cases} (\mathcal{S}_1) : \sigma_1 = \sigma'_1 \\ (\mathcal{S}_2) : \sigma_2 = \sigma'_2 \\ (\mathcal{S}_3) : 3\sigma_1^3 - 3\sigma_1\sigma_2 + \sigma_3 + \sigma'_3 = 0, \end{cases}$$

then the rational numbers x_k and y_k , $1 \leq k \leq 6$, defined by the following set of equalities:

$$(3) \quad \begin{cases} x_1 = a_1 + \sigma_1 & x_2 = a_2 + \sigma_1 & x_3 = a_3 + \sigma_1 \\ y_1 = a_1 - \sigma_1 & y_2 = a_2 - \sigma_1 & y_3 = a_3 - \sigma_1 \\ x_4 = a'_1 - \sigma'_1 & x_5 = a'_2 - \sigma'_1 & x_6 = a'_3 - \sigma'_1 \\ y_4 = a'_1 + \sigma'_1 & y_5 = a'_2 + \sigma'_1 & y_6 = a'_3 + \sigma'_1 \end{cases}$$

yield a solution of the following system:

$$(4) \quad (\mathcal{T}) : \begin{cases} (\mathcal{T}_1) : x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = y_1 + y_2 + y_3 + y_4 + y_5 + y_6, \\ (\mathcal{T}_2) : x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 = y_1^2 + y_2^2 + y_3^2 + y_4^2 + y_5^2 + y_6^2, \\ (\mathcal{T}_3) : x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 + x_6^3 = y_1^3 + y_2^3 + y_3^3 + y_4^3 + y_5^3 + y_6^3, \\ (\mathcal{T}_9) : x_1^9 + x_2^9 + x_3^9 + x_4^9 + x_5^9 + x_6^9 = y_1^9 + y_2^9 + y_3^9 + y_4^9 + y_5^9 + y_6^9. \end{cases}$$

Proof. First, make the assumption $(\mathcal{S}_1) : \sigma_1 = \sigma'_1$. Then the system (3) immediately implies (\mathcal{T}_1) and (\mathcal{T}_2) .

For any non-negative integer p , put $d_p = \sum_{k=1}^3 (a_k^p - a'_k{}^p)$. Then

$$\sum_{k=1}^6 (x_k^3 - y_k^3) = 6\sigma_1 d_2$$

and

$$\sum_{k=1}^6 (x_k^9 - y_k^9) = 6\sigma_1 (12d_2\sigma_1^6 + 42d_4\sigma_1^4 + 28d_6\sigma_1^2 + 3d_8).$$

The further assumption $(\mathcal{S}_2) : \sigma_2 = \sigma'_2$ implies $d_2 = 0$, hence (\mathcal{T}_3) .

The assumptions (\mathcal{S}_1) , (\mathcal{S}_2) also imply the following:

$$\begin{cases} d_4 = 4(\sigma_3 - \sigma'_3)\sigma_1, \\ d_6 = 3(\sigma_3 - \sigma'_3)(2\sigma_1^3 - 4\sigma_1\sigma_2 + \sigma_3 + \sigma'_3), \\ d_8 = 4(\sigma_3 - \sigma'_3)(2\sigma_1(\sigma_1^2 - \sigma_2)(\sigma_1^2 - 3\sigma_2) + (\sigma_3 + \sigma'_3)(3\sigma_1^2 - 2\sigma_2)). \end{cases}$$

From this we deduce:

$$(5) \quad \sum_{k=1}^6 (x_k^9 - y_k^9) = 144\sigma_1(\sigma_3 - \sigma'_3)(5\sigma_1^2 - \sigma_2)(3\sigma_1^3 - 3\sigma_1\sigma_2 + \sigma_3 + \sigma'_3).$$

Consequently, from the assumption $(\mathcal{S}_3) : 3\sigma_1^3 - 3\sigma_1\sigma_2 + \sigma_3 + \sigma'_3 = 0$, we obtain (\mathcal{T}_9) . \square

Remark 1. Equating to 0 other factors in (5) leads to trivial solutions.

For if $\sigma_1 = 0$, we obtain a trivial solution of (\mathcal{T}) satisfying $x_i = y_i, i = 1, \dots, 6$.

If $\sigma_3 = \sigma'_3$, then, taking into consideration the equalities $\sigma_1 = \sigma'_1$ and $\sigma_2 = \sigma'_2$, we deduce $a_i = a'_i, i = 1, 2, 3$, which leads to a trivial solution of (\mathcal{T}) satisfying

$$(x_1, x_2, x_3, x_4, x_5, x_6) = (y_4, y_5, y_6, y_1, y_2, y_3).$$

If $5 \sigma_1^2 - \sigma_2 = 0$, then necessarily a_1, a_2, a_3 , and a'_1, a'_2, a'_3 are zero, which again gives a trivial solution of (\mathcal{T}) .

Remark 2. The sextuple $(a_1, a_2, a_3, a'_1, a'_2, a'_3) = (17, -18, 5, -19, 9, 14)$ is a solution of the system (\mathcal{S}) at (2), with $\sigma_1 = 4$. The numbers x_k and y_k from (3) give the following solution of the system (\mathcal{T}) at (4): $(21, -14, 9, -23, 5, 10, 13, -22, 1, -15, 13, 18)$. This permutation of the solution at (1) now satisfies the equation (\mathcal{T}_2) .

Henceforth we shall focus attention on the system (\mathcal{S}) . Observe that if $\sigma_1 = 0$, then $\sigma'_1 = 0$, and the system (3) shows that we obtain only trivial solutions of (\mathcal{T}) , satisfying $x_i = y_i, i = 1, \dots, 6$. We shall refer to solutions of (\mathcal{S}) which satisfy $\sigma_1 = 0$ as **trivial** solutions of the system (\mathcal{S}) .

Remark 3. The only solution of (\mathcal{S}) such that $a_1 = a_2 = a_3$ is the zero solution. For if in (\mathcal{S}) we replace a_2 and a_3 by a_1 , we obtain:

$$a'_1 + a'_2 + a'_3 = 3 a_1, \quad a'_2 a'_3 + a'_3 a'_1 + a'_1 a'_2 = 3 a_1^2, \quad a'_1 a'_2 a'_3 = -55 a_1^3,$$

so that the three numbers a'_1, a'_2 and a'_3 are the solutions of the following equation: $x^3 - 3 a_1 x^2 + 3 a_1^2 x + 55 a_1^3 = 0$. This may also be written as $(x - a_1)^3 = -7 (2a_1)^3$. Since x and a_1 are rational, we necessarily have $x = a_1$ and $a_1 = 0$, which implies $(a_1, a_2, a_3, a'_1, a'_2, a'_3) = (0, 0, 0, 0, 0, 0)$.

3. AN ELLIPTIC SURFACE

We first parametrize the two equations $(\mathcal{S}_1), (\mathcal{S}_2)$, at (2).

Proposition 2. *Let $a = (a_1, a_2, a_3, a'_1, a'_2, a'_3)$ be a sextuple of rational numbers. The following two statements are equivalent:*

I: a satisfies the system:
$$\begin{cases} a_1 + a_2 + a_3 = a'_1 + a'_2 + a'_3, \\ a_2 a_3 + a_3 a_1 + a_1 a_2 = a'_2 a'_3 + a'_3 a'_1 + a'_1 a'_2. \end{cases}$$

II: There exist triples of rational numbers $p = (p_1, p_2, p_3)$ and $q = (q_1, q_2, q_3)$ such that

$$(6) \quad p_1 + p_2 + p_3 = 0, \quad q_1 + q_2 + q_3 = 0,$$

and

$$(7) \quad \begin{cases} a_1 - a'_1 = p_1 q_1 \\ a_2 - a'_2 = p_2 q_1 \\ a_3 - a'_3 = p_3 q_1, \end{cases} \quad \begin{cases} a_2 - a'_3 = p_1 q_2 \\ a_3 - a'_1 = p_2 q_2 \\ a_1 - a'_2 = p_3 q_2, \end{cases} \quad \begin{cases} a_3 - a'_2 = p_1 q_3 \\ a_1 - a'_3 = p_2 q_3 \\ a_2 - a'_1 = p_3 q_3. \end{cases}$$

Proof. It is easily verified that II implies I. Let us prove that I implies II. For this purpose observe first that if $a_1 + a_2 + a_3 = a'_1 + a'_2 + a'_3$, then each of the following

six numbers:

$$\begin{aligned} & (a_1 - a'_1) (a_3 - a'_1) - (a_2 - a'_2) (a_2 - a'_3) \\ & (a_2 - a'_2) (a_1 - a'_2) - (a_3 - a'_3) (a_3 - a'_1) \\ & (a_3 - a'_3) (a_2 - a'_3) - (a_1 - a'_1) (a_1 - a'_2) \\ & -(a_1 - a'_1) (a_1 - a'_3) + (a_2 - a'_2) (a_3 - a'_2) \\ & -(a_2 - a'_2) (a_2 - a'_1) + (a_3 - a'_3) (a_1 - a'_3) \\ & -(a_3 - a'_3) (a_3 - a'_2) + (a_1 - a'_1) (a_2 - a'_1) \end{aligned}$$

is equal to $a_2 a_3 + a_3 a_1 + a_1 a_2 - (a'_2 a'_3 + a'_3 a'_1 + a'_1 a'_2)$.

Consequently, if a satisfies the two equalities in I, we obtain the six equalities:

$$\begin{aligned} (a_1 - a'_1) (a_3 - a'_1) &= (a_2 - a'_2) (a_2 - a'_3) \\ (a_2 - a'_2) (a_1 - a'_2) &= (a_3 - a'_3) (a_3 - a'_1) \\ (a_3 - a'_3) (a_2 - a'_3) &= (a_1 - a'_1) (a_1 - a'_2) \\ (a_1 - a'_1) (a_1 - a'_3) &= (a_2 - a'_2) (a_3 - a'_2) \\ (a_2 - a'_2) (a_2 - a'_1) &= (a_3 - a'_3) (a_1 - a'_3) \\ (a_3 - a'_3) (a_3 - a'_2) &= (a_1 - a'_1) (a_2 - a'_1). \end{aligned}$$

Now consider the matrix:

$$M = \begin{bmatrix} a_1 - a'_1 & a_2 - a'_3 & a_3 - a'_2 \\ a_2 - a'_2 & a_3 - a'_1 & a_1 - a'_3 \\ a_3 - a'_3 & a_1 - a'_2 & a_2 - a'_1 \end{bmatrix}.$$

From the preceding six equalities, the rank of M is 0 or 1. As is well known, this implies the existence of two triples of rational numbers $p = (p_1, p_2, p_3)$ and $q = (q_1, q_2, q_3)$ such that

$$\begin{array}{lll} a_1 - a'_1 = p_1 q_1 & a_2 - a'_3 = p_1 q_2 & a_3 - a'_2 = p_1 q_3 \\ a_2 - a'_2 = p_2 q_1 & a_3 - a'_1 = p_2 q_2 & a_1 - a'_3 = p_2 q_3 \\ a_3 - a'_3 = p_3 q_1 & a_1 - a'_2 = p_3 q_2 & a_2 - a'_1 = p_3 q_3. \end{array}$$

Since $a_1 + a_2 + a_3 = a'_1 + a'_2 + a'_3$, we obtain (6), (7), the equations defining II, as required. \square

Using the parametrization in (6), (7), of the equations (\mathcal{S}_1) , (\mathcal{S}_2) , we make the following substitution into the equations (\mathcal{S}) :

$$(8) \quad \begin{aligned} a_1 &= a'_3 + p_2(-q_1 - q_2), & a_2 &= a'_3 + p_1 q_2, & a_3 &= a'_3 + (-p_1 - p_2)q_1, \\ a'_1 &= a'_3 + p_2(-q_1 - q_2) - p_1 q_1, & a'_2 &= a'_3 + p_1 q_2 - p_2 q_1, \end{aligned}$$

where for the purpose of homogeneity we set $a'_3 = p_2 q_4$, say. The third equation (\mathcal{S}_3) delivers a homogeneous cubic equation in $\{q_1, q_2, q_4\}$, with coefficients homogeneous of degree 3 in p_1, p_2 . Writing

$$t = p_1/p_2,$$

this equation defines a curve of genus 1 over the function field $\mathbb{Q}(t)$, that is, an elliptic surface, and it is readily checked that the curve contains a point at

$(q_1, q_2, q_4) = (1, 1, 1)$, so that the curve is actually elliptic over $\mathbb{Q}(t)$. Specifically, its equation has the form

$$C_t: -3(2+t)(3+3t+t^2)q_1^3 + (-27-2t+16t^2+6t^3)q_1^2q_2 + 7(11+11t+3t^2)q_1^2q_4 \\ - (15-16t+2t^2+6t^3)q_1q_2^2 - 7(-11+5t+5t^2)q_1q_2q_4 - 56(2+t)q_1q_4^2 \\ + 3(-1+t)(1-t+t^2)q_2^3 + 7(3-5t+3t^2)q_2^2q_4 + 56(-1+t)q_2q_4^2 + 56q_4^3 = 0.$$

Points on the curve C_t can be pulled back to solutions of the system (\mathcal{T}) by means of (3) and (8), namely:

$$\begin{aligned} x_1 : x_2 : x_3 : x_4 : x_5 : x_6 : y_1 : y_2 : y_3 : y_4 : y_5 : y_6 = \\ - (3+t)q_1 + (-2+t)q_2 + 4q_4 : & - (2+t)q_1 + (-1+2t)q_2 + 4q_4 : \\ - (3+2t)q_1 + (-1+t)q_2 + 4q_4 : & q_1 - tq_2 - 2q_4 : \\ (1+t)q_1 + q_2 - 2q_4 : & (2+t)q_1 + (1-t)q_2 - 2q_4 : \\ (1+t)q_1 - tq_2 - 2q_4 : & (2+t)q_1 + q_2 - 2q_4 : \\ q_1 + (1-t)q_2 - 2q_4 : & - (3+2t)q_1 + (-2+t)q_2 + 4q_4 : \\ (9) \quad - (3+t)q_1 + (-1+2t)q_2 + 4q_4 : & - (2+t)q_1 + (-1+t)q_2 + 4q_4. \end{aligned}$$

Symmetries of the solutions of (\mathcal{S}) under permutation of the a_i, a'_i induce symmetries of the underlying curve C_t , and the effect is that the curves corresponding to the parameters

$$(10) \quad t, -1 - \frac{1}{t}, -\frac{1}{1+t}, -\frac{t}{1+t}, -1-t, \frac{1}{t}$$

are isomorphic. A Weierstrass form for the elliptic curve C_t is discovered to be the following:

$$E_t: Y^2 = X^3 + 1323(1+t+t^2)(23+69t-54t^2-223t^3-54t^4+69t^5+23t^6)X \\ - 2646(947+5682t+16143t^2+28630t^3+22734t^4-18342t^5-46761t^6 \\ (11) \quad - 18342t^7+22734t^8+28630t^9+16143t^{10}+5682t^{11}+947t^{12}).$$

This latter curve has torsion group $\mathbb{Z}/3\mathbb{Z}$ over $\mathbb{Q}(t)$ with point of order 3 given by

$$(147(1+t+t^2)^2, 756(1-t+t^2)(3+3t+t^2)(1+3t+3t^2)),$$

corresponding to the torsion point on C_t ,

$$T_0 = (1, -2, t) \quad (\text{and where } -T_0 = (2, -1, 1+t)).$$

The condition that we derive a trivial solution of (\mathcal{S}) has become

$$(t+2)q_1 - (t-1)q_2 - 3q_4 = 0,$$

and this line cuts C_t in precisely the three points of finite order. Thus only the torsion points of C_t return trivial solutions of (2).

4. NUMERICAL RESULTS

We can use standard computer software to determine values of the parameter t for which the rational rank of the curve (11) is positive. The group of points on the associated curve C_t is then infinite, and we are able to deduce an infinity of solutions to the system (\mathcal{S}) at (2), and hence an infinity of solutions to the system (\mathcal{T}) at (4).

The effect of the symmetries is that it suffices to search with $1 < t$, and in the range $3 \leq \text{numerator}(t) + \text{denominator}(t) \leq 20$ we obtain Table 1, which lists the rational rank of (11) along with corresponding points (q_1, q_2, q_4) on C_t . All computations were performed with Magma [3].

Example. We compute solutions of the system (\mathcal{T}) at (4) corresponding to $t = 3$.

The curve C_3 has rank 1 with point of infinite order $Q = (1, 8, -5)$, so that a point on C_3 is of type $nQ + \epsilon T_0$, $n \in \mathbb{Z}$, $\epsilon = 0, \pm 1$. The pullbacks of the three points corresponding to $\epsilon = 0, \pm 1$ give the same solution of the system (4) up to permutation, and so it is only necessary to consider $\epsilon = 0$. Furthermore, the points $\pm nQ$ deliver the same solution up to permutation, and so we can further restrict ourselves to considering only the pullbacks of the points nQ , $n > 0$. Table 2 lists the pullback points for $n = 1, \dots, 5$.

A similar chain of solutions can be computed for each of the t corresponding to entries in Table 1, with a double infinity of solutions when the rank is 2, and a triple infinity of solutions at $t = 9/4$. We list some “small” solutions arising in this latter instance. Write $Q_1 = (106, -282, 279)$, $Q_2 = (124, -272, 293)$, $Q_3 = (239, -181, 396)$ for the points from Table 1.

Note that the assertion at (10) is equivalent to the fact that the curves C_{p_i/p_j} , $1 \leq i \neq j \leq 3$, are isomorphic. The symmetry in the parametrization (7) interchanging a'_2 and a'_3 is equivalent to the symmetry interchanging p_i with q_i ($i = 1, 2, 3$). Thus when C_{p_1/p_2} has positive rank with point $(q_1, q_2, a'_3/p_2)$, there is a corresponding point $(p_1, p_2, a'_2/q_2)$ on C_{q_1/q_2} , and hence the isomorphic curves C_{q_i/q_j} , $1 \leq i \neq j \leq 3$ also have positive rank.

5. PARAMETRIZATIONS OVER $\mathbb{Q}(\omega)$

It would be useful to know the $\mathbb{Q}(t)$ rank of the curve E_t at (11). If this were positive, then corresponding points on C_t would pull back to solutions of the system (\mathcal{T}) that are polynomials in $\mathbb{Z}[t]$. Unfortunately, we are unable to determine whether or not E_t contains any non-torsion points over $\mathbb{Q}(t)$. The large points in Table 1 for $t = 15/2, 12/7, 11/9$, together with several specializations of E_t that have rank 0, would suggest that the parametrization of a point is unlikely, and therefore that the $\mathbb{Q}(t)$ -rank is 0. The equation E_t is that of an elliptic $K3$ -surface, and as such, methods of Shioda [6] allow us to compute an upper bound for the rank of E_t over $\mathbb{C}(t)$, and this turns out to be 3. We are able to spot *two* independent points P_1, P_2 of infinite order on C_t over $\mathbb{C}(t)$, but have not determined whether or not there is a third:

$$P_1 = (6, -4 - 2\omega, 3 + 3t), \quad P_2 = (-\omega, 1, \frac{1}{2}(1 - \omega)),$$

where $\omega = \frac{1}{2}(-1 + \sqrt{-3})$ is a complex cube root of unity. Summing each point with its Galois conjugate:

$$P_1 + \bar{P}_1 = (1, -2, t), \quad P_2 + \bar{P}_2 = (1, 1, 1),$$

we obtain points of finite order. Thus no non-trivial rational polynomial parametrization of (\mathcal{T}) arises in this way. The two points P_1, P_2 do generate infinitely many polynomial parametrizations of (\mathcal{T}) , but of course with coefficients over $\mathbb{Q}(\omega)$. For example, the pullbacks of P_1 and P_2 result in trivial solutions of the system (2);

TABLE 1. Values of t with C_t of positive rank

t	independent point(s) on C_t	rank
3	(1, 8, -5)	1
4	(1446, 14711, -15299)	1
5/2	(356, 1882, -647)	1
4/3	(25, 278, -12)	1
7	(207, -52, 785)	1
8	(1, 3, -5)	1
7/2	(82, -664, 821)	1
5/4	(2648, 539468, -54847)	1
7/3	(543, 1656, -125), (9137158, 2514571, 13080130)	2
10	(59616, 83123, -33345)	1
6/5	(6549648283004459, -22665359828265916, 8749022109376884)	1
11	(479, -2916, 13421), (14033, -43851, 222497)	2
12	(19, 64, -190)	1
11/2	(-27783615238, -228089845712, 349090862393)	1
10/3	(94087753, 101830636, 85790044)	1
9/4	(106, -282, 279), (124, -272, 293), (239, -181, 396)	3
8/5	(36194817, 128001892, 11175862)	1
7/6	(3049, -8401, 3738), (35004, -61398, 40157),	2
13	(4724, 1055987, -5026528)	1
9/5	(121419260, -1377907645, 613796368)	1
11/4	(126758321, -30905647, 231572729)	1
8/7	(14818297849097, 12821985982426, 14953926106708)	1
13/3	(150351, 100917, 224711), (13051968, -12150687, 37914218)	2
11/5	(154090, 443445, -3052)	1
9/7	(742, -2709, 1116), (2455971, 3614233, 2299483)	2
16	(18, 179, -963), (958, 6731, -34507)	2
15/2	(134207609048965376073388156958141987, 28950557364798896116286088178331879, 422719433838369207424332656027729381)	1
14/3	(359601, 464652, 185656), (12153, -8250, 35330)	2
13/4	(25702, 74362, -22943)	1
11/6	(1532, -16154, 7563)	1
9/8	(27787, -37949, 30240), (21756703708, -959147282732, 73870303329)	2
11/7	(11754403807181, -239314804762243, 71755024964941)	1
18	(7901044, -22041517, 187889479)	1
17/2	(153, 599, -1262), (1874, 2748, -1015)	2
15/4	(4688, 8548, -97), (880834, -4824818, 6950465)	2
14/5	(23, 108, -44), (1287580662737, 1391088002287, 1201351021327)	2
13/6	(4093367511, 4633565061, 3798630686)	1
12/7	(4607882348502503129769514, 7355497055510069300923039, 3688741999490895912106389)	1
11/8	(2720488, 6913952, 1992365)	1
10/9	(343956623832196, -126385184284523, 362777970896793)	1
17/3	(127, -596, 1405)	1
11/9	(888221912602727881028999424599, -492896695539027689262279545737, 987459994593608883597631915735)	1

TABLE 2. Solutions to the system (4) arising from $t = 3$

n	1	2	3	4	5
x_1	18	13825	-176607027	-295503813128476	16781116061381923831545
x_2	-15	-9157	154786818	163694336414557	7388891819632091442261
x_3	13	1092	134510017	453392718089919	-21580737831236008171454
x_4	13	4495	-22242470	129024868182881	-17757024654750594094103
x_5	-22	-14187	168678401	198009483059676	-4986911769004750233739
x_6	1	6812	-202780835	-487825971930557	21449301398866340776666
y_1	10	10945	-232951931	-456295433816476	15486481036492920280369
y_2	-23	-12037	98441914	2902715726557	6094256794743087891085
y_3	5	-1788	78165113	292601097401919	-22875372856125011722630
y_4	21	7375	34102434	289816488870881	-16462389629861590542927
y_5	-14	-11307	225023305	358801103747676	-3692276744115746682563
y_6	9	9692	-146435931	-327034351242557	22743936423755344327842

TABLE 3. Solutions to the system (4) arising from the rank 3 curve

point on $C_{9/4}$	x_1	x_2	x_3	x_4	x_5	x_6
	y_1	y_2	y_3	y_4	y_5	y_6
Q_2	453	150	-307	-455	-98	281
	429	174	-331	-431	-122	305
Q_1	978	365	-643	-991	-63	490
	842	501	-779	-855	-199	626
Q_3	-1136	583	261	785	1739	-1800
	-1568	1015	-171	1217	1307	-1368
$Q_1 + Q_2$	-3147	2120	75	1927	4832	-4927
	-4027	3000	-805	2807	3952	-4047
$Q_1 + Q_3$	6190	-18943	17613	3265	-12363	9958
	470	-13223	11893	8985	-18083	15678
$Q_1 + Q_2 + Q_3$	159999	99268	-117995	-175511	67884	21299
	105055	154212	-172939	-120567	12940	76243
$Q_2 - Q_3$	319333	-455078	333993	-72523	-428926	415401
	207133	-342878	221793	39677	-541126	527601

however, the pullbacks of $2P_1$ and $2P_2$ result in the following cubic parametrizations, respectively:

$$\begin{aligned}
 x_1 : x_2 : x_3 : x_4 : x_5 : x_6 : y_1 : y_2 : y_3 : y_4 : y_5 : y_6 = \\
 & (3\omega + 3)t^3 + (\omega + 4)t^2 + (-\omega + 3)t - 3\omega : \\
 & (-2\omega - 1)t^3 + (3\omega + 3)t^2 + (4\omega + 4)t + \omega + 2 : \\
 & 3\omega t^3 + (2\omega - 1)t^2 + (-9\omega - 6)t - 2\omega - 1 : \\
 & (-\omega - 2)t^3 + (-5\omega - 6)t^2 + (5\omega - 1)t + \omega - 1 : \\
 & (2\omega + 1)t^3 + (9\omega + 3)t^2 + (-2\omega - 3)t - 3\omega - 3 :
 \end{aligned}$$

$$\begin{aligned}
& (-\omega + 1)t^3 - 4\omega t^2 - 3\omega t + 2\omega + 1 : \\
& (-\omega + 1)t^3 + (-5\omega + 1)t^2 + (5\omega + 6)t + \omega + 2 : \\
& (2\omega + 1)t^3 + (9\omega + 6)t^2 + (-2\omega + 1)t - 3\omega : \\
& (-\omega - 2)t^3 + (-4\omega - 4)t^2 + (-3\omega - 3)t + 2\omega + 1 : \\
& 3\omega t^3 + (\omega - 3)t^2 + (-\omega - 4)t - 3\omega - 3 : \\
& (-2\omega - 1)t^3 + 3\omega t^2 + 4\omega t + \omega - 1 : \\
& (3\omega + 3)t^3 + (2\omega + 3)t^2 + (-9\omega - 3)t - 2\omega - 1;
\end{aligned}$$

and

$$\begin{aligned}
& x_1 : x_2 : x_3 : x_4 : x_5 : x_6 : y_1 : y_2 : y_3 : y_4 : y_5 : y_6 = \\
& (2\omega + 4)t^3 + (4\omega + 6)t^2 + (5\omega + 8)t + 3\omega + 3 : \\
& -2t^3 + (3\omega + 2)t^2 + (3\omega + 1)t + \omega - 1 : \\
& (-4\omega - 6)t^3 + (-4\omega - 4)t^2 + (\omega + 7)t + 3 : \\
& (-2\omega - 2)t^3 + (-5\omega - 8)t^2 + (-6\omega - 9)t - \omega - 2 : \\
& (-2\omega - 6)t^3 + (-6\omega - 14)t^2 - 3t + \omega + 2 : \\
& (4\omega + 8)t^3 + (5\omega + 12)t^2 + 2t - 2\omega - 1 : \\
& (4\omega + 8)t^3 + (7\omega + 12)t^2 + (2\omega + 2)t + \omega - 1 : \\
& (-2\omega - 6)t^3 - 4t^2 + (6\omega + 7)t + 3\omega + 3 : \\
& (-2\omega - 2)t^3 + (-\omega + 2)t^2 + (-2\omega + 1)t - 2\omega - 1 : \\
& (-4\omega - 6)t^3 + (-8\omega - 14)t^2 + (-3\omega - 3)t + \omega + 2 : \\
& -2t^3 + (-3\omega - 8)t^2 + (-3\omega - 9)t - \omega - 2 : \\
& (2\omega + 4)t^3 + (2\omega + 6)t^2 + (3\omega + 8)t + 3.
\end{aligned}$$

ACKNOWLEDGEMENTS

The second author warmly thanks Dr. Simon J. Agou for his kindness and encouragement, and Dr. François Vellutini and Yves Mannheimer for reading an initial version of the manuscript.

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