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APPROXIMATION OF THE DISCONTINUITIES OF A FUNCTION BY ITS CLASSICAL ORTHOGONAL POLYNOMIAL FOURIER COEFFICIENTS

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Abstract. In the present paper, we generalize the method suggested in an earlier paper by the author and overcome its main deficiency.

First, we modify the well-known Prony method, which subsequently will be utilized for recovering exactly the locations of jump discontinuities and the associated jumps of a piecewise constant function by means of its Fourier coefficients with respect to any system of the classical orthogonal polynomials.

Next, we will show that the method is applicable to a wider class of functions, namely, to the class of piecewise smooth functions—for functions which piecewise belong to $C^2[-1,1]$, the locations of discontinuities are approximated to within $O(1/n)$ by means of their Fourier-Jacobi coefficients. Unlike the previous one, the generalized method is robust, since its success is independent of whether or not a location of the discontinuity coincides with a root of a classical orthogonal polynomial. In addition, the error estimate is uniform for any $[c,d] \subset (-1,1)$.

To the end, we discuss the accuracy, stability, and complexity of the method and present numerical examples.

1. Introduction

Truncated Fourier series of functions with jump discontinuous are known to exhibit the Gibbs phenomenon, which makes these partial sums a poor approximation tool. However, if the locations of the singularities and the associated jumps of the function are known, then a number of spectral methods for the reconstruction of the function are already available. Thus, it is essential to accurately recover the locations of singularities and magnitudes of jumps utilizing only Fourier coefficients of a function.

This problem was studied by several authors and the methods suggested by them are applicable if a finite number of Fourier coefficients of a discontinuous function with respect to the trigonometric or Jacobi polynomial system are known. (See Banerjee and Geer [2], Bauer [3], Cai *et al.* [4], Eckhoff [5, 6, 7], Gelb and Tadmor [10, 11], Kvernadze [13, 14], Mhaskar and Prestin [16, 17, 18], and the indicated references.)

The first explicit method for recovering the singularities of a piecewise smooth function by means of the Fourier coefficients with respect to an orthonormal system

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of algebraic polynomials, namely, the Chebyshev system, was introduced by Eckhoff [5]. According to the proposed method, if a function has a finite number, M, of jump discontinuities, then approximations to the locations of singularities are found as solutions of a certain Mth degree polynomial equation. An accuracy of the order $O(1/n)$ for recovery of the locations of discontinuities was predicted and numerically confirmed.

In [13] we obtained an identity determining the jumps of a piecewise continuous function of bounded harmonic variation by means of its Fourier partial sums with respect to a generalized Jacobi system of polynomials. In particular,

Theorem 1.1. *Let $r = 0, 1, \cdots$, $\alpha > -1$, $\beta > -1$, and $f \in HBV$. Then the identity*

$$\lim_{n\to\infty} \frac{(S_n^{(\alpha,\beta)})^{(2r+1)}(f,x)}{n^{2r+1}} = \frac{(-1)^r(1-x^2)^{-r-1/2}}{(2r+1)\pi}[f](x) \tag{1.1}$$

is valid for each fixed $x \in (-1,1)$.

Here, and elsewhere, $S_n^{(\alpha,\beta)}(f,x)$ is the nth partial sum of the Fourier series of the function f with respect to the Jacobi system of orthonormal polynomials (see Table 1), and $[f](x) \equiv f(x+) - f(x-)$. For the exact definition of HBV, the class of functions of harmonic bounded variation, consult [22].

According to the identity (1.1), for a fixed r and sufficiently large n, the largest local maximum of the absolute value of the differentiated partial sums of the Fourier-Jacobi series occurs in the vicinity of the actual points of discontinuity of the function. Hence, the locations of discontinuities may be identified and approximated "graphically", i.e., looking for relative sharp local spikes of the graph of a differentiated Fourier-Jacobi partial sum.

Later, Gelb and Tadmor [11], and Mhaskar and Prestin [17] proposed two new methods. In [11] concentration kernels $K_\epsilon(\cdot)$ were introduced, depending on the small parameter ϵ. The kernels satisfy the condition $K_\epsilon * f(x) = [f](x) + O(\epsilon)$ and thus recover both the location and amplitude of all singularities. In particular, the authors have considered concentration kernels with respect to the Fourier-Gegenbauer (with nonpositive indices) partial sums. The authors studied the accuracy of approximations. For example,

$$\left|\frac{\pi\sqrt{1-x^2}}{n}(S_n^{(\alpha,\alpha)})'(f,x) - [f](x)\right| \le \text{Const}\,\frac{\log n}{n(1-x^2)^{\alpha/2+1/4}},$$

where $-1 < \alpha \le 0$ and $-1 + \text{Const}/n^2 < x < 1 - \text{Const}/n^2$.

Mhaskar and Prestin [17] proposed a class of algebraic polynomial frames that can be used to detect discontinuities in derivatives of all orders of a function. A rate of convergence of the frame operators has been studied at the vicinity of and away from the singularity points. The locations of singularities are approximated to within $O(1/n)$.

In the present paper, we generalize the method suggested by us in [15] and overcome its main deficiency.

First, we modify the well-known Prony method [19], which subsequently will be utilized for recovering exactly the locations of jump discontinuities and the associated jumps of a piecewise constant function by means of its Fourier coefficients with respect to any system of the classical orthogonal polynomials. For the trigonometric system the method essentially matches the Prony method.

Next, we will show that the method is applicable to a wider class of functions, namely, to the class of piecewise smooth functions—for functions which piecewise belong to $C^2[-1,1]$, the locations of discontinuities are approximated to within $O(1/n)$ by means of their Fourier-Jacobi coefficients. For functions with continuous derivatives between the jump discontinuities the accuracy of approximation is by an order better, $O(1/n^2)$.

Unlike the previous one, the generalized method is robust, since its success is independent of whether or not a location of the singularity of a function coincides with a root of a classical orthogonal polynomial. In addition, the error estimate is uniform for any $[c,d] \subset (-1,1)$.

To the end, we discuss the accuracy, stability, and complexity of the method and present numerical examples.

2. Preliminaries

Throughout this paper we use the following general notation: $\mathbb{N}$, $\mathbb{Z}_+$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ are the sets of positive integers, nonnegative integers, integers, real numbers, and complex numbers, respectively.

If $\mathbf{b} = (b_1, b_2, \ldots, b_M) \in \mathbb{C}^M$ is a column vector, then $||\mathbf{b}|| = \max_{1\le m\le M}|b_m|$ is its ℓ_∞ norm. By $||\mathbf{A}|| = \sup_{||\mathbf{b}||=1} ||\mathbf{A}\mathbf{b}||/||\mathbf{b}||$, we denote the natural (induced) ℓ_∞ norm of the matrix $\mathbf{A} = (a_{ij})_{i,j=1}^M$.

By $C^{-1}[a,b]$ we denote the space of bounded functions that may have only a finite number of jump discontinuities and are normalized by the condition $f(x) \equiv (f(x+) + f(x-))/2$ (here and elsewhere $f(x+)$ and $f(x-)$ denote the right-hand and left-hand side limits of a function f at a point x). By $C^r[a,b]$, $r \in \mathbb{Z}_+$, we denote the space of r-times continuously differentiable functions on $[a,b]$, where $C^0[a,b] \equiv C[a,b]$ is the class of continuous functions on $[a,b]$.

By $[f](x) \equiv f(x+) - f(x-) \neq 0$, we denote the jump of the function $f \in C^{-1}[a,b]$ at the point $a < x < b$. By $M \equiv M_0 \equiv M(f)$, we denote the number of discontinuities of the function $f \in C^{-1}[a,b]$ and by $x_m \equiv x_m^{(0)} \equiv x_m(f)$, $m = 1, 2, \ldots, M$, we denote the points of discontinuity of the function $f \in C^{-1}[a,b]$ arranged in increasing order. For simplicity, $M_i \equiv M(f^{(i)})$ and $x_m^{(i)} \equiv x_m(f^{(i)})$, $m = 1, 2, \ldots, M_i$.

By K we denote constants, possibly depending on some fixed parameters and in general distinct in different formulas. Sometimes the important arguments of K will be written explicitly in the expressions for it. For quantities A_n and B_n, possibly depending on some other variables as well, we write $A_n = o(B_n)$ or $A_n = O(B_n)$, if $\lim_{n\to\infty} A_n/B_n = 0$ or $\sup_{n\in N}|A_n/B_n| < \infty$, respectively.

2.1. **The classical orthogonal polynomials.**

A function w is called a weight (function) on $[a,b]$ if $w(x) \ge 0$ for $x \in [a,b]$ and $(n \in \mathbb{Z}_+)$

$$0 < \int_a^b x^n w(x)dx < \infty.$$

We say that a system of polynomials $\sigma(w) \equiv (P_n(w,x))_{n=0}^\infty \equiv (P_n(x))_{n=0}^\infty$, degree$(P_n(w,x)) = n$, with positive leading coefficients, is orthogonal with respect to the weight w if $(n \neq m)$

$$\int_a^b P_n(w,x)P_m(w,x)w(x)dx = 0.$$

It is well known [21, p. 42, Theorem 3.2.1] that all orthogonal polynomials satisfy the recurrence formula: There exist constants $A_n(w)$, $B_n(w)$, and $C_n(w)$ such that

$$xP_n(w,x) = A_n(w)P_{n+1}(w,x) + B_n(w)P_n(w,x) + C_n(w)P_{n-1}(w,x) \tag{2.1}$$

for $n \in \mathbb{N}$ and $x \in \mathbb{R}$.

Let $A(x)$ be a polynomial degree of no more than one and let $B(x)$ be a polynomial degree of no more than two. The weight functions which satisfy the boundary value problem ($x \in [a,b]$)

$$\frac{w'(x)}{w(x)} = \frac{A(x)}{B(x)}, \tag{2.2}$$

$$w(a+)B(a+) = w(b-)B(b-) = 0 \tag{2.3}$$

are called the classical weights [20, Section 1, p. 44]. Up to a linear transformation, they are the Jacobi, Laguerre, and Hermite weights. The systems of polynomials orthogonal with respect to the classical weights are named correspondingly; see Table 1 (cf. [21, pp. 71, 101, and 106]).

Table 1. The classical orthogonal polynomials.

System	*Jacobi*	*Laguerre*	*Hermite*
$A(x)$ *in* (2.2)	$(\beta-\alpha)-(\alpha+\beta)x$	$\alpha - x$	$-2x$
$B(x)$ *in* (2.2)	$1-x^2$	x	1
$w(x)$ *in* (2.2)	$(1-x)^\alpha(1+x)^\beta, \alpha>-1, \beta>-1$	$x^\alpha e^{-x}, \alpha>-1$	e^{-x^2}
$[a,b]$	$[-1,1]$	$[0,\infty)$	$(-\infty,\infty)$
$(P_n(w,x))_{n=0}^\infty$	$(P_n^{(\alpha,\beta)}(x))_{n=0}^\infty$	$(L_n^{(\alpha)}(x))_{n=0}^\infty$	$(H_n(x))_{n=0}^\infty$
$c_n(w)$ *in* (2.4)	$\frac{(-1)^n}{n!2^n}$	$\frac{1}{n!}$	$(-1)^n$
$A_n(w)$ *in* (2.1)	$\frac{2(n+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}$	$-n-1$	$\frac{1}{2}$
$B_n(w)$ *in* (2.1)	$\frac{\beta^2-\alpha^2}{(2n+\alpha+\beta+2)(2n+\alpha+\beta)}$	$2n+\alpha+1$	0
$C_n(w)$ *in* (2.1)	$\frac{2(n+\alpha)(n+\beta)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta)}$	$-n-\alpha$	n

In what follows, we always assume that $(P_n(w,x))_{n=0}^\infty$ is a system of classical polynomials, orthogonal on $[a,b] \equiv [-1,1], [0,\infty)$, or $(-\infty,\infty)$, with respect to a corresponding classical weight w, unless it is mentioned otherwise.

Following are the results well known for the classical orthogonal polynomials.

Rodrigues's formula [20, Theorem 2.2, p. 55] ($n \in \mathbb{Z}_+$) is

$$w(x)P_n(w,x) = c_n(w)\frac{d^n}{dx^n}[w(x)B^n(x)], \tag{2.4}$$

where $c_n(w)$ is specified in Table 1.

It is easy to check (see Table 1) that if $(P_n(w,x))_{n=0}^\infty$ is a system of the classical polynomials orthogonal with respect to a weight w, then the system of polynomials ($i \in \mathbb{N}$)

$$(P_{i,n}(x))_{n=0}^\infty \equiv (P_{i,n}(w,x))_{n=0}^\infty, \tag{2.5}$$

orthogonal with respect the weight $w_i(x) \equiv w(x)B^i(x)$, is also classical. In fact, $P_{i,n}^{(\alpha,\beta)}(x) = P_n^{(\alpha+i,\beta+i)}(x)$, $L_{i,n}^{(\alpha)}(x) = L_n^{(\alpha+i)}(x)$, and $H_{i,n}(x) = H_n(x)$. The coefficients in recurrence formula (2.1) for the system $(P_{i,n}(w,x))_{n=0}^{\infty}$ will be denoted by $A_{i,n} \equiv A_{i,n}(w)$, etc.

Besides, by virtue of (2.4), (2.5), and Table 1

$$w(x)P_n(w,x) = \frac{c_n(w)}{c_{n-i}(w)} \frac{d^i}{dx^i}[w_i(x)P_{i,n-i}(w,x)] \tag{2.6}$$

and

$$\frac{d^k}{dx^k}[w_i(x)P_{i,n}(w,x)]|_{x=a+} = \frac{d^k}{dx^k}[w_i(x)P_{i,n}(w,x)]|_{x=b-} = 0 \tag{2.7}$$

for $i, n \in \mathbb{N}$ and $k < i$.

If $x_{k,n}(w)$, $k = 1, 2, \ldots, n$, are the zeros of the polynomial $P_n(w,x)$, then [21, Theorem 3.3.2, p. 46]

$$x_{k,n+1}(w) < x_{k,n}(w) < x_{k+1,n+1}(w). \tag{2.8}$$

The estimate

$$|P_{n-1}^{(\alpha,\beta)}(x)| < K(\alpha,\beta)n^{-1/2}((1-x)^{1/2}+n^{-1})^{-\alpha-1/2}((1+x)^{1/2}+n^{-1})^{-\beta-1/2} \tag{2.9}$$

holds for $x \in [-1,1]$ and $n \in \mathbb{N}$ (cf. [1, p. 226]).

$$P_n^{(\beta,\alpha)}(x) = (-1)^n P_n^{(\alpha,\beta)}(-x) \tag{2.10}$$

for $n \in \mathbb{N}$ [21, p. 59].

The asymptotic formula [21, Theorem 8.21.8, p. 196]

$$P_n^{(\alpha,\beta)}(\cos\tau) = n^{-1/2}\kappa(\alpha,\beta,\tau)\cos(\tilde{n}\tau+\gamma) + O(n^{-3/2}) \tag{2.11}$$

holds as $n \to \infty$, where $\kappa(\alpha,\beta,\tau) = \pi^{-1/2}\sin^{-\alpha-1/2}(\tau/2)\cos^{-\beta-1/2}(\tau/2)$, $\tilde{n} = n+(\alpha+\beta+1)/2$, $\gamma = -(2\alpha+1)\pi/4$, and $0 < \tau < \pi$. The bound for the error term holds uniformly in the interval $[\epsilon, \pi-\epsilon]$, $0 < \epsilon < \pi/2$.

Lemma 2.1. *Let* $[c,d] \subset (-1,1)$. *Then*

$$((P_n^{(\alpha,\beta)}(x))^2 + (P_{n+1}^{(\alpha,\beta)}(x))^2)^{-\frac{1}{2}} = \pi^{\frac{1}{2}}2^{-\frac{\alpha+\beta}{2}}n^{\frac{1}{2}}(w^{(-\alpha+1/2,-\beta+1/2)}(x) + O(n^{-1}))^{-\frac{1}{2}} \tag{2.12}$$

for all $x \in [c,d]$, *where* $w^{(\alpha,\beta)}(x) \equiv (1-x)^\alpha(1+x)^\beta$.

Proof. Without loss of generality, due to (2.10), let us assume that $x \in [0,d]$. Next, by (2.11) we have ($x \equiv \cos\tau$)

$$\kappa^2(\alpha,\beta,\tau) = \pi^{1/2}2^{\alpha+\beta+1}w^{(-\alpha-1/2,-\beta-1/2)}(x) \tag{2.13}$$

and

$$(P_n^{(\alpha,\beta)}(x))^2 = n^{-1}\kappa^2(\alpha,\beta,\tau)\cos^2(\tilde{n}\tau+\gamma) + O(n^{-2}) \tag{2.14}$$

uniformly for $x \in [0,d]$. Thus,

$$\begin{aligned}(P_n^{(\alpha,\beta)}(x))^2 + (P_{n+1}^{(\alpha,\beta)}(x))^2 &= \kappa^2(\alpha,\beta,\tau)\\ &\quad\times [n^{-1}\cos^2(\tilde{n}\tau+\gamma) + ((n+1)^{-1} - n^{-1} + n^{-1})\cos^2((\tilde{n}+1)\tau+\gamma)] + O(n^{-2})\\ &= n^{-1}\kappa^2(\alpha,\beta,\tau)[\cos^2(\tilde{n}\tau+\gamma) + \cos^2((\tilde{n}+1)\tau+\gamma)] + O(n^{-2}).\end{aligned} \tag{2.15}$$

Since

$$\begin{aligned}2 &\geq \cos^2(\tilde{n}\tau+\gamma)+\cos^2((\tilde{n}+1)\tau+\gamma)\\ &= 1+\frac{1}{2}\cos 2(\tilde{n}\tau+\gamma)+\frac{1}{2}\cos 2((\tilde{n}+1)\tau+\gamma)\\ &= 1+\cos\tau\cos((2\tilde{n}+1)\tau+2\gamma)\geq 1-\cos\tau = 1-x,\end{aligned}\tag{2.16}$$

combining (2.13)–(2.16), we obtain (2.12). □

Finally, if $\int_a^b f^2(x)w(x)dx<\infty$, then by

$$a_n(f)\equiv a_n(w,f)\equiv\int_a^b f(t)P_n(w,t)w(t)dt\tag{2.17}$$

we denote the nth Fourier coefficient of the function f.

3. A modified Prony method

In this section we describe a method of how to recover the values x_m, $m=1,2,\cdots,M$, if the sequence $a_n^{(0)}\equiv\sum_{m=1}^M\lambda_m f_n(x_m)$, $n\in\mathbb{Z}$, is given. (We are assuming that the functions f_n are given and λ_m, $m=1,2,\cdots,M$, are unknown, but fixed numbers in the sum.) The method essentially is based on Prony's idea [19]. We will show that the suggested technique is applicable to a wide class of functions $(f_n)_{n=-\infty}^{\infty}$. Subsequently, this method will be utilized to recover exactly the locations of discontinuities and the associated jumps of a piecewise constant function by means of its Fourier coefficients with respect to a system of the classical orthogonal polynomials.

Lemma 3.1. *Let $(a_n^{(0)})_{n=-\infty}^{\infty}$ and $(A_n^{(l)})_{n=-\infty}^{\infty}$, $l=-L,-L+1,\cdots,L$, for a given $L\in\mathbb{Z}_+$, be sequences of complex numbers. Suppose for each given $n\in\mathbb{Z}$, $a_n^{(k)}(t_1,t_2,\ldots,t_k)$, $k\in\mathbb{N}$, is generated by the recurrence formula*

$$a_n^{(k)}(t_1,t_2,\ldots,t_k)\equiv t_k a_n^{(k-1)}(t_1,t_2,\ldots,t_{k-1})+\sum_{l=-L}^{L}A_n^{(l)}a_{n+l}^{(k-1)}(t_1,t_2,\ldots,t_{k-1}),\tag{3.1}$$

and let

$$a_n^{(k)}\equiv a_n^{(k)}(0,0,\cdots,0).\tag{3.2}$$

Then

$$a_n^{(k)}(t_1,t_2,\ldots,t_k)=\sum_{i=-\infty}^{\infty}q_i^{(k)}a_n^{(i)},\tag{3.3}$$

where $q_i^{(k)}\equiv q_i^{(k)}(t_1,t_2,\cdots,t_k)$ are defined as follows:

$$(x+t_1)(x+t_2)\cdots(x+t_k)=\sum_{i=-\infty}^{\infty}q_i^{(k)}x^i.\tag{3.4}$$

Proof. Let us establish a simple relation between $q_i^{(k+1)}$ and $q_i^{(k)}$. Since

$$(x+t_1)(x+t_2)\cdots(x+t_k)(x+t_{k+1})=[(x+t_1)(x+t_2)\cdots(x+t_k)](x+t_{k+1}),$$

by virtue of (3.4) we have $\sum_{i=-\infty}^{\infty}q_i^{(k+1)}x^i=(x+t_{k+1})\sum_{i=-\infty}^{\infty}q_i^{(k)}x^i$, i.e.,

$$q_i^{(k+1)}=q_{i-1}^{(k)}+q_i^{(k)}t_{k+1}\tag{3.5}$$

for $i\in\mathbb{Z}$.

We will prove identity (3.3) by mathematical induction. Let $k = 1$. Then (3.4) implies that $q_1^{(1)} = 1$, $q_0^{(1)} = t_1$, and $q_i^{(1)} = 0$ for $i < 0$ and $i > 1$.

From (3.1) and (3.2), on the other hand, we have ($n \in \mathbb{Z}$)

$$a_n^{(1)}(t_1) = t_1 a_n^{(0)} + \sum_{l=-L}^{L} A_n^{(l)} a_{n+l}^{(0)} = t_1 a_n^{(0)} + a_n^{(1)} = q_0^{(1)} a_n^{(0)} + q_1^{(1)} a_n^{(1)} = \sum_{i=-\infty}^{\infty} q_i^{(1)} a_n^{(i)}.$$

Now we assume that identity (3.3) holds for k and we will prove it for $k+1$. According to (3.1)–(3.3) and (3.5) we have

$$\begin{aligned}
&a_n^{(k+1)}(t_1, \dots, t_{k+1}) \\
&\quad = t_{k+1} a_n^{(k)}(t_1, t_2, \cdots, t_k) + \sum_{l=-L}^{L} A_n^{(l)} a_{n+l}^{(k)}(t_1, t_2, \cdots, t_k) \\
&\quad = t_{k+1} \sum_{i=-\infty}^{\infty} q_i^{(k)} a_n^{(i)} + \sum_{l=-L}^{L} A_n^{(l)} \sum_{i=-\infty}^{\infty} q_i^{(k)} a_{n+l}^{(i)} \\
&\quad = \sum_{i=-\infty}^{\infty} q_i^{(k)} (t_{k+1} a_n^{(i)} + \sum_{l=-L}^{L} A_n^{(l)} a_{n+l}^{(i)}) = \sum_{i=-\infty}^{\infty} q_i^{(k)} (t_{k+1} a_n^{(i)} + a_n^{(i+1)}) \\
&\quad = \sum_{i=-\infty}^{\infty} (q_i^{(k)} t_{k+1} a_n^{(i)} + q_i^{(k)} a_n^{(i+1)}) = \sum_{i=-\infty}^{\infty} ((q_i^{(k+1)} - q_{i-1}^{(k)}) a_n^{(i)} + q_i^{(k)} a_n^{(i+1)}) \\
&\quad = \sum_{i=-\infty}^{\infty} q_i^{(k+1)} a_n^{(i)} - \sum_{i=-\infty}^{\infty} q_{i-1}^{(k)} a_n^{(i)} + \sum_{i=-\infty}^{\infty} q_i^{(k)} a_n^{(i+1)} = \sum_{i=-\infty}^{\infty} q_i^{(k+1)} a_n^{(i)},
\end{aligned}$$

since the second and the third sums cancel out (i shifted to $i+1$), and that completes the proof. □

Lemma 3.2. *Let $(x_m)_{m=1}^M$, $(\lambda_m)_{m=1}^M$, and $(A_n^{(l)})_{n=-\infty}^{\infty}$, $l = -L, -L+1, \cdots, L$, for given $M \in \mathbb{N}$ and $L \in \mathbb{Z}_+$, be sequences of complex numbers. Also, suppose the functions g and $(f_n)_{n=-\infty}^{\infty}$, with domains containing the set $\{x_1, x_2, \cdots, x_M\}$, satisfy the identity ($n \in \mathbb{Z}$)*

$$g(x) f_n(x) = \sum_{l=-L}^{L} A_n^{(l)} f_{n+l}(x). \tag{3.6}$$

If

$$a_n^{(0)} \equiv \sum_{m=1}^{M} \lambda_m f_n(x_m), \qquad n \in \mathbb{Z}, \tag{3.7}$$

then

$$a_n^{(k)}(t_1, t_2, \dots, t_k) = \sum_{m=1}^{M} \lambda_m f_n(x_m) \prod_{s=1}^{k} (t_s + g(x_m)) \tag{3.8}$$

for $n \in \mathbb{Z}$ and $k \in \mathbb{N}$, where $a_n^{(k)}(t_1, t_2, \dots, t_k)$ is defined by (3.1).

Proof. By virtue of (3.1), (3.6), and (3.7) we have

$$\begin{aligned} a_n^{(1)}(t_1) =& t_1 a_n^{(0)} + \sum_{l=-L}^{L} A_n^{(l)} a_{n+l}^{(0)} = \sum_{m=1}^{M} \lambda_m (t_1 f_n(x_m) + \sum_{l=-L}^{L} A_n^{(l)} f_{n+l}(x_m)) \\ =& \sum_{m=1}^{M} \lambda_m f_n(x_m)(t_1 + g(x_m)) \end{aligned}$$

for $n \in \mathbb{Z}$.

The rest may be easily completed by mathematical induction. □

Now, in Theorem 3.3, we describe the recovery process.

Theorem 3.3. *Let, for a given $M \in \mathbb{N}$, $(x_m)_{m=1}^M$ be a sequence of distinct complex numbers and $(\lambda_m)_{m=1}^M$ be a sequence of nonzero complex numbers. In addition, let $(A_n^{(l)})_{n=-\infty}^{\infty}$, $l = -L, -L+1, \cdots, L$, for a given $L \in \mathbb{Z}_+$, be sequences of complex numbers. Suppose the functions g and $(f_n)_{n=-\infty}^{\infty}$, with domains containing the set $\{x_1, x_2, \cdots, x_M\}$, satisfy the identity* (3.6), *where the function g is invertible and*

$$\prod_{m=1}^{M} f_n(x_m) \neq 0 \tag{3.9}$$

for some $n \in \mathbb{Z}$.

If $a_n^{(0)} \equiv \sum_{m=1}^M \lambda_m f_n(x_m)$ and $a_n^{(k)}$, $k \in \mathbb{N}$, are defined by (3.1) *and* (3.2), *then the system of linear equations, for $n \in \mathbb{Z}$ guaranteeing* (3.9), *($j = 0, 1, \cdots M-1$)*

$$\sum_{i=0}^{M-1} a_n^{(i+j)} q_i^{(M)}(n) + a_n^{(M+j)} = 0 \tag{3.10}$$

has the unique solution $(q_0^{(M)}(n), q_1^{(M)}(n), \cdots, q_{M-1}^{(M)}(n))$, and the roots of the polynomial equation

$$x^M + \sum_{i=0}^{M-1} q_{M-i}^{(M)}(n) x^{M-i} = 0 \tag{3.11}$$

are $-g(x_m)$, $m = 1, 2, \cdots, M$.

Proof. First, let us show that coefficient matrix $\mathbf{A}_n^{(M)} \equiv (a_n^{(i+j)})_{i,j=0}^{M-1}$ of the linear system (3.10) is nonsingular. Indeed, by virtue of (3.2) and (3.8) ($k \in \mathbb{N}$)

$$a_n^{(k)} = \sum_{m=1}^{M} \lambda_m f_n(x_m) g^k(x_m). \tag{3.12}$$

Hence, (3.12) implies ($\delta_{i,j} = 1$ if $i = j$ and $\delta_{i,j} = 0$ if $i \neq j$)

$$\mathbf{A}_n^{(M)} = (g^j(x_{i+1}))_{j,i=0}^{M-1} \cdot (\lambda_i \delta_{i,j})_{i,j=1}^{M} \cdot (f_n(x_i)\delta_{i,j})_{i,j=1}^{M} \cdot (g^j(x_{i+1}))_{i,j=0}^{M-1}. \tag{3.13}$$

The matrix $(g^j(x_{i+1}))_{j,i=0}^{M-1}$ is a Vandermonde type, and it is easy to verify that

$$\mathrm{Det}(g^j(x_{i+1}))_{j,i=0}^{M-1} = \prod_{1 \leq i < j \leq M} (g(x_j) - g(x_i)).$$

Therefore, invertibility of g guarantees its nonsingularity. Invertibility of the remaining matrices in the product (3.13) follows from the conditions (3.9) and $\lambda_m \neq 0$, $m = 1, 2, \ldots, M$.

Next, by (3.5)

$$q_i^{(M+1)}(t_1, t_2, \cdots, t_M, 0) = q_{i-1}^{(M)}(t_1, t_2, \cdots, t_M)$$

or

$$q_i^{(M+j)}(t_1, t_2, \cdots, t_M, 0, \cdots, 0) = q_{i-j}^{(M)}(t_1, t_2, \cdots, t_M). \tag{3.14}$$

Now, (3.3), (3.4), and (3.14) imply

$$\begin{aligned} a_n^{(M+j)}(t_1, t_2, \cdots, t_M, 0, \cdots, 0) &= \sum_{i=-\infty}^{\infty} a_n^{(i)} q_i^{(M+j)}(t_1, t_2, \cdots, t_M, 0, \cdots, 0) \\ &= \sum_{i=-\infty}^{\infty} a_n^{(i)} q_{i-j}^{(M)}(t_1, t_2, \cdots, t_M) \\ &= \sum_{i=-\infty}^{\infty} a_n^{(i+j)} q_i^{(M)}(t_1, t_2, \cdots, t_M) \\ &= \sum_{i=0}^{M-1} a_n^{(i+j)} q_i^{(M)}(t_1, \cdots, t_M) + a_n^{(M+j)}. \end{aligned} \tag{3.15}$$

On the other hand, (3.8) yields

$$a_n^{(M+j)}(-g(x_1), -g(x_2), \cdots, -g(x_M), 0, \cdots, 0) = 0,$$

which, combined with (3.15), implies the following identities ($j = 0, 1, \cdots, M-1$):

$$\begin{aligned} &\sum_{i=0}^{M-1} a_n^{(i+j)} q_i^{(M)}(-g(x_1), -g(x_2), \cdots, -g(x_M)) + a_n^{(M+j)} \\ &\qquad = a_n^{(M+j)}(-g(x_1), -g(x_2), \cdots, -g(x_M), 0, \cdots, 0) = 0. \end{aligned} \tag{3.16}$$

Thus, according to the identities (3.16), the system (3.10) has the unique solution $q_i^{(M)}(n) = q_i^{(M)}(-g(x_1), -g(x_2), \cdots, -g(x_M))$, $i = 0, 1, \cdots, M-1$; and, the solutions of polynomial equation (3.11), i.e., (3.4), are $-g(x_m)$, $m = 1, 2, \cdots, M$.

Besides, due to (3.8),

$$a_n^{(M-1)}(-g(x_2), \cdots, -g(x_M)) = \lambda_1 f_n(x_1) \prod_{s=2}^{M} (g(x_1) - g(x_s)).$$

Hence, λ_1, and similarly λ_m, $m = 2, 3, \cdots, M$, can be recovered exactly as well.

In the end, let us mention that once the sequence (3.12) is generated, recovering $g(x_m)$, $m = 1, 2, \ldots, M$, via the linear system (3.10) and the polynomial equation (3.11) closely follows Prony's method. □

Let us give examples of functions g and $(f_n)_{n=-\infty}^{\infty}$ satisfying the identity (3.6).

• For any given invertible function $g(x)$, the system of functions $(g^n(x))_{n=0}^{\infty}$ obviously will satisfy (3.6). In particular, if $g(x) = x$, then $f_n(x) = x^n$, or if $g(x) = e^{\imath x}$, then $f_n(x) = e^{\imath n x}$, $\imath \equiv (0, 1)$, i.e., the trigonometric system. Let us add, that Theorem 3.3 essentially matches Prony's method for the trigonometric system.

• As is well known, any system of orthogonal polynomials $(P_n(w, x))_{n=0}^{\infty}$ satisfies the recurrence relation (2.1). Hence, $g(x) = x$ in this case.

• If g is a polynomial degree L, monotone on a segment containing the set $\{x_1, x_2, \cdots, x_M\}$, then $(f_n)_{n=0}^{\infty}$ may be any basis for the set of polynomials.

4. Main results

Now, we will show how to apply the recovery process to a system of the classical orthogonal polynomials.

4.1. **Recovering the discontinuities of a piecewise constant function.** Essentially, we have to show that (normalized) Fourier coefficients with respect to the classical orthogonal polynomials of a piecewise constant function can be represented in the form (3.7) and that the classical orthogonal polynomials satisfy the identity (3.6).

Utilizing integration by parts, it is easy to check that (see (2.5)–(2.7) and (2.17)) a Fourier coefficient of the function ($i \in \mathbb{Z}_+$)

$$\chi_i(x,t) \equiv \begin{cases} 0, & \text{if } a < t < x, \\ \frac{(t-x)^i}{i!}, & \text{if } x < t < b, \end{cases}$$

with respect to a system $\sigma(w)$ of the classical orthogonal polynomials can be expressed as follows:

$$a_n(w, \chi_i(x,\cdot)) = (-1)^{i+1} \frac{c_n(w)}{c_{n-i-1}(w)} w_{i+1}(x) P_{i+1,n-i-1}(w,x) \tag{4.1}$$

for $n > i$.

If the function f is piecewise constant on $[a,b]$, with jump discontinuities at the points x_m, $m = 1, 2, \cdots, M$, then it may be represented as

$$f(x) = \sum_{m=1}^{M} [f](x_m)\chi_0(x_m, x) + f(a+). \tag{4.2}$$

Next, by (4.1) and (4.2), a normalized Fourier coefficient of the function f with respect to a system of the classical orthogonal polynomials is represented as (compare to (3.7))

(4.3)

$$\begin{aligned} a_n^{(0)} \equiv a_n^{(0)}(f) &\equiv -\frac{c_n(w)}{c_{n+1}(w)} a_{n+1}(w,f) = -\frac{c_n(w)}{c_{n+1}(w)} \sum_{m=1}^{M} [f](x_m) a_{n+1}(w, \chi_0(x_m,\cdot)) \\ &= -\frac{c_n(w)}{c_{n+1}(w)} \sum_{m=1}^{M} [f](x_m)(-1)\frac{c_{n+1}(w)}{c_n(w)} w_1(x_m) P_{1,n}(x_m) \\ &= \sum_{m=1}^{M} [f](x_m) w_1(x_m) P_{1,n}(x_m). \end{aligned}$$

Besides, due to (2.1),

$$xP_{1,n}(w,x) = A_{1,n}(w)P_{1,n+1}(w,x) + B_{1,n}(w)P_{1,n}(w) + C_{1,n}(w)P_{1,n-1}(w,x), \tag{4.4}$$

i.e., the classical orthogonal polynomials satisfy the identity (3.6) with $g(x) = x$.

Therefore, if $a_n^{(0)}$, $n \in \mathbb{N}$, is a normalized Fourier coefficient of a piecewise constant function f, defined by (4.3), then due to (4.4), $a_n^{(k)}(t_1, t_2, \cdots, t_k)$ will be generated as follows (compare to (3.1)):

(4.5)
$$\begin{aligned} a_n^{(k)}(t_1, t_2, \cdots, t_k) \equiv t_k a_n^{(k-1)}(t_1, t_2, \cdots, t_{k-1}) &+ A_{1,n}(w) a_{n+1}^{(k-1)}(t_1, t_2, \cdots, t_{k-1}) \\ &+ B_{1,n}(w) a_n^{(k-1)}(t_1, t_2, \cdots, t_{k-1}) \\ &+ C_{1,n}(w) a_{n-1}^{(k-1)}(t_1, t_2, \cdots, t_{k-1}). \end{aligned}$$

Now, by virtue of Theorem 3.3, if

$$\prod_{m=1}^{M} P_{1,n}(w, x_m) \neq 0 \tag{4.6}$$

for some $n \in \mathbb{N}$, then the discontinuity locations $x_1, x_2, \cdots, x_M$, as well as the magnitudes of associated jumps, can be recovered exactly. It is easy to check that knowledge of any consecutive $4M - 1$ Fourier coefficients is sufficient to perform the proposed method, subject to the condition (4.6).

Obviously, the recovery process depends on whether or not a singularity location represents a root of a classical polynomial; see (4.6). To avoid this dependency, we consider a modified linear system of equations (compare to (3.10))

$$\sum_{i=0}^{M-1} b_n^{(i+j)} q_i^{(M)}(n) + b_n^{(M+j)} = 0, \tag{4.7}$$

where $b_n^{(i+j)} \equiv a_n^{(i+j)} + \imath a_{n+1}^{(i+j)}$.

Then, as is easy to verify (see (3.13)), the coefficient matrix $\mathbf{B}_n^{(M)} \equiv \mathbf{B}_n^{(M)}(f) \equiv (b_n^{(i+j)})_{i,j=0}^{M-1}$ of the system (4.7) now may be factored as

$$\begin{aligned} \mathbf{B}_n^{(M)} =& ((x_{i+1}^j))_{j,i=0}^{M-1} \cdot ([f](x_i) w_1(x_i) \delta_{i,j})_{i,j=1}^{M} \\ & \cdot ((P_{1,n}(w, x_i) + \imath P_{1,n+1}(w, x_i))\delta_{i,j})_{i,j=1}^{M} \cdot ((x_{i+1}^j))_{i,j=0}^{M-1} \end{aligned} \tag{4.8}$$

and due to (2.8), it is never singular. Thus, the linear system of equations (4.7) is consistent regardless of the locations of singularities of a function.

Summarizing all the above, we have the following theorem.

Theorem 4.1. *Let f be a piecewise constant function defined on a segment $[a, b]$ with a finite number, M, of discontinuities. Then the solutions of the polynomial equation* (3.11)*, the coefficients of which are the solution of the linear system* (4.7)*, represent the discontinuity locations of the function f.*

4.2. **Approximating the discontinuities of a piecewise smooth function.** If a given function is not piecewise constant, with jump discontinuities at $x_1, x_2, \cdots, x_M$, then $(j = 0, 1, \cdots, M-1)$

$$a_n^{(M+j)}(-x_1, -x_2, \cdots, -x_M, 0, 0, \cdots, 0) \neq 0$$

(see (3.16)). However, we will show that for functions which piecewise belong to $C^2[-1, 1]$,

$$\lim_{n\to\infty} a_n^{(M+j)}(-x_1, -x_2, \cdots, -x_M, 0, 0, \cdots, 0) = 0.$$

Thus, solving the homogeneous linear system (4.7) for sufficiently large n, instead of $(j = 0, 1, \cdots, M-1)$

$$\begin{aligned}\sum_{i=0}^{M-1} b_n^{(i+j)} q_i^{(M)}(-x_1, -x_2, \cdots, -x_M) + b_n^{(M+j)} \\ = b_n^{(M+j)}(-x_1, -x_2 \cdots, -x_M, 0, \cdots, 0) \approx 0,\end{aligned}$$

we find the coefficients of the polynomial equation (3.11) approximately. Correspondingly, the solutions of (3.11) will represent approximations to x_m, $m = 1, 2, \cdots, M$.

The following theorem addresses the accuracy of the approximation to the locations of discontinuities for a piecewise smooth function.

Theorem 4.2. *Suppose the function f piecewise belongs to $C^2[-1,1]$ with a finite number, M, of jump discontinuities at the points x_m, $m = 1, 2, ..., M$. In addition, let us assume that the coefficient matrix $\mathbf{B}_n^{(M)}$ of the linear system of equations* (4.7) *is determined by means of Fourier coefficients of the function f with respect to the Jacobi system $\sigma^{(\alpha,\beta)}$. Then*

$$\begin{aligned}x_m(n) = x_m + O(\frac{1}{n^{3/2}}) \frac{||(\mathbf{B}_n^{(M)})^{-1}||}{\prod_{i,j=1;i<j}^{M}(x_i - x_j)} \\ \times (\sum_{m=1}^{M_1} |[f'](x_m^{(1)})| w^{(\alpha/2+3/4, \beta/2+3/4)}(x_m^{(1)}) + o(1))\end{aligned} \tag{4.9}$$

and

$$\begin{aligned}||(\mathbf{B}_n^{(M)})^{-1}|| = O(n^{1/2}) \max_{1\le m\le M} \frac{(w^{(-\alpha-1/2,-\beta-1/2)}(x_m) + O(n^{-1}))^{-1/2}}{|[f](x_m)| w^{(\alpha+1,\beta+1)}(x_m)} \\ \times \max_{1\le m\le M} \prod_{k=1;k\neq m}^{M} \frac{1}{(x_k - x_m)^2}\end{aligned} \tag{4.10}$$

where $x_m(n)$, $m = 1, 2, \ldots, M$, are the roots of the polynomial equation (3.11) *with the coefficients being the solution of the linear system of equations* (4.7) *and $w^{(\alpha,\beta)}(x) = (1-x)^\alpha (1+x)^\beta$.*

For functions with continuous derivative between the points of singularity, i.e., $x_m = x_m^{(1)}$, $M = M_1$, $O(n^{-3/2})$ should be changed by $O(n^{-5/2})$ in estimate (4.9).

Proof. Since the function f piecewise belongs to $C^2[-1,1]$, then

$$f(x) = \sum_{i=0}^{2} \sum_{m=1}^{M_i} [f^{(i)}](x_m^{(i)}) \chi_i(x_m^{(i)}, x) + f_c(x) \equiv \chi(x) + F_c(x), \tag{4.11}$$

where $f_c \in C^2[-1,1]$, $\chi(x) \equiv \sum_{m=1}^{M} [f](x_m) \chi_0(x_m, x)$, and $F_c \in C[-1,1]$.

Due to (2.5), (4.1), (4.3), (4.4), (4.11), and Table 1, we have

$$
\begin{aligned}
a_n^{(0)} &\equiv a_n^{(0)}(f) \equiv 2(n+1)a_{n+1}^{(\alpha,\beta)}(f) \\
&= \sum_{m=1}^{M} [f](x_m) w^{(\alpha+1,\beta+1)}(x_m) P_n^{(\alpha+1,\beta+1)}(x_m) \\
&\qquad + \frac{1}{2n} \sum_{m=1}^{M_1} [f'](x_m^{(1)}) w^{(\alpha+2,\beta+2)}(x_m^{(1)}) P_{n-1}^{(\alpha+2,\beta+2)}(x_m^{(1)}) \\
&\qquad + \frac{1}{4n(n-1)} \sum_{m=1}^{M_2} [f''](x_m^{(2)}) w^{(\alpha+3,\beta+3)}(x_m^{(2)}) P_{n-2}^{(\alpha+3,\beta+3)}(x_m^{(2)}) \\
&\qquad + a_n^{(0)}(f_c) \\
&= a_n^{(0)}(\chi) + a_n^{(0)}(F_c),
\end{aligned}
\tag{4.12}
$$

where $a_n^{(0)}(\chi) \equiv \sum_{m=1}^{M} [f](x_m) w^{(\alpha+1,\beta+1)}(x_m) P_n^{(\alpha+1,\beta+1)}(x_m)$.

Furthermore, by virtue of (2.6) and integration by parts (twice), we have

$$
\begin{aligned}
a_n^{(0)}(f_c) &= 2(n+1)a_{n+1}^{(\alpha,\beta)}(f_c) = 2(n+1)\int_{-1}^{1} f_c(t) P_{n+1}^{(\alpha,\beta)}(t) w^{(\alpha,\beta)}(t)dt \\
&= -\int_{-1}^{1} f_c(t) d[P_n^{(\alpha+1,\beta+1)}(t) w^{(\alpha+1,\beta+1)}(t)] \\
&= -f_c(t) P_n^{(\alpha+1,\beta+1)}(t) w^{(\alpha+1,\beta+1)}(t)|_{-1}^{1} \\
&\quad + \int_{-1}^{1} P_n^{(\alpha+1,\beta+1)}(t) w^{(\alpha+1,\beta+1)}(t) df_c(t) \\
&= \int_{-1}^{1} f_c'(t) P_n^{(\alpha+1,\beta+1)}(t) w^{(\alpha+1,\beta+1)}(t)dt \\
&= -\frac{1}{2n}\int_{-1}^{1} f_c'(t) d[P_{n-1}^{(\alpha+2,\beta+2)}(t) w^{(\alpha+2,\beta+2)}(t)] \\
&= \frac{1}{2n}\int_{-1}^{1} P_{n-1}^{(\alpha+2,\beta+2)}(t) w^{(\alpha+2,\beta+2)}(t) df_c'(t) \\
&= \frac{1}{2n}\int_{-1}^{1} f_c''(t) P_{n-1}^{(\alpha+2,\beta+2)}(t) w^{(\alpha+2,\beta+2)}(t)dt.
\end{aligned}
\tag{4.13}
$$

Next, it is well known that [21, p. 68] ($n \in \mathbb{N}$)

$$
\hat{P}_n^{(\alpha,\beta)}(t) \equiv \sqrt{\frac{(2n+\alpha+\beta+1)\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}} P_n^{(\alpha,\beta)}(t)
\tag{4.14}
$$

represents the orthonormal system of Jacobi polynomials and

$$
K_1 n^{1/2} < \sqrt{\frac{(2n+\alpha+\beta+1)\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}} < K_2 n^{1/2}.
\tag{4.15}
$$

Since $f_c'' \in C[-1,1]$, then, say, by Bessel's inequality

$$
a_{n-1}^{(\alpha+2,\beta+2)}(f_c'') = \int_{-1}^{1} f_c''(t) \hat{P}_{n-1}^{(\alpha+2,\beta+2)}(t) w^{(\alpha+2,\beta+2)}(t)dt = o(1),
$$

which combined with (4.13), (4.14), and (4.15) yields

$$
\begin{aligned}
a_n^{(0)}(f_c) =& \frac{1}{2n}\int_{-1}^{1} f_c''(t)P_{n-1}^{(\alpha+2,\beta+2)}(t)w^{(\alpha+2,\beta+2)}(t)dt \\
=& O(n^{-3/2})\int_{-1}^{1} f_c''(t)\hat{P}_{n-1}^{(\alpha+2,\beta+2)}(t)w^{(\alpha+2,\beta+2)}(t)dt = o(n^{-3/2}).
\end{aligned}
\tag{4.16}
$$

By (4.4), (4.5), and (4.12) we have

(4.17)

$$
\begin{aligned}
a_n^{(1)}(-x_1) =& (-x_1)a_n^{(0)} + A_n^{(\alpha+1,\beta+1)}a_{n+1}^{(0)} + B_n^{(\alpha+1,\beta+1)}a_n^{(0)} + C_n^{(\alpha+1,\beta+1)}a_{n-1}^{(0)} \\
=& \sum_{m=2}^{M}[f](x_m)w^{(\alpha+1,\beta+1)}(x_m)P_n^{(\alpha+1,\beta+1)}(x_m)(x_m - x_1) \\
&+ \frac{1}{2}\sum_{m=1}^{M_1}[f'](x_m^{(1)})w^{(\alpha+2,\beta+2)}(x_m^{(1)}) \\
&\times [\frac{A_n^{(\alpha+1,\beta+1)}}{n+1}P_n^{(\alpha+2,\beta+2)}(x_m^{(1)}) + \frac{B_n^{(\alpha+1,\beta+1)} - x_1}{n}P_{n-1}^{(\alpha+2,\beta+2)}(x_m^{(1)}) \\
&+ \frac{C_n^{(\alpha+1,\beta+1)}}{n-1}P_{n-2}^{(\alpha+2,\beta+2)}(x_m^{(1)})] \\
&+ \ldots + a_n^{(1)}(f_c, -x_1) \equiv I_1 + I_2 + I_3 + a_n^{(1)}(f_c, -x_1).
\end{aligned}
$$

Since (see Table 1)

$$
|A_n^{(\alpha+1,\beta+1)}|, |B_n^{(\alpha+1,\beta+1)}|, |C_n^{(\alpha+1,\beta+1)}| = O(1),
\tag{4.18}
$$

due to (2.9), we get

(4.19)

$$
\begin{aligned}
|I_2| = O(\frac{1}{n})& \sum_{m=1}^{M_1}|[f'](x_m^{(1)})|w^{(\alpha+2,\beta+2)}(x_m^{(1)}) \\
&\times \max(|A_n^{(\alpha+1,\beta+1)}|, |B_n^{(\alpha+1,\beta+1)}|, |C_n^{(\alpha+1,\beta+1)}|) \\
&\times \sum_{i=n-2}^{n}|P_i^{(\alpha+2,\beta+2)}(x_m^{(1)})| = O(\frac{1}{n^{3/2}})\sum_{m=1}^{M_1}|[f'](x_m^{(1)})|w^{(\alpha/2+3/4,\beta/2+3/4)}(x_m^{(1)}).
\end{aligned}
$$

Analogously, $I_3 = O(n^{-5/2})$, and $|a_n^{(1)}(f_c, -x_1)| = o(n^{-3/2})$ by virtue of (4.16) and (4.18).

Combining (4.17) and (4.19), we obtain

$$
\begin{aligned}
a_n^{(1)}(-x_1) =& \sum_{m=2}^{M}[f](x_m)w^{(\alpha+1,\beta+1)}(x_m)P_n^{(\alpha+1,\beta+1)}(x_m)(x_m - x_1) \\
&+ O(\frac{1}{n^{3/2}})(\sum_{m=1}^{M_1}|[f'](x_m^{(1)})|w^{(\alpha/2+3/4,\beta/2+3/4)}(x_m^{(1)}) + o(1)).
\end{aligned}
$$

Following the arguments presented above, we have

$$a_n^{(M+j)}(-x_1,\ldots,-x_M,0,...,0) = O(\frac{1}{n^{3/2}})(\sum_{m=1}^{M_1}|[f'](x_m^{(1)})|w^{(\alpha/2+3/4,\beta/2+3/4)}(x_m^{(1)})+o(1)). \tag{4.20}$$

By virtue of (3.16) and (4.20), we conclude that the coefficients $q_i^{(M)}$, $i = 0,1,\ldots,M-1$, of the polynomial (3.4) satisfy the system of linear equations ($j = 0,1,\cdots,M-1$)

$$\sum_{i=0}^{M-1} a_n^{(i+j)}q_i^{(M)}+a_n^{(M+j)} = a_n^{(M+j)}(-x_1,\ldots,-x_M,0,...,0) = O(\frac{1}{n^{3/2}})(\sum_{m=1}^{M_1}|[f'](x_m^{(1)})|w^{(\alpha/2+3/4,\beta/2+3/4)}(x_m^{(1)})+o(1)) \tag{4.21}$$

or

$$\sum_{i=0}^{M-1} b_n^{(i+j)}q_i^{(M)}+b_n^{(M+j)} = b_n^{(M+j)}(-x_1,\ldots,-x_M,0,...,0) = O(\frac{1}{n^{3/2}})(\sum_{m=1}^{M_1}|[f'](x_m^{(1)})|w^{(\alpha/2+3/4,\beta/2+3/4)}(x_m^{(1)})+o(1)). \tag{4.22}$$

Hence, in view of (4.7) and (4.22) we obtain

$$\mathbf{B}_n^{(M)}(\mathbf{q}^{(M)}(n)-\mathbf{q}^{(M)}) = \mathbf{r}(n)$$

or due to $||\mathbf{A}\cdot\mathbf{B}|| \leq ||\mathbf{A}||||\mathbf{B}||$ (cf. [12, p. 70])

$$||\mathbf{q}^{(M)}(n)-\mathbf{q}^{(M)}|| = ||(\mathbf{B}_n^{(M)})^{-1}\mathbf{r}(n)|| \leq ||(\mathbf{B}_n^{(M)})^{-1}||||\mathbf{r}(n)||, \tag{4.23}$$

where $\mathbf{q}^{(M)}(n) \equiv (q_0^{(M)}(n),\ldots,q_{M-1}^{(M)}(n))$, $\mathbf{q}^{(M)} \equiv (q_0^{(M)},\ldots,q_{M-1}^{(M)})$, and (see (4.22))

$$||\mathbf{r}(n)|| = O(\frac{1}{n^{3/2}})(\sum_{m=1}^{M_1}|[f'](x_m^{(1)})|w^{(\alpha/2+3/4,\beta/2+3/4)}(x_m^{(1)})+o(1)). \tag{4.24}$$

If $F(x_1,x_2,\cdots,x_M) = (q_0^{(M)},q_1^{(M)},\cdots,q_{M-1}^{(M)})$ is the function mapping the real distinct roots of a monic polynomial on its coefficients, with the domain $\{(x_1,x_2,\cdots,x_M) : -1 < x_1 < x_2 < \cdots < x_M < 1\}$, then it is easy to obtain the estimate (cf. [15])

$$||\mathbf{x}-\mathbf{x}(n)|| < K(M)\frac{||\mathbf{q}^{(M)}-\mathbf{q}^{(M)}(n)||}{\prod_{i,j=1;i<j}^{M}(x_j-x_i)}, \tag{4.25}$$

where $\mathbf{x} \equiv (x_1,x_2,\ldots,x_M)$ and $\mathbf{x}(n) \equiv (x_1(n),x_2(n),\ldots,x_M(n))$. Combining (4.23), (4.24), and (4.25), we obtain (4.9).

Now let us estimate $||(\mathbf{B}_n^{(M)})^{-1}||$. Due to (4.12), the coefficient matrix $\mathbf{B}_n^{(M)}$ can be represented as

$$\mathbf{B}_n^{(M)} = \mathbf{B}_n^{(M)}(f) = \mathbf{B}_n^{(M)}(\chi)+\mathbf{B}_n^{(M)}(F_c). \tag{4.26}$$

Equations (4.12) and (2.9) imply

(4.27)

$$a_n^{(0)}(F_c) = O(\frac{1}{n^{3/2}})(\sum_{m=1}^{M_1} |[f'](x_m^{(1)})|w^{(\alpha/2+3/4,\beta/2+3/4)}(x_m^{(1)}) + o(1)) = O(n^{-3/2}).$$

Then, by (4.5), (4.18), and (4.27) we obtain

$$b_n^{(k)}(F_c) = a_n^{(k)}(F_c) + \imath a_{n+1}^{(k)}(F_c) = O(n^{-3/2})$$

for $k \in \mathbb{N}$.

As is known, $||\mathbf{A}|| = \max_{1\le i\le M} \sum_{j=1}^{M}|a_{ij}|$ (cf. [12, p. 70]). Therefore,

$$||\mathbf{B}_n^{(M)}(F_c)|| = \max_{1\le i\le M} \sum_{j=1}^{M} |b_n^{(i+j)}(F_c)| = O(n^{-3/2}). \tag{4.28}$$

On the other hand, by (4.12)

$$b_n^{(0)}(\chi) = \sum_{m=1}^{M} [f](x_m)w^{(\alpha+1,\beta+1)}(x_m)(P_n^{(\alpha+1,\beta+1)}(x_m) + \imath P_{n+1}^{(\alpha+1,\beta+1)}(x_m)).$$

Thus, in view of (4.8),

$$\begin{aligned} \mathbf{B}_n^{(M)}(\chi) = ((x_{i+1})^j)_{j,i=0}^{M-1}([f](x_i)w^{(\alpha+1,\beta+1)}(x_i)(P_n^{(\alpha+1,\beta+1)}(x_i) \\ + \imath P_{n+1}^{(\alpha+1,\beta+1)}(x_i))\delta_{i,j})_{i,j=1}^{M}((x_{i+1})^j)_{i,j=0}^{M-1} \\ \equiv \mathbf{X}\cdot\mathbf{FP}_n\cdot\mathbf{X}^t, \end{aligned} \tag{4.29}$$

and therefore

$$\begin{aligned} ||\mathbf{X}||^{-1}||\mathbf{FP}_n||^{-1}||\mathbf{X}^t||^{-1} &\le ||\mathbf{B}_n^{(M)}(\chi)||^{-1} \\ &\le ||(\mathbf{B}_n^{(M)}(\chi))^{-1}|| \le ||\mathbf{X}^{-1}||||\mathbf{FP}_n^{-1}||||(\mathbf{X}^t)^{-1}||. \end{aligned} \tag{4.30}$$

It is known [8] that

$$||\mathbf{X}^{-1}|| \le \max_{1\le m\le M} \prod_{k=1;k\ne m} \frac{1+|x_k|}{|x_m - x_k|}. \tag{4.31}$$

On the other hand, by virtue of (2.12)

$$||\mathbf{FP}_n^{-1}|| = O(n^{1/2}) \max_{1\le m\le M} \frac{(w^{(-\alpha-1/2,-\beta-1/2)}(x_m) + O(n^{-1}))^{-1/2}}{|[f](x_m)|w^{(\alpha+1,\beta+1)}(x_m)}. \tag{4.32}$$

Now, the Perturbation Lemma [12, p. 74] combined with (4.26), (4.28), and (4.32) yields that $\mathbf{B}_n^{(M)}(f)$ is invertible as well and

$$||(\mathbf{B}_n^{(M)}(f))^{-1}|| \le \frac{||(\mathbf{B}_n^{(M)}(\chi))^{-1}||}{1 - ||(\mathbf{B}_n^{(M)}(\chi))^{-1}||||\mathbf{B}_n^{(M)}(F_c)||}. \tag{4.33}$$

Therefore, (4.28), (4.30), (4.31), (4.32), and (4.33) lead to (4.10).

For the estimate for functions with continuous derivatives between the points of singularities, we refer to the proof of Theorem 3.2 in [15] as it is essentially the same. □

5. Description of the algorithm and numerical examples

Let us give a detailed description of the suggested algorithm utilizing the Fourier-Jacobi coefficients of a given piecewise smooth function. As we will show below, the method is unstable for the Laguerre and Hermite systems.

Algorithm

(1) Given a finite number of Fourier-Jacobi coefficients $a_n^{(\alpha,\beta)}(f)$ of the function f which piecewise belongs to $C^2[-1,1]$, we set $a_n^{(0)} \equiv 2(n+1)a_{n+1}^{(\alpha,\beta)}(f)$ and generate $a_n^{(k)}$ via the recurrence formula

$$a_n^{(k)} \equiv A_n^{(\alpha+1,\beta+1)} a_{n+1}^{(k-1)} + B_n^{(\alpha+1,\beta+1)} a_n^{(k-1)} + C_n^{(\alpha+1,\beta+1)} a_{n-1}^{(k-1)}. \tag{5.1}$$

Then $b_n^{(k)} \equiv a_n^{(k)} + \imath a_{n+1}^{(k)}$.

(2) In order to identify the number, M, of discontinuities, following Eckhoff, we pick a trial number $\tilde{M}$, large enough to guarantee that $\tilde{M} > M$. Then the rank of the matrix $\mathbf{B}_n^{(\tilde{M})} = (b_n^{(i+j)})_{i,j=0}^{\tilde{M}-1}$ will equal M. In addition, this number may be checked against the sharp local relative spikes of differentiated Fourier-Jacobi partial sums; see (1.1).

(3) Next, solve the linear system of equations $(j = 0, 1, \cdots, M-1)$

$$\sum_{i=0}^{M-1} b_n^{(i+j)} q_i^{(M)}(n) + b_n^{(M+j)} = 0. \tag{5.2}$$

(4) Use the solution of (5.2) as the coefficients of the polynomial equation

$$x^M + \sum_{i=0}^{M-1} q_{M-i}^{(M)}(n) x^{M-i} = 0 \tag{5.3}$$

and solve it.

(5) The roots $x_m(n)$ of the polynomial equation (5.3) represent approximations to the locations of jump discontinuities x_m, $m = 1, 2, \cdots, M$, of the function f.

Stability. If, in evaluating $b_n^{(0)}$, we encounter roundoff error e_n, then our computed value $\tilde{b}_n^{(k)}$ by formula (5.1) yields

$$\begin{aligned} |e_n^{(k)}| \equiv |\tilde{b}_n^{(k)} - b_n^{(k)}| &= |A_n^{(\alpha+1,\beta+1)} e_{n+1}^{(k-1)} + B_n^{(\alpha+1,\beta+1)} e_n^{(k-1)} + C_n^{(\alpha+1,\beta+1)} e_{n-1}^{(k-1)}| \\ &\leq (|A_n^{(\alpha+1,\beta+1)}| + |B_n^{(\alpha+1,\beta+1)}| + |C_n^{(\alpha+1,\beta+1)}|) \max(|e_{n+1}^{(k-1)}|, |e_n^{(k-1)}|, |e_{n-1}^{(k-1)}|). \end{aligned} \tag{5.4}$$

It is easy to check that $\lim_{n\to\infty}(|A_n^{(\alpha+1,\beta+1)}| + |B_n^{(\alpha+1,\beta+1)}| + |C_n^{(\alpha+1,\beta+1)}|) = 1$; in particular, for the Gegenbauer polynomials (see Table 1), $|A_n^{(\alpha+1,\beta+1)}| + |B_n^{(\alpha+1,\beta+1)}| + |C_n^{(\alpha+1,\beta+1)}| < 1$ for all $n \in \mathbb{N}$.

If we assume that the roundoff errors $e_n^{(0)}$, $n \in \mathbb{N}$, are bounded by some constant $\epsilon > 0$, then (5.4) leads to $|b_n^{(k)} - \tilde{b}_n^{(k)}| \leq \epsilon$, $k, n \in \mathbb{N}$; thus the method is stable, generating the coefficients of the linear system of equations (5.2). Although the coefficient matrix $\mathbf{B}_n^{(M)}$ of the linear system (5.2) is symmetric, it in not necessarily, say, positive definite [12, p. 61], which would guarantee the stability of Gaussian elimination computations with respect to the growth of roundoff errors.

For the Laguerre and Hermite systems, however, $|A_{1,n}|+|B_{1,n}|+|C_{1,n}|>n$; see Table 1. Thus, the method is unstable with respect to those systems.

Complexity. Since the matrix $(a_n^{(i+j)})_{i,j=0}^{M-1}$ is symmetric, calculating $a_n^{(2M-1)}$, we calculate all the entries of the matrix: a total of $12M^2-8M+2$ multiplications and $8M^2-8M+2$ additions. The rest of the steps of the algorithm are standard.

In order to test the theoretical results, i.e., the exact recovery of the locations of discontinuities and the associated jumps of a piecewise constant function by means of its Fourier coefficients with respect to a system of orthogonal polynomials, we utilized *Mathematica*. We have tested several piecewise constant functions with a wide variety of a number of discontinuities and jump magnitudes. In some examples, the singularities clustered within 10^{-4} distance and the jumps ranging from 10^{-2} to 100. All discontinuity locations, as well as the associated jumps, of the function have been recovered exactly symbolically using its Fourier coefficients with respect to various systems of the classical orthogonal polynomials.

The examples below illustrate the application of the method, Theorem 4.2, to various piecewise smooth functions. All computations are performed in double precision $((-k)\equiv 10^{-k})$.

The function (5.5) has two jump discontinuities at $x_1=-3/5$ and $x_2=1/5$. Below we will illustrate a step-by-step application of the algorithm to the function.

$$f_1(x)=\begin{cases}0 & \text{if } -1<x<-3/5,\\ (2x+3)^{1/3} & \text{if } -3/5<x<1/5,\\ 0 & \text{if } 1/5<x<1.\end{cases} \tag{5.5}$$

In order to identify the number, M, of singularities of the function f_1, we pick $\tilde{M}=5$ and apply QR factorization to the coefficient matrix of the system of linear equations (5.2).

The following is the triangular matrix of QR factorization of the matrix (5.2) for $\tilde{M}=5$:

$$\begin{pmatrix} 0.34 & 0.02-0.12\imath & 0.05-0.04\imath & 0.02-0.03\imath & 0.01-0.01\imath \\ 0. & 0.13 & 0.05-7.8(-8)\imath & 0.03-3.7(-7)\imath & 0.02-3.2(-7)\imath \\ 0. & 0. & -2.1(-7) & -2.1(-7)+7.4(-8)\imath & -1.8(-7)+6.4(-8)\imath \\ 0. & 0. & 0. & -1.5(-7) & -1.2(-7)+1.1(-8)\imath \\ 0. & 0. & 0. & 0. & 2.0(-10) \end{pmatrix}$$

It is plausible that $M=2$.

Now, the system of linear equations (5.2) is solved and the results are given in Table 2.

TABLE 2. The solutions of the system of linear equations (5.2) for various values of n with errors in the estimates to the coefficients $q_0=-3/25$ and $q_1=-2/5$ of the polynomial (5.3).

n	32	64	128
$q_0(n)$	$-0.119943-4.5(-7)\imath$	$-0.119984+1.6(-6)\imath$	$-0.119996-1.9(-9)\imath$
$\lvert q_0(n)-q_0\rvert$	$5.6(-5)$	$1.5(-5)$	$3.8(-6)$
$q_1(n)$	$-0.400378+1.0(-6)\imath$	$-0.400093+2.5(-7)\imath$	$-0.400024+1.0(-7)\imath$
$\lvert q_1(n)-q_1\rvert$	$3.7(-4)$	$9.3(-5)$	$2.4(-5)$

Finally, the polynomial equation (5.3) is solved with the coefficients presented in Table 2. The final results are presented in Table 3.

TABLE 3. The solutions of the polynomial equation (5.3) for various values of n with errors in the estimates to the discontinuity locations $x_1 = -3/5$ and $x_2 = 1/5$ for function (5.5).

n	32	64	128
$x_1(n)$	$-0.600213 + 1.8(-7)\imath$	$-0.600050 + 2.2(-6)\imath$	$-0.600013 + 7.4(-8)\imath$
$\lvert x_1 - x_1(n)\rvert$	$2.1(-4)$	$5.0(-5)$	$1.3(-5)$
$x_2(n)$	$0.199834 + 8.2(-7)\imath$	$0.199957 - 2.0(-6)\imath$	$0.199988 + 2.8(-8)\imath$
$\lvert x_2 - x_2(n)\rvert$	$1.6(-4)$	$4.2(-5)$	$1.1(-5)$

The following is a piecewise smooth function with three jump discontinuities:

$$f_2(x) = \begin{cases} -x - 2/3 & \text{if } -1 < x < -2/3, \\ (x+2/3)^2 & \text{if } -2/3 < x < -1/4, \\ e^x & \text{if } -1/4 < x < 1/3, \\ \sin x & \text{if } 1/3 < x < 2/3, \\ \ln x & \text{if } 2/3 < x < 1. \end{cases} \tag{5.6}$$

Below we present the absolute values of the errors in the estimation of the points of discontinuity of function (5.6) obtained by applying the suggested method and summarized in Table 4. Since the singularity locations are real, we consider only the real part of $x_m(n)$ for approximating x_m. The results are obviously better, but somewhat irregular.

TABLE 4. Errors in the estimates to the discontinuity locations for function (5.6) using its Fourier-Legendre coefficients.

n	32	64	128	256
$x_1 = -1/4$	$1.4(-2)$	$1.0(-2)$	$9.5(-3)$	$3.6(-3)$
$x_2 = 1/3$	$1.6(-1)$	$4.1(-2)$	$1.7(-2)$	$1.6(-2)$
$x_3 = 2/3$	$2.9(-2)$	$1.1(-2)$	$6.6(-3)$	$4.1(-3)$

This is a piecewise polynomial function with five jump discontinuities:

$$f_3(x) = \begin{cases} x^2 + 2x/3 + 10/9 & \text{if } -1 < x < -1/3, \\ x/6 + 7/144 & \text{if } -1/3 < x < -1/4, \\ x^3 - 1 & \text{if } -1/4 < x < -1/10, \\ -1001/1000 & \text{if } -1/10 < x < 0, \\ 4x^4 - 4x + 1 & \text{if } 0 < x < 1/4, \\ (x-1/4)^6 + 1/64 & \text{if } 1/4 < x < 1/2, \\ 9x^5 - 12x^3 + 5 & \text{if } 1/2 < x < 2/3, \\ 3/4 - x & \text{if } 2/3 < x < 3/4, \\ x^2 - 3x/2 + 9/16 & \text{if } 3/4 < x < 1. \end{cases} \tag{5.7}$$

TABLE 5. Errors in the estimates to the discontinuity locations for function (5.7) using its Fourier-Legendre coefficients.

n	32	64	128	256
$x_1 = -1/3$	1.6(−2)	6.3(−3)	4.4(−3)	1.4(−3)
$x_2 = -1/4$	6.3(−2)	1.8(−2)	7.4(−4)	5.9(−3)
$x_3 = 0$	7.4(−3)	7.6(−3)	3.8(−3)	1.1(−3)
$x_4 = 1/2$	1.2(−4)	5.0(−4)	7.5(−4)	6.0(−4)
$x_5 = 2/3$	6.9(−4)	1.4(−3)	1.9(−4)	4.9(−4)

The absolute value of the errors in the computed singularity locations for function (5.7) is given in Table 5.

As expected, due to (4.9) and (4.10), the locations of singularities for both functions f_2 and f_3 were approximated to within $O(1/n)$. The accuracy of approximation is by an order better for the function f_1, $O(1/n^2)$, since its derivative is continuous between the points of singularities.

6. Conclusion

We have studied a new method for approximating the jump discontinuity locations of a piecewise smooth function, if a finite number of its Fourier coefficients with respect to a system of the classical orthogonal polynomials are known. The method is based on a general but relatively simple modified Prony recovery technique—a three-term linear recurrence formula, a system of linear equations, and a polynomial equation. A utilization of this technique leads to a single unified method for *exact* recovery of the locations of singularities, as well as the magnitudes of associated jumps, of a piecewise constant function by means of its Fourier coefficients with respect to any system of the classical orthogonal polynomials and the trigonometric system.

Although unstable for piecewise smooth functions in general, the method still exactly identifies the locations of discontinuities of a piecewise constant function by means of Fourier coefficients with respect to the polynomial systems orthogonal on unbounded regions, namely, the Laguerre and Hermite systems.

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