

ON THE POINCARÉ-FRIEDRICHS INEQUALITY  
FOR PIECEWISE  $H^1$  FUNCTIONS IN ANISOTROPIC  
DISCONTINUOUS GALERKIN FINITE ELEMENT METHODS

HUO-YUAN DUAN AND ROGER C. E. TAN

ABSTRACT. The purpose of this paper is to propose a proof for the Poincaré-Friedrichs inequality for piecewise  $H^1$  functions on *anisotropic* meshes. By verifying suitable assumptions involved in the newly proposed proof, we show that the Poincaré-Friedrichs inequality for piecewise  $H^1$  functions holds independently of the aspect ratio which characterizes the shape-regular condition in finite element analysis. In addition, under the maximum angle condition, we establish the Poincaré-Friedrichs inequality for the Crouzeix-Raviart nonconforming linear finite element. Counterexamples show that the maximum angle condition is only sufficient.

1. INTRODUCTION

In discontinuous Galerkin finite element methods (including nonconforming methods) [8, 6, 19, 21, 7, 17], the following Poincaré-Friedrichs inequality (in two-dimensions) for piecewise polynomials is frequently used:

$$(1.1) \quad \|v\|_0 \leq C_{PF} \left( \sum_{D \in \mathcal{P}} |v|_{1,D}^2 + \sum_{f \in \mathcal{F}} |f|^{-1} \int_f |[v]|^2 \right)^{\frac{1}{2}},$$

where  $\mathcal{P} = \{D\}$  is a nonoverlapped partition of a given bounded domain  $\Omega$  in  $\mathbb{R}^2$ ,  $\mathcal{F}$  denotes the set of all sides in  $\mathcal{P}$ , and  $v$  is of piecewise  $H^1$  functions whose restrictions to each  $D \in \mathcal{P}$  are in  $H^1(D)$  with jump  $[v] = v|_{D_2} - v|_{D_1}$  across  $f = \partial D_1 \cap \partial D_2$  for  $D_1, D_2 \in \mathcal{P}$ . Inequality (1.1), proven in [6] (see also a recent work [15]), is a discrete version of the well-known Poincaré-Friedrichs inequality (see [25]):

$$(1.2) \quad \|v\|_0 \leq C |\Omega|^{\frac{1}{2}} |v|_1 \quad \forall v \in H_0^1(\Omega).$$

However, the inequality (1.1) was proven [6, 15] under the so-called shape-regular condition. This condition says that the aspect ratio  $\sigma_D$  of the diameter  $h_D$  of the sub-domain  $D$  and the supremum  $\rho_D$  of the diameters of all balls contained in  $D$  must be bounded from above (cf. [18, 23]), i.e., there exists a constant  $\sigma > 0$  such

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that

$$(1.3) \quad \sigma_D := \frac{h_D}{\rho_D} \leq \sigma \quad \forall D \in \mathcal{P} \quad (\text{shape-regular condition}).$$

As a matter of fact, the proof in [6] relies on the local trace theorem, e.g., for triangles  $D$  with side  $f$ :

$$(1.4) \quad |f| \int_f v^2 \leq C \frac{|f|^2}{|D|} \left( \|v\|_{0,D}^2 + h_D^2 \|\nabla v\|_{0,D}^2 \right) \quad \forall v \in H^1(D),$$

where the ratio  $\frac{|f|^2}{|D|}$  may be proportional to  $\sigma_D$  by (since  $D$  is triangle, there are sides  $f$  whose lengths take the diameter  $h_D$ , the length of the longest side of the triangle  $D$ )

$$(1.5) \quad \frac{4}{3} \sigma_D \leq \frac{h_D^2}{|D|} \leq 4 \sigma_D.$$

While the key step in the proofs in [15, 16] is the following estimation for linear functions  $v$  on triangle  $D$ :

$$(1.6) \quad |v|_{1,D}^2 \leq C \frac{h_D^2}{|D|} \sum_{i=1}^3 |v(a_i)|^2 \quad \text{or} \quad |v|_{1,D}^2 \leq C \frac{h_D^2}{|D|} \sum_{i=1}^3 |v(m_i)|^2,$$

where  $a_i$ ,  $1 \leq i \leq 3$ , is the  $i$ th vertex of the triangle  $D$ , and  $m_i$ ,  $1 \leq i \leq 3$ , is the mid-point of the  $i$ th side of the triangle  $D$ . Thus, both proofs in [6, 15] cannot deal with the case of *anisotropic meshes* in which the aspect ratio  $\sigma_D$  (or equivalently,  $h_D^2/|D|$ ) grows to infinity on some  $D$  sub-domains when the global mesh size  $h = \max_{D \in \mathcal{P}} h_D \rightarrow 0$  or some parameter such as the width of boundary layer tends to zero [5]. So, it gives rise to a naturally important question: Does (1.1) hold in the case of anisotropic meshes where the shape-regular condition is violated? Since (1.1) plays a prerequisite role in the stability analysis, such a question must be resolved in advance when using anisotropic discontinuous Galerkin finite element methods which are fundamentally instrumental in the treatment of corner and edge singularities, boundary and interior layers and adaptive algorithms; cf. [5, 20, 17].

In this note, we propose a new proof of (1.1) for general partitions of the domain. The newly proposed proof shows that (1.1) holds independently of the *aspect ratio* under Hypothesis H) and Condition C), and as a consequence, (1.1) holds on anisotropic meshes. As far as we know, this is the first proof which can be used to establish the Poincaré-Friedrichs inequality of piecewise  $H^1$  functions on anisotropic meshes.

Specifically, we show that (1.1) can be directly obtained from Hypothesis H) which states the local version of the Poincaré-Friedrichs inequality with mean value zero on the edge (a part of the sub-domain boundary) (cf. [25]) when the sub-domains of the partition can be rearranged to satisfy an essentially local condition labeled as Condition C). We remark that such a proof is elementary and that the local Poincaré-Friedrichs inequality labeled as Hypothesis H) can be easily verified through the well-known scaling argument [18, 23], while Condition C) can be fulfilled by most anisotropic meshes in [5] which are usually graded meshes. As a by-product, an explicit estimate can be obtained on the constant  $C_{PF}$ , which takes  $C|\Omega|^{\frac{1}{d}}$  in most cases, where  $C$  does not depend on  $\Omega \subset \mathbb{R}^d$ ,  $d = 2, 3$ . This estimation is consistent with (1.2). We should note that  $C_{PF}$  in (1.1) was in general

very difficult to determine and that an explicit estimate on  $C_{PF}$  plays a physically important role for practical purposes; see [10, 12, 28, 30] and the cited references.

In addition, under the maximum angle condition [27], we establish the Poincaré-Friedrichs inequality for the Crouzeix-Raviart (CR) nonconforming linear finite element [21], which is of particular interest in mixed methods for problems like the Stokes problem or the Reissner-Mindlin plate problem [7]. We remark that the maximum angle condition is much weaker than the usual shape-regular condition (equivalently, the minimum angle condition [14]) and is widely adopted in finite element analysis; see [5, 3, 4, 1, 2, 22]. But, counterexamples show that the maximum angle condition is only sufficient. Further, it is not clear whether the general inequality (1.1) with jumps term holds or not under the maximum angle condition.

The outline of this paper is as follows. In Section 2, Condition C) in two dimensions is stated and is verified in anisotropic meshes. Section 3 is devoted to the establishment of the Poincaré-Friedrichs inequalities in both two and three dimensions for piecewise  $H^1$  functions under Hypothesis H) and Condition C), and the verification of the three-dimensional Condition C) in anisotropic meshes, and the verification of Hypothesis H) for simplexes. In the last section, the Poincaré-Friedrichs inequality is established for the CR nonconforming linear element under the maximum angle condition.

## 2. CONDITION C) IN TWO DIMENSIONS AND ITS VERIFICATION IN ANISOTROPIC MESHES

We state Condition C) mentioned in the Introduction and verify it in a set of examples of anisotropic meshes. To fix the idea, we consider first only the two-dimensional case.

Given a bounded domain  $\Omega \subset \mathbb{R}^2$  with boundary  $\partial\Omega$ . Let  $\mathcal{P}^N$ ,  $N = 1, 2, \dots$ , denote a family of partitions of  $\Omega$ :  $\mathcal{P}^N = \{D_i^N; 1 \leq i \leq n^N\}$ ,  $\bar{\Omega} = \bigcup_{i=1}^{n^N} \bar{D}_i^N$ . We denote by  $\mathcal{F}^N$  the collection of all sides in  $\mathcal{P}^N$ . In what follows, when we mention a generic partition or sub-domain, we shall omit its superscripts and (or) subscripts. We denote by  $|f|$  and  $|D|$  the  $d-1$  and  $d$ -dimensional volumes of  $f \in \mathcal{F}$  and  $D \in \mathcal{P}$ , respectively, and by  $h_D$  the diameter of  $D$ , and by  $[v]$  the jump of  $v$  across an  $f \in \mathcal{F}$ :  $[v] = v|_{D_2} - v|_{D_1}$  along  $f \subseteq \bar{D}_2 \cap \bar{D}_1$ , and  $[v] = v|_D$  if  $f \subseteq \bar{D} \cap \partial\Omega$ . The  $L^2$ -norm is denoted by  $\|\cdot\|_{0,D}$ , with  $\|\cdot\|_0 := \|\cdot\|_{0,\Omega}$ , and  $|v|_{1,D} = \|\nabla v\|_{0,D}$  for  $v \in H^1(D) := \{v \in L^2(D), \nabla v \in (L^2(D))^2\}$ , with  $|v|_1 := |v|_{1,\Omega}$ . We set the global mesh-size  $h := \max_{D \in \mathcal{P}} h_D$ .

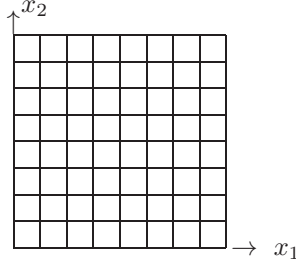
Below we state Condition C). To that goal, we rearrange the sub-domains of  $\mathcal{P}$  in the following way of ‘level decomposition’.

**Level decomposition.** Let  $\mathcal{P}$  be divided into  $K$  levels such that each level  $1 \leq k \leq K$  has  $m_k$  sub-domains:  $D_{i;k}$ ,  $1 \leq i \leq m_k$ , and these sub-domains are connected by a subset  $\mathcal{F}_k^c = \{f_{i-1,i;k} \in \mathcal{F}; 1 \leq i \leq m_k\}$ , such that  $D_{1;k}$  has a side  $f_{0,1;k}$  on  $\partial\Omega$  and  $f_{i-1,i;k} \subset \partial D_{i-1;k}$  and  $f_{i-1,i;k} \subset \bar{D}_{i;k}$ , for  $2 \leq i \leq m_k$ . Let  $\mathcal{F}^c$  denote the union of  $\mathcal{F}_1^c, \dots, \mathcal{F}_K^c$ .

**Condition C).** We require that the above ‘level decomposition’ holds for all levels  $1 \leq k \leq K$  with  $m = m_k$ :

$$(2.1) \quad \sum_{i=2}^m \sum_{r=1}^{i-1} \frac{|D_i| h_{D_r}^2}{|D_r|} \leq C_1(\Omega),$$

where  $C_1(\Omega)$  depends on  $\Omega$ , but it does not depend on  $m$  and  $k$ ,  $1 \leq k \leq K$ .

FIGURE 1. A partition with  $N = 3$ 

*Remark 1.* The level decomposition aims at decomposing the two-dimensional partitions into a sequence of ‘one-dimensional’ partitions, so that we can obtain the Poincaré-Friedrichs inequality in two dimensions directly from the sum of its ‘one-dimensional’ version on each ‘one-dimensional’ level.

Regarding Condition C), if all the sub-domains in  $\mathcal{P}$  have comparable areas, it would be roughly stated as  $m \sum_{i=1}^m h_{D_i}^2 \leq C(\Omega)$ , which essentially says that

$$(2.2) \quad \sum_{D \in \mathcal{P}} h_D^2 \leq C(\Omega),$$

since we may have  $m \sum_{i=1}^m h_{D_i}^2 \leq C \sum_{D \in \mathcal{P}} h_D^2$ . For triangles we see that if the shape-regular condition (1.3) holds, i.e., if  $h_D^2/|D| \leq C$  for all  $D \in \mathcal{P}$  holds (see (1.5)), we immediately have (2.2) from the following inequality which holds trivially:

$$(2.3) \quad \sum_{D \in \mathcal{P}} |D| \leq C |\Omega| \quad (\sum_{D \in \mathcal{P}} |D| = |\Omega| \text{ for nonoverlapped partitions}).$$

So, roughly speaking, the shape-regular condition (1.3) implies Condition C). But, the converse is in general not true. In fact, we will see that Condition C) itself (or the rough form (2.2)) is quite general and can hold even if the partitions do not satisfy any known shape-regular conditions such as minimum-angle condition (i.e., (1.3)) and maximum-angle condition [27].  $\square$

Here we give a very simple example so that readers can obtain some intuitive observations about the level decomposition and the constant  $C_1(\Omega)$  in Condition C). We usually have

$$(2.4) \quad C_1(\Omega) = C_1 |\Omega| \quad \text{or} \quad C_1 \mu^2 \quad \text{or} \quad C_1 \mathfrak{b}^2,$$

where  $\mu$  denotes the diameter of  $\Omega$  and  $\mathfrak{b}$  represents the directional diameter of  $\Omega$  along some direction, say along the  $x_1$  direction in the  $O - x_1 x_2$  coordinates system. Let us consider the example. Let  $\Omega = [0, 1]^2$  be partitioned into  $n = (2^N)^2$  ( $N = 1, 2, \dots$ ) squares (cf. Fig. 1). Along the  $x_2$  direction we have  $K = 2^N$  levels and each level has  $m_k = 2^N = K$  sub-domains (along the  $x_1$  direction). Clearly,  $|D| = h^2$  for all  $D$ , with  $h = 1/K$ , and  $|\Omega| = h^2 n = h^2 K^2$ . Condition C) holds with  $C_1(\Omega) = |\Omega|$ , since  $\sum_{i=1}^{m_k} |D_{i;k}| = m_k h^2 \leq |\Omega|$  and  $\sum_{i=2}^{m_k} (i-1) |D_{i;k}| = \sum_{i=2}^{m_k} \sum_{r=1}^{i-1} |D_{i;k}| h_{D_{r;k}}^2 / |D_{r;k}| = h^2 m_k (m_k - 1) / 2 \leq m_k^2 h^2 / 2 = |\Omega| / 2$ .

Before studying a set of examples of anisotropic meshes where the shape-regular condition (1.3) does not hold but Condition C) does, we give two variants of Condition C).

(V1) Assume that the sub-domains  $D_i$ ,  $1 \leq i \leq m$ , in *each level* have comparable areas, i.e.,  $|D_i| \approx$  the same (up to multiplicative constants), for  $1 \leq i \leq m$ , or satisfy  $|D_i| \leq C |D_r|$  for  $1 \leq r \leq i$  and  $1 \leq i \leq m$  for each level. Condition C) can be stated as follows:

**Condition C\*)** We assume that there holds for all levels:

$$(2.5) \quad \sum_{i=2}^m \sum_{r=1}^{i-1} h_{D_r}^2 \leq C_1^*(\Omega).$$

(V2) Assume that the sub-domains of each level have comparable  $\rho_D$  as defined in (1.3). For triangles we have  $|D| \approx h_D \rho_D$ , and we have the following variant of Condition C) for triangles:

**Condition C\*\*)** We assume that there holds for all levels:

$$(2.6) \quad \sum_{i=2}^m \sum_{r=1}^{i-1} h_{D_i} h_{D_r} \leq C_1^{**}(\Omega).$$

*Remark 2.* Conditions C\*) or C\*\*) often holds for graded meshes [5], the most widely used meshes in anisotropic finite element methods. Moreover, we may introduce a sequence of numbers,  $\delta_{i,r} := |D_i|/|D_r|$ , and  $\gamma_{i,r} := \rho_{D_i}/\rho_{D_r}$ , these numbers would be less than one, and the general forms  $\sum_{i=2}^m \sum_{r=1}^{i-1} \delta_{i,r} h_{D_r}^2 \leq C$  and  $\sum_{i=2}^m \sum_{r=1}^{i-1} \gamma_{i,r} h_{D_i} h_{D_r} \leq C$  would be useful.

In addition, for general but nongraded partitions, the general rule for checking Condition C) is to choose  $K \gg m_k$  for all  $1 \leq k \leq K$ , so that the estimate on the sum involved in Condition C) could be more easily done (cf. Example 4 below). Sometimes, a better level decomposition like the one in the above example as shown in Fig. 1 also helps to check Condition C). For this reason, it would be desirable to introduce a rectangle  $R$  containing  $\Omega$ , with edge length  $\mu$  being the diameter of  $\Omega$ , we may partition the outside of  $\Omega$  referring to the partition  $\mathcal{P}$  of  $\Omega$ . Then we check Condition C) on  $R$ , with some constant  $C_1(R)$  (but  $|R| = \mu^2$ ). Note that the piecewise  $H^1$  function defined on  $\Omega$  can be extended to  $R$  with values zero outside  $\Omega$ , so that the Poincaré-Friedrichs inequality for the  $u$  on  $\Omega$  can be obtained from the one for the extended  $u$  on  $R$ . □

We are now in a position to verify Condition C) for anisotropic meshes by giving the following set of Examples 1-4. The prototype of these anisotropic meshes can be found in [5].

**Example 1** (Babuška-Aziz-Dobrowolski’s partition). Let  $\Omega = [0, a]^2$  be partitioned into triangles to form a family of partitions  $\mathcal{P}^N$ ,  $N = 1, 2, \dots$ , with  $h_{x_2} = a/4^N$  and  $h_{x_1} = a/2^N$  and with the global mesh size  $h = \frac{a}{2^N}$ ; see Fig. 2.

The above partition is quite a representative example of anisotropic partitions, and does not fulfill any shape-regular conditions known in the literature, such as the minimum and maximum angle conditions:

$$\begin{aligned} \theta_{\min} &= \arctan(h/a) \rightarrow 0, \\ \theta_{\max} &= \arccos(1 - 2/(1 + h^2/a^2)) \rightarrow \arccos(-1) = \pi \quad \text{as } h \rightarrow 0, \end{aligned}$$

while the aspect ratio  $\sigma_D = \frac{h_D}{\rho_D} \approx \frac{2}{\theta_{\min}} \rightarrow \infty$  as  $h \rightarrow 0$ . Moreover, the two angles of those triangles having the bottom sides parallel to the  $x_2$  direction approximate  $\pi/2$  as  $h \rightarrow 0$ , i.e.  $\theta = \arccos \frac{h}{\sqrt{a^2+h^2}} \rightarrow \arccos 0 = \frac{\pi}{2}$  as  $h \rightarrow 0$ .

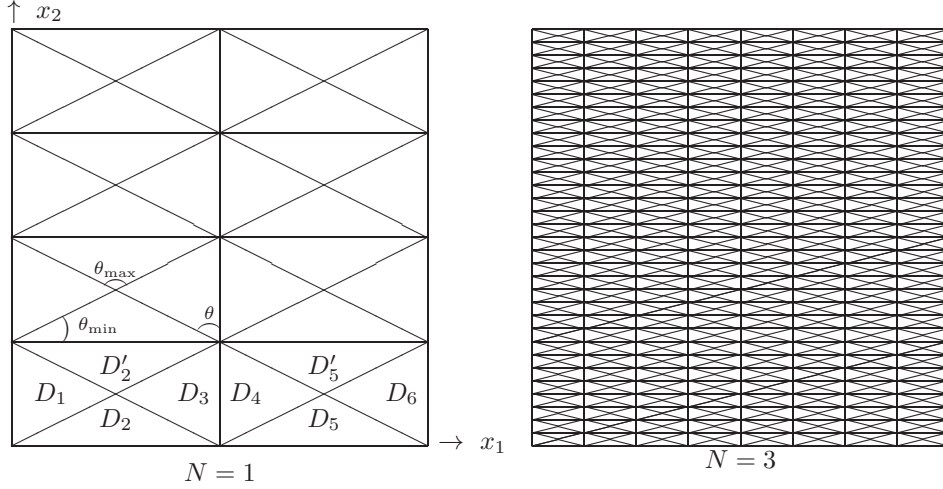


FIGURE 2. Babuška-Aziz-Dobrowolski's partition

The level decomposition is taken as  $K = 2 \times 4^N$  (along the  $x_2$  direction) and  $m_k = 2^N \times 3$  (along the  $x_1$  direction),  $1 \leq k \leq K$ , for example, with  $N = 1$  in Fig. 2, we have  $K = 2 \times 4$ ,  $m_k = 2 \times 3$ , and the first two consecutive (overlapped) levels are:  $D_1, D_2, D_3, D_4, D_5, D_6$  and  $D_1, D'_2, D_3, D_4, D'_5, D_6$ . Since all sub-domains are triangles and have the same areas  $\frac{a^2}{2^{3N+2}}$ , we can verify Condition C\*) with

$$\sum_{i=2}^{m_k} \sum_{r=1}^{i-1} h_{D_{r,k}}^2 \leq \sum_{i=2}^{m_k} (i-1) \frac{a^2}{2^{2N}} = \frac{m_k(m_k-1)}{2} \frac{a^2}{2^{2N}} \leq \frac{9}{2} a^2 = \frac{9}{2} |\Omega|.$$

**Example 2** (An anisotropic partition for boundary layers). Let  $\Omega = [0, a]^2$  be partitioned into four rectangular domains:  $[0, a - \varepsilon_0] \times [0, \varepsilon_0], [a - \varepsilon_0, a] \times [0, \varepsilon_0], [a - \varepsilon_0, a] \times [\varepsilon_0, a], [0, a - \varepsilon_0] \times [\varepsilon_0, a]$ , where  $\varepsilon_0 > 0$  represents some parameter which is usually chosen as  $\varepsilon |\ln \varepsilon|$  with  $0 < \varepsilon \ll 1$  being the width of the boundary layer. Note that the boundary layer phenomenon occurs in the advection-diffusion problem or the Reissner-Mindlin plate problem and some numerical methods; cf. [5]. These four rectangle domains are uniformly hierarchically refined as follows: they are first partitioned into  $(2^N)^2$  rectangles and each rectangle is then partitioned into 2 triangles; see Fig. 3. Such meshes are related to the Bakhvalov-Shishkin mesh [5]. The partition of  $\Omega$  is anisotropic in the sense that the aspect ratio for the rectangle domain  $[0, a - \varepsilon_0] \times [0, \varepsilon_0]$  is  $\sigma_D = \frac{h_D}{\rho_D} \approx \frac{a - \varepsilon_0}{\varepsilon_0} \rightarrow \infty$  as  $\varepsilon_0 \rightarrow 0$ .

For this example we choose a level decomposition as follows:  $K = 2 \times 2^N$  (along the  $x_2$  direction) and  $m_k = m_{k1} + m_{k2}$ , with  $m_{k1} = m_{k2} = 2 \times 2^N$  (from left to right along the  $x_1$  direction), for  $1 \leq k \leq K$ . Since  $|D_i| \leq |D_r|$  for all  $r \leq i$ , where  $1 \leq i \leq m_k$ ,  $1 \leq k \leq K$ , we can easily check Condition C\*) with  $C_1^*(\Omega) = C |\Omega|$ , where  $C$  does not depend on the parameter  $\varepsilon_0$ .

**Example 3** (An anisotropically graded partition for singularities). Let  $\Omega = [-a, a]^2 \setminus [0, a] \times [-a, 0]$  be partitioned by applying the cross product of the graded

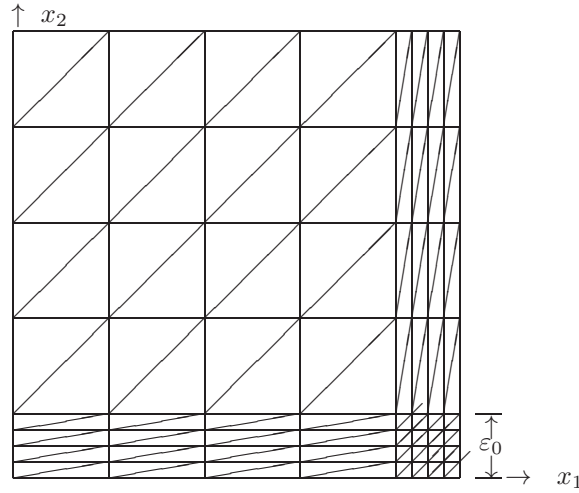


FIGURE 3. An anisotropic partition in boundary layers

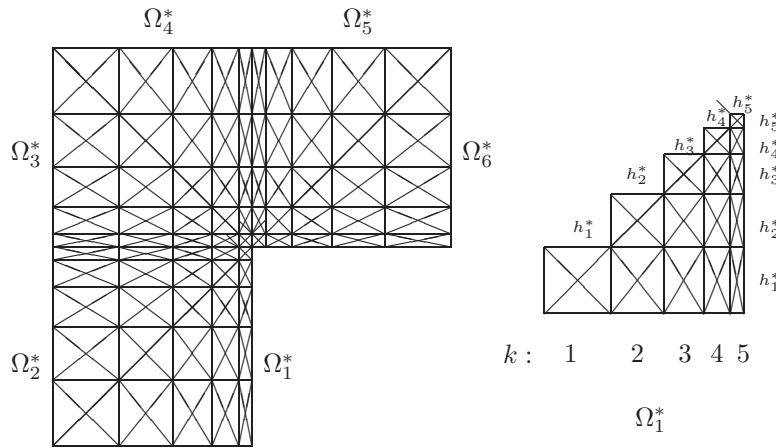


FIGURE 4. An anisotropically graded partition for corner and edge singularities

one-dimensional meshes. The graded mesh for the interval  $[0, a]$  is given by  $h_1^* > \dots > h_N^*$ ; see Fig. 4 for the case  $N = 5$ .

The whole domain  $\Omega$  can be partitioned into the union of  $\Omega_j^*$ ,  $1 \leq j \leq 6$ , as shown in Fig. 4, where  $\Omega_1^*$  and  $\Omega_2^*$  are overlapped along the diagonal direction and the same for the other two pairs  $(\Omega_3^*, \Omega_4^*)$  and  $(\Omega_5^*, \Omega_6^*)$ . The graded partition is anisotropic, since in the finer region (e.g.,  $k = 5$ , see  $\Omega_1^*$  in Fig. 4), the aspect ratio  $h_1^*/h_N^*$  grows to infinity as the global mesh size  $\rightarrow 0$ . The Poincaré-Friedrichs inequality for the whole  $\Omega$  can be obtained by combining those for the six sub-domains  $\Omega_j^*$ ,  $1 \leq j \leq 6$ , since, the jumps term in the Poincaré-Friedrichs inequalities for all  $\Omega_j^*$ , for example, for  $\Omega_1^*$  as shown in Fig. 4, does not involve the top part (along the diagonal direction) and the left and right parts of the boundary of  $\Omega_1^*$ ; see Remark 4 later on. The level decomposition for each  $\Omega_j^*$  is chosen as  $K = N$ , with each level sub-divided into two sub-levels which are overlapped in the same way as shown in

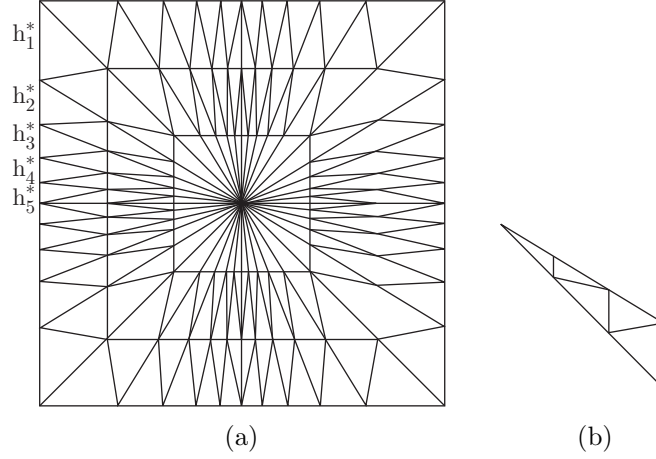


FIGURE 5. (a) An anisotropic partition with a vertex (center point) shared by triangles, the number of which tends to infinity as the partition is refined. (b) Each level has five fixed triangles.

Fig. 2, and each sub-level has  $k \times 3$  triangles,  $1 \leq k \leq K$ . With the above level decomposition, we know that the triangles of each sub-level have comparable  $\rho_D$ . In fact,  $\rho_D \approx h_k^*$  for each level  $1 \leq k \leq K$ . Noting that for each level  $1 \leq k \leq K$  we have  $h_{D_{i,k}} \approx h_1^*$  for  $1 \leq i \leq 3$ ,  $h_{D_{i,k}} \approx h_2^*$  for  $4 \leq i \leq 6$ , and so on, we can easily verify Condition C<sup>\*\*</sup>) with  $C_1^{**}(\Omega) = C|\Omega|$ .

**Example 4** (An anisotropic partition). Given  $\Omega = [0, 1]^2$ . Let two interior squares be fixed, respectively, at  $1/3$  and  $2/3$  of the half-diagonal line of  $\Omega$ . The three squares are mutually parallel. Along the boundary  $\partial\Omega$ , the graded one-dimensional partition is applied, where the graded mesh for the interval  $[0, 1/2]$  is given by  $h_1^*, \dots, h_N^*$ , and then the partition of  $\Omega$  is generated by connecting the center point  $(1/2, 1/2)$  of  $\Omega$  with the points on  $\partial\Omega$ , where each resulting quadrilateral is further bisected into two triangles by connecting two opposite vertices. See Fig. 5 for the case  $N = 5$ . The partition is obviously anisotropic because the number of the triangles sharing the center point of  $\Omega$  tends to infinity as the partition is being refined. The level decomposition is chosen as  $K = 8N$  (along  $\partial\Omega$ ), and the number of triangles in each level is fixed, i.e.,  $m_k = 5$ , for  $k = 1, \dots, K = 8N$  and for  $N = 1, 2, \dots$ . It is thus very easy to verify Condition C) with  $C_1(\Omega) = C|\Omega|$ .

### 3. POINCARÉ-FRIEDRICHS INEQUALITIES IN TWO AND THREE DIMENSIONS

We establish the Poincaré-Friedrichs inequality for piecewise  $H^1$  functions with their restrictions to each  $D \in \mathcal{P}$  being in  $H^1(D)$ , under Condition C) and the following hypothesis:

**Hypothesis H) (A local Poincaré-Friedrichs inequality)** For all  $\mathcal{P}^N$ ,  $N = 1, 2, \dots$ , we assume that for any  $v \in H^1(D_i^N)$  and any  $f \in \mathcal{F}^N$  with  $f \subset \bar{D}_i^N$  (Closure of  $D_i^N$ ),  $1 \leq i \leq n^N$ , if  $\int_f v = 0$ , there exists a constant  $C_2 > 0$  independent of  $f, D_i^N$  and  $v$  such that

$$(3.1) \quad \|v\|_{0,D_i^N}^2 \leq C_2 h_{D_i^N}^2 |v|_{1,D_i^N}^2.$$



*Remark 3.* Hypothesis H) is a local Poincaré-Friedrichs inequality. It plays a key role in proving Lemma 1 below. This lemma essentially establishes one-dimensional Poincaré-Friedrichs inequalities along each ‘one-dimensional’ level; what is really needed is (3.13) resulting from Hypothesis H). (3.13) just says (for example, for linear polynomials)

$$(3.2) \quad (u(p_1) - u(p_2))^2 \leq |p_1 - p_2|^2 |\nabla u|^2,$$

where  $p_1$  and  $p_2$  are two opposite points on two opposite sides  $f_1$  and  $f_2$  of some sub-domain  $D$ , such that  $u(p_j) = \int_{f_j} u/|f_j|$ ,  $j = 1, 2$ .

**Theorem 1.** *Assume that Hypothesis H) and Condition C) hold. Then, for any piecewise  $H^1$  function  $u$  defined on  $\mathcal{P}$  (the partition of  $\Omega \subset \mathbb{R}^2$ ), there exists a constant  $C(\Omega) = \sqrt{\max(32 C_2 C_1(\Omega) + 2 C_2 h^2, 8 C_1(\Omega), 4 C|\Omega|)}$  such that*

$$(3.3) \quad \|u\|_0 \leq C(\Omega) \left( \sum_{D \in \mathcal{P}} |u|_{1,D}^2 + \sum_{f \in \mathcal{F}^c} |f|^{-2} \left( \int_f [u] \right)^2 \right)^{\frac{1}{2}}.$$

Let  $u$  be any given piecewise  $H^1$  function, whose restrictions to each sub-domain of  $\mathcal{P}$  are denoted as  $u_{i;k}$ ,  $1 \leq i \leq m_k$ ,  $1 \leq k \leq K$ , in the same way as the level decomposition. We define two piecewise constant functions  $\bar{u}^\pm \in L^2(\Omega)$  sub-domain-by-sub-domain in the same way as the level decomposition as follows:

$$(3.4) \quad \bar{u}_{i-1;k}^+|_{D_{i;k}} := \frac{1}{|f_{i-1,i;k}|} \int_{f_{i-1,i;k}} u_{i;k}, \quad 1 \leq i \leq m_k, 1 \leq k \leq K,$$

$$(3.5) \quad \bar{u}_{0;k}^- = 0, \quad \bar{u}_{i;k}^-|_{D_{i;k}} := \frac{1}{|f_{i,i+1;k}|} \int_{f_{i,i+1;k}} u_{i;k}, \quad 1 \leq i \leq m_k - 1, 1 \leq k \leq K.$$

**Lemma 1.** *Assume that Hypothesis H) and Condition C) hold. Given  $u$ , a piecewise  $H^1$  function defined on  $\mathcal{P}$ , with  $\bar{u}^\pm$  defined by (3.4) and (3.5). We have*

$$(3.6) \quad \begin{aligned} \|\bar{u}^+\|_0^2 &\leq 2C|\Omega| \sum_{k=1}^K (\bar{u}_{0;k}^+)^2 + 16C_2 C_1(\Omega) \sum_{D \in \mathcal{P}} |u|_{1,D}^2 \\ &+ 4C_1(\Omega) \sum_{k=1}^K \sum_{i=1}^{m_k-1} (\bar{u}_{i;k}^+ - \bar{u}_{i;k}^-)^2. \end{aligned}$$

*Proof.* We first see that for  $1 \leq i \leq m_k$  and  $1 \leq k \leq K$ ,

$$(3.7) \quad \bar{u}_{i-1;k}^+ = \bar{u}_{0;k}^+ + \sum_{r=1}^{i-1} (\bar{u}_{r;k}^+ - \bar{u}_{r-1;k}^+),$$

where

$$(3.8) \quad \bar{u}_{r;k}^+ - \bar{u}_{r-1;k}^+ = \bar{u}_{r;k}^+ - \bar{u}_{r;k}^- + \bar{u}_{r;k}^- - \bar{u}_{r-1;k}^+,$$

$$(3.9) \quad \bar{u}_{r;k}^- - \bar{u}_{r-1;k}^+ = \bar{u}_{r;k}^- - u_{r;k} + u_{r;k} - \bar{u}_{r-1;k}^+.$$

Since from (3.4) and (3.5),

$$(3.10) \quad \int_{f_{r,r+1;k}} (\bar{u}_{r;k}^- - u_{r;k}) = 0, \quad \int_{f_{r-1,r;k}} (u_{r;k} - \bar{u}_{r-1;k}^+) = 0,$$

we have from Hypothesis H)

$$(3.11) \quad \|\bar{u}_{r;k}^- - u_{r;k}\|_{0,D_{r;k}}^2 \leq C_2 h_{D_{r;k}}^2 |u_{r;k}|_{1,D_{r;k}}^2,$$

$$(3.12) \quad \|u_{r;k} - \bar{u}_{r-1;k}^+\|_{0,D_{r;k}}^2 \leq C_2 h_{D_{r;k}}^2 |u_{r;k}|_{1,D_{r;k}}^2,$$

and we have from (3.9), (3.11), (3.12) and the Cauchy inequality

$$(3.13) \quad \|\bar{u}_{r;k}^- - \bar{u}_{r-1;k}^+\|_{0,D_{r;k}}^2 \leq 4 C_2 h_{D_{r;k}}^2 |u_{r;k}|_{1,D_{r;k}}^2.$$

We thus have from the Cauchy inequality, (3.8) and (3.13),

$$(3.14) \quad \begin{aligned} & \left( \sum_{r=1}^{i-1} (\bar{u}_{r;k}^+ - \bar{u}_{r-1;k}^+) \right)^2 \leq 2 \left( \sum_{r=1}^{i-1} (\bar{u}_{r;k}^+ - \bar{u}_{r;k}^-) \right)^2 \\ & \quad + 2 \left( \sum_{r=1}^{i-1} (\bar{u}_{r;k}^- - \bar{u}_{r-1;k}^+) \right)^2 \\ & \leq 2 \left( \sum_{r=1}^{i-1} \frac{h_{D_{r;k}}^2}{|D_{r;k}|} \right) \left( \sum_{r=1}^{i-1} \frac{|D_{r;k}|}{h_{D_{r;k}}^2} (\bar{u}_{r;k}^+ - \bar{u}_{r;k}^-)^2 \right) \\ & \quad + 2 \left( \sum_{r=1}^{i-1} \frac{h_{D_{r;k}}^2}{|D_{r;k}|} \right) \left( \sum_{r=1}^{i-1} \frac{|D_{r;k}|}{h_{D_{r;k}}^2} (\bar{u}_{r;k}^- - \bar{u}_{r-1;k}^+)^2 \right) \\ & = 2 \left( \sum_{r=1}^{i-1} \frac{h_{D_{r;k}}^2}{|D_{r;k}|} \right) \left( \sum_{r=1}^{i-1} \frac{|D_{r;k}|}{h_{D_{r;k}}^2} (\bar{u}_{r;k}^+ - \bar{u}_{r;k}^-)^2 \right) \\ & \quad + 2 \left( \sum_{r=1}^{i-1} \frac{h_{D_{r;k}}^2}{|D_{r;k}|} \right) \left( \sum_{r=1}^{i-1} h_{D_{r;k}}^{-2} \|\bar{u}_{r;k}^- - \bar{u}_{r-1;k}^+\|_{0,D_{r;k}}^2 \right) \\ & \leq 2 \left( \sum_{r=1}^{i-1} \frac{h_{D_{r;k}}^2}{|D_{r;k}|} \right) \left( \sum_{r=1}^{i-1} (\bar{u}_{r;k}^+ - \bar{u}_{r;k}^-)^2 \right) \\ & \quad + 8 C_2 \left( \sum_{r=1}^{i-1} \frac{h_{D_{r;k}}^2}{|D_{r;k}|} \right) \left( \sum_{r=1}^{i-1} |u_{r;k}|_{1,D_{r;k}}^2 \right), \end{aligned}$$

where, in obtaining the first part of the last inequality we have used the following obvious fact: For any  $D \in \mathcal{P}$  in two dimensions since  $D$  can be contained in the square with side length being  $h_D$  we have

$$(3.15) \quad |D| \leq h_D^2.$$

Hence, we have from (3.7), the Cauchy inequality and (3.14),

$$(3.16) \quad \begin{aligned} & \sum_{i=1}^{m_k} \|\bar{u}_{i-1;k}^+\|_{0,D_{i;k}}^2 = \sum_{i=1}^{m_k} (\bar{u}_{i-1;k}^+)^2 |D_{i;k}| \\ & \leq 2 \sum_{i=1}^{m_k} (\bar{u}_{0;k}^+)^2 |D_{i;k}| + 4 \sum_{i=1}^{m_k} |D_{i;k}| \left( \sum_{r=1}^{i-1} \frac{h_{D_{r;k}}^2}{|D_{r;k}|} \right) \left( \sum_{r=1}^{i-1} (\bar{u}_{r;k}^+ - \bar{u}_{r;k}^-)^2 \right) \\ & \quad + 16 C_2 \sum_{i=1}^{m_k} |D_{i;k}| \left( \sum_{r=1}^{i-1} \frac{h_{D_{r;k}}^2}{|D_{r;k}|} \right) \left( \sum_{r=1}^{i-1} |u_{r;k}|_{1,D_{r;k}}^2 \right) \\ & \leq 2 (\bar{u}_{0;k}^+)^2 \sum_{i=1}^{m_k} |D_{i;k}| + 4 \left( \sum_{i=2}^{m_k} |D_{i;k}| \sum_{r=1}^{i-1} \frac{h_{D_{r;k}}^2}{|D_{r;k}|} \right) \sum_{i=1}^{m_k-1} (\bar{u}_{i;k}^+ - \bar{u}_{i;k}^-)^2 \\ & \quad + 16 C_2 \left( \sum_{i=2}^{m_k} |D_{i;k}| \sum_{r=1}^{i-1} \frac{h_{D_{r;k}}^2}{|D_{r;k}|} \right) \sum_{i=1}^{m_k-1} |u_{i;k}|_{1,D_{i;k}}^2. \end{aligned}$$

We finally obtain from (3.16),

$$\begin{aligned}
(3.17) \quad & \sum_{k=1}^K \sum_{i=1}^{m_k} \|\bar{u}_{i-1;k}^+\|_{0,D_{i;k}}^2 \leq 2 \sum_{k=1}^K (\bar{u}_{0;k}^+)^2 \sum_{i=1}^{m_k} |D_{i;k}| \\
& + 4 \sum_{k=1}^K \left( \sum_{i=2}^{m_k} \sum_{r=1}^{i-1} \frac{|D_{i;k}| h_{D_{r;k}}^2}{|D_{r;k}|} \right) \sum_{i=1}^{m_k-1} (\bar{u}_{i;k}^+ - \bar{u}_{i;k}^-)^2 \\
& + 16 C_2 \sum_{k=1}^K \left( \sum_{i=2}^{m_k} \sum_{r=1}^{i-1} \frac{|D_{i;k}| h_{D_{r;k}}^2}{|D_{r;k}|} \right) \sum_{i=1}^{m_k-1} |u_{i;k}|_{1,D_{i;k}}^2.
\end{aligned}$$

Therefore, we conclude that (3.6) holds from the above inequality and (2.1) in Condition C) and the obvious fact in (2.3):  $\sum_{i=1}^{m_k} |D_{i;k}| \leq C |\Omega|$ .  $\square$

*Proof of Theorem 1.* Since from (3.4),

$$(3.18) \quad \int_{f_{i-1,i;k}} (u_{i;k} - \bar{u}_{i-1;k}^+) = 0,$$

we have from Hypothesis H) and the Cauchy inequality

$$(3.19) \quad \|u_{i;k}\|_{0,D_{i;k}}^2 \leq 2 C_2 h_{D_{i;k}}^2 |u_{i;k}|_{1,D_{i;k}}^2 + 2 \|\bar{u}_{i-1;k}^+\|_{0,D_{i;k}}^2,$$

and we have

$$\begin{aligned}
(3.20) \quad \|u\|_0^2 &= \sum_{k=1}^K \sum_{i=1}^{m_k} \|u_{i;k}\|_{0,D_{i;k}}^2 \leq 2 C_2 \sum_{k=1}^K \sum_{i=1}^{m_k} h_{D_{i;k}}^2 |u_{i;k}|_{1,D_{i;k}}^2 \\
&+ 2 \sum_{k=1}^K \sum_{i=1}^{m_k} \|\bar{u}_{i-1;k}^+\|_{0,D_{i;k}}^2,
\end{aligned}$$

where

$$\sum_{k=1}^K \sum_{i=1}^{m_k} h_{D_{i;k}}^2 |u_{i;k}|_{1,D_{i;k}}^2 \leq \max_{D \in \mathcal{P}} h_D^2 \left( \sum_{k=1}^K \sum_{i=1}^{m_k} |u_{i;k}|_{1,D_{i;k}}^2 \right) = h^2 \sum_{D \in \mathcal{P}} |u|_{1,D}^2.$$

We then have from Lemma 1,

$$\begin{aligned}
(3.21) \quad \|u\|_0^2 &\leq (32 C_2 C_1(\Omega) + 2 C_2 h^2) \sum_{D \in \mathcal{P}} |u|_{1,D}^2 \\
&+ 8 C_1(\Omega) \sum_{k=1}^K \sum_{i=1}^{m_k-1} |f_{i,i+1;k}|^{-2} \left( \int_{f_{i,i+1;k}} [u] \right)^2 \\
&+ 4 C |\Omega| \sum_{k=1}^K |f_{0,1;k}|^{-2} \left( \int_{f_{0,1;k}} u \right)^2.
\end{aligned}$$

The proof is finished.  $\square$

*Remark 4.* Note that the jumps term in the Poincaré-Friedrichs inequality obtained in Theorem 1 involves only some subsets of the whole set of sides/faces. Considering the example in Fig. 1, we see that only approximately 50% of the sides enter into the jumps term, and that those sides on the bottom, top and right (boundary) of  $\Omega$  are not involved. In addition, our proof does not require the partitions to be

nonoverlapped. Overlapped partitions may arise from Mortar and domain decomposition methods and elsewhere [13, 31, 26]. We point out that the proofs in [6, 15] do not cover the case of overlapped partitions.  $\square$

For three-dimensional domains  $\Omega \subset \mathbb{R}^3$ , by stating similar Condition C) and Hypothesis H), we can establish the Poincaré-Friedrichs inequality for piecewise  $H^1$  functions defined with respect to the partitions  $\mathcal{P}$  of  $\Omega$ . In fact, for three-dimensional domains, Hypothesis H) has the same form as that for two dimensions. To state Condition C), we need level decomposition as follows:

**3D Level decomposition.** Let  $\mathcal{P} = \{D\}$ , the partition of  $\Omega$ , be firstly divided into  $L$  levels and then each level  $1 \leq l \leq L$  be divided into  $K_l$  levels such that each level  $1 \leq k \leq K_l$  has  $m_{k,l}$  sub-domains:  $D_{i;k,l}$ ,  $1 \leq i \leq m_{k,l}$ , and these sub-domains are connected by a subset  $\mathcal{F}_{k,l}^c = \{f_{i-1,i;k,l} \in \mathcal{F}; 1 \leq i \leq m_{k,l}\}$ , such that  $D_{1;k,l}$  has a face  $f_{0,1;k,l}$  on  $\partial\Omega$  and  $f_{i-1,i;k,l} \subset \partial D_{i-1;k,l}$  and  $f_{i-1,i;k,l} \subset \bar{D}_{i;k,l}$ , for  $2 \leq i \leq m_{k,l}$ . Let  $\mathcal{F}^c$  denote the union of  $\mathcal{F}_{k,l}^c$ ,  $1 \leq k \leq K_l$ ,  $1 \leq l \leq L$ .

**3D Condition C).** We require that the above ‘level decomposition’ makes

$$(3.22) \quad \max \left( \sum_{i=2}^m \sum_{r=1}^{i-1} \frac{|D_i| h_{D_r}^2}{|D_r|}, \sum_{i=2}^m \sum_{r=1}^{i-1} \frac{|D_i|}{|f_{r,r+1}|^{\frac{1}{2}}}, \sum_{i=1}^m \frac{|D_i|}{|f_{0,1}|^{\frac{1}{2}}} \right) \leq C_1(\Omega)$$

for all levels  $1 \leq k \leq K_l$  and  $1 \leq l \leq L$  with  $m = m_{k,l}$ , where  $C_1(\Omega)$  depends on  $\Omega$ , but it does not depend on  $m$  and  $k, l$ ,  $1 \leq k \leq K_l$ ,  $1 \leq l \leq L$ . We usually have  $C_1(\Omega) = C_1 |\Omega|^{\frac{2}{3}}$ , or  $C_1 \mu^2$ , or  $C_1 \mathfrak{b}^2$ , where  $\mu$  denotes the diameter of  $\Omega$  and  $\mathfrak{b}$  represents the directional diameter of  $\Omega$  along some direction.

Similar to the two-dimensional case (See (3.14) and (3.15) in proving Lemma 1 for two dimensions), if

$$(3.23) \quad \frac{|D|}{h_D^2} \leq C |f|^{\frac{1}{2}} \quad \text{for } f \in \mathcal{F} \text{ with } f \subset \bar{D},$$

the above Condition C) can then be replaced by the following unified form as in two dimensions:

$$(3.24) \quad \sum_{i=2}^m \sum_{r=1}^{i-1} \frac{|D_i| h_{D_r}^2}{|D_r|} \leq C_1(\Omega).$$

Equation (3.23) is obviously true for those commonly used triangulations (i.e., conforming partitions [18]) composed of tetrahedra and hexahedra, since  $|D| \leq C |f| h_D$  (the bottom area  $\times$  the height)  $\leq C |f|^{1/2} h_D^2$ .

For the three-dimensional case, just following the same argument as in proving Theorem 1, we have

**Theorem 2.** *Assume that Hypothesis H) in three dimensions and 3D Condition C) hold. Then, for any piecewise  $H^1$  function  $u$  defined on  $\mathcal{P}$  (the partition of  $\Omega \subset \mathbb{R}^3$ ), there exists a constant  $C(\Omega) = \sqrt{\max(32 C_2 C_1(\Omega) + 2 C_2 h^2, 8 C_1(\Omega))}$  such that*

$$(3.25) \quad \|u\|_0 \leq C(\Omega) \left( \sum_{D \in \mathcal{P}} |u|_{1,D}^2 + \sum_{f \in \mathcal{F}^c} |f|^{-\frac{3}{2}} \left( \int_f [u] \right)^2 \right)^{\frac{1}{2}}.$$

In what follows we consider two examples of anisotropic meshes in three dimensions to verify Condition C) as in (3.24).

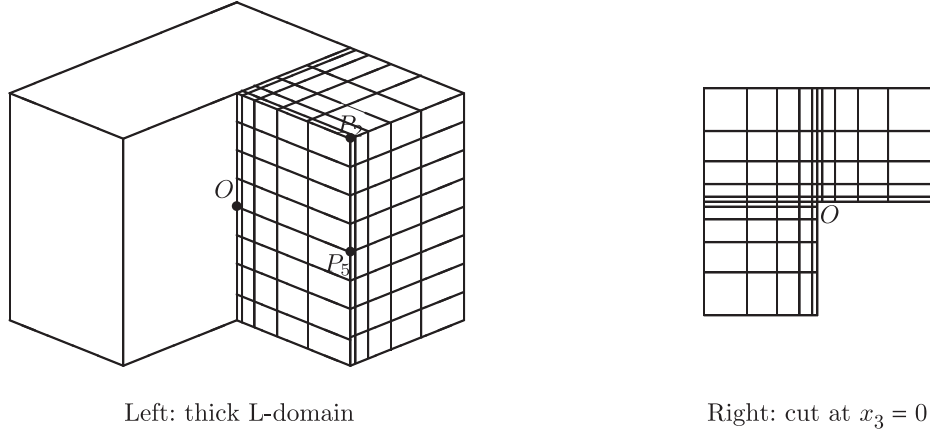


FIGURE 6. An anisotropic partition for a thick  $L$ -domain with a ‘singular’ edge

**Example 5** (An anisotropically graded partition for edge singularities). Let  $\Omega = ([-a, a]^2 \setminus [0, a] \times [-a, 0]) \times [-a, a]$  be the thick  $L$ -domain in  $\mathbb{R}^3$ .  $\Omega$  has a ‘singular’ edge. Let  $\Omega$  be partitioned as follows (See Fig. 6): along the plane  $(x_1, x_2)$  the cross product of the anisotropically graded one-dimensional meshes for the interval  $[0, a]$  with  $h_1^* > h_2^* > \dots > h_N^*$ , satisfying  $h_1^*/h_N^* \rightarrow \infty$  as  $N \rightarrow \infty$ , while along the  $x_3$  direction the uniform coarser partition of the interval  $[0, a]$  with  $h_z = a/N_z$ , with some  $N_z$  such that

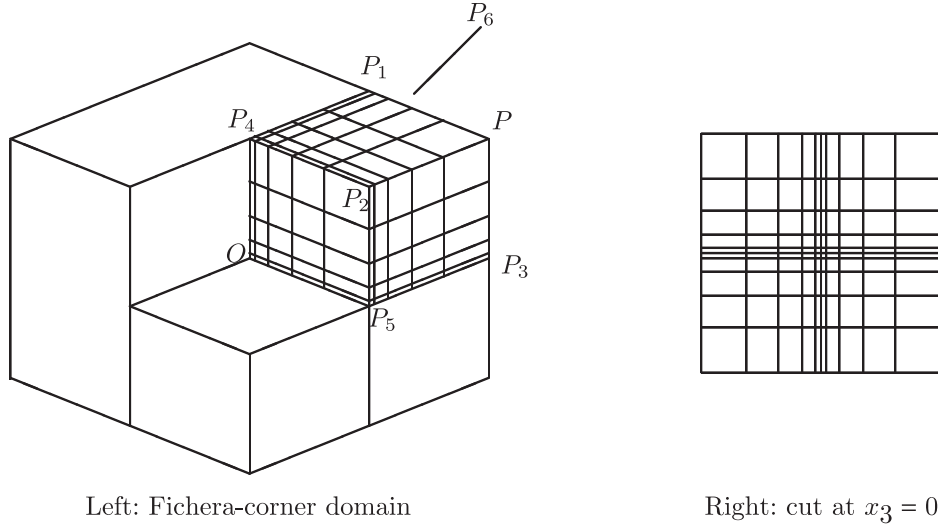
$$(3.26) \quad h_z \geq C h_1^*.$$

For simplicity, we do not further sub-partition each cuboid into six tetrahedra. Due to the symmetry, we need only consider the part of  $[0, a]^3$ . The level decomposition is chosen as follows: along the  $x_2$  (opposite) direction,  $l = 1, 2, \dots, N$ , and then along the  $x_1$  (opposite) direction  $k = 1, 2, \dots, N$ , and then along the  $x_3$  (opposite) direction  $i = 1, 2, \dots, N_z$ . Each ‘one-dimensional’ level has  $N_z$  cuboids, and all of the cuboids have the same volumes and have the same diameters  $\approx h_z$  (because of (3.26)). For example, along the level  $P_2P_5$  (i.e.,  $l = N = 5$  and  $k = 1$ ) there are  $N_z = 4$  cuboids of the same volume  $h_N^* \times h_1^* \times h_z$  and the same diameter  $\approx h_z$  (because of  $h_z \leq \sqrt{(h_N^*)^2 + (h_1^*)^2} + h_z \leq C h_z$ ). We verify (3.24) by computing with  $m = N_z$ ,

$$\sum_{i=2}^m \sum_{r=1}^{i-1} \frac{|D_i| h_{D_r}^2}{|D_r|} = \sum_{i=2}^{N_z} \sum_{r=1}^{i-1} h_z^2 = h_z^2 \frac{N_z(N_z - 1)}{2} \leq h_z^2 N_z^2 = a^2 \leq C |\Omega|^{2/3}.$$

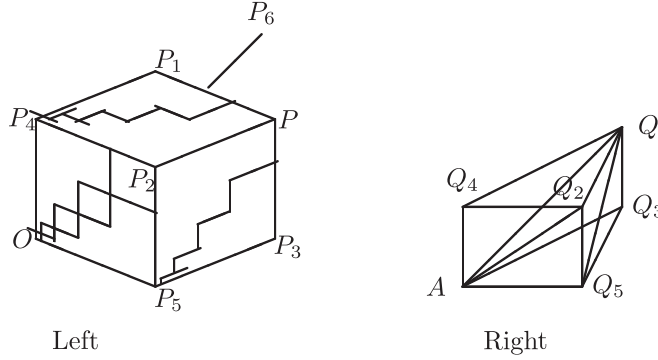
If considering tetrahedra partitions, just sub-partitioning each cuboid into 6 tetrahedra of the same volumes, we have two sub-levels of tetrahedra in each level of cuboids from the above level decomposition for cuboids, where each cuboid contains three consecutively connected tetrahedra of each sub-level; see the right-hand side figure in Fig. 8 below. All the tetrahedra in each sub-level have the same volume and the same diameter, so we have the same estimates as above, up to a multiplicative constant.

**Example 6** (An anisotropically graded partition for corner and edge singularities). Let  $\Omega$  be the Fichera-corner domain:  $\Omega = [-a, a]^3 \setminus ([0, a] \times [-a, 0] \times [0, a])$ . Let



Left: Fichera-corner domain Right: cut at  $x_3 = 0$

FIGURE 7. An anisotropic partition for Fichera-corner domain with ‘singular’ corners and edges.



Left: Two ‘tetrahedra’:  $P - P_2P_4O$  and  $P - P_2P_5O$ , with overlapped cuboids along the common plane  $PP_2O$ .  
 Right: Three consecutively connected tetrahedra in a cuboid:  $Q - Q_2Q_4A$ ,  $Q - Q_2Q_5A$ , and  $Q - Q_3Q_5A$ .

$\mathcal{P}$  of  $\Omega$  be the tensor cross product (all the directions of  $x_1, x_2$  and  $x_3$ ) of the anisotropically graded one-dimensional partition of the interval  $[0, a]$  with  $h_1^* > h_2^* > \dots > h_N^*$ , satisfying  $\sum_{i=1}^N h_i^* = a$  and  $h_1^*/h_N^* \rightarrow \infty$  as  $N \rightarrow \infty$ ; see Fig. 7.

This example is basically analyzed following the same argument as in Example 3. We only consider part of  $[0, a]^3$ :  $PP_2P_4P_1 - P_3P_5OP_6$ . We further divide  $PP_2P_4P_1 - P_3P_5OP_6$  into 6 overlapped ‘tetrahedra’:  $P - P_2P_5O$ ,  $P - P_2P_4O$ , and  $P - P_3P_5O$ ,  $P - P_3P_6O$ , and  $P - P_1P_4O$ ,  $P - P_1P_6O$ , where each cuboid in the overlapped part is overlapped at most 5 times, and where  $P - P_2P_5O$  and  $P - P_2P_4O$  are overlapped along the common plane  $PP_2O$ , see the left-hand side in Fig. 8, and the same for the other two groups of ‘tetrahedra’ (i.e.,  $P - P_3P_5O$  and  $P - P_3P_6O$  are overlapped along the plane  $PP_3O$ , and  $P - P_1P_4O$  and  $P - P_1P_6O$

are overlapped along the plane  $PP_1O$ ). It suffices to verify Condition C) for each ‘tetrahedron’, say  $P - P_2P_4O$ , and to establish the Poincaré-Friedrichs inequality on such ‘tetrahedron’. The Poincaré-Friedrichs inequality on  $\Omega$  is obtained just by summing all the ones on these  $7 \times 6 = 42$  ‘tetrahedra’.

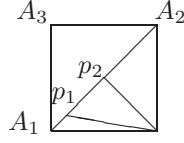
To verify Condition C) for ‘tetrahedron’  $P - P_2P_4O$ , we have the following level decomposition: along  $x_2$  (opposite) direction  $l = 1, 2, \dots, N$ , and then along  $x_1$  (opposite) direction  $k = 1, 2, \dots, l$ , and then along  $x_3$  (opposite) direction  $i = 1, 2, \dots, k$ . In each ‘one-dimensional’ level, there are  $k$  cuboids in all, and each cuboid  $D_i$  has the volume  $h_l^* \times h_k^* \times h_i^*$  and has the diameter  $\approx h_i^*$ , where the diameter  $\approx h_i^*$  is calculated by  $h_i^* \leq \sqrt{(h_l^*)^2 + (h_k^*)^2 + (h_i^*)^2} \leq \sqrt{3} (h_i^*) = \sqrt{3} h_i^*$ , because  $l \geq k \geq i$  and  $h_l^* \leq h_k^* \leq h_i^*$ . We verify Condition C) as in (3.24) by computing with  $m = k$

$$\begin{aligned} \sum_{i=2}^m \sum_{r=1}^{i-1} \frac{|D_i| h_{D_r}^2}{|D_r|} &= \sum_{i=2}^k \sum_{r=1}^{i-1} \frac{h_l^* \times h_k^* \times h_i^*}{h_l^* \times h_k^* \times h_r^*} (h_r^*)^2 \\ &= \sum_{i=2}^k \sum_{r=1}^{i-1} h_i^* h_r^* \leq \left( \sum_{i=1}^N h_i^* \right)^2 = a^2 \leq C |\Omega|^{2/3}. \end{aligned}$$

For other tetrahedra, we can verify Condition C) in the same way as above and we have the same estimates, but with different level decompositions. For  $P - P_2P_5O$  along the  $x_2$  (opposite) direction  $l = 1, 2, \dots, N$ , and then along the  $x_3$  (opposite) direction  $k = 1, 2, \dots, l$ , and then along the  $x_1$  (opposite) direction  $i = 1, 2, \dots, k$ ; for  $P - P_3P_5O$ : along the  $x_3$  (opposite) direction  $l = 1, 2, \dots, N$ , and then along the  $x_2$  (opposite) direction  $k = 1, 2, \dots, l$ , and then along the  $x_1$  (opposite) direction  $i = 1, 2, \dots, k$  and for  $P - P_3P_6O$ : along the  $x_3$  (opposite) direction  $l = 1, 2, \dots, N$ , and then along the  $x_1$  (opposite) direction  $k = 1, 2, \dots, l$ , and then along the  $x_2$  (opposite) direction  $i = 1, 2, \dots, k$ ; for  $P - P_1P_6O$ , along the  $x_1$  (opposite) direction  $l = 1, 2, \dots, N$ , and then along the  $x_3$  (opposite) direction  $k = 1, 2, \dots, l$ , and then along the  $x_2$  (opposite) direction  $i = 1, 2, \dots, k$  and for  $P - P_1P_4O$ , along the  $x_1$  (opposite) direction  $l = 1, 2, \dots, N$ , and then along the  $x_2$  (opposite) direction  $k = 1, 2, \dots, l$ , and then along the  $x_3$  (opposite) direction  $i = 1, 2, \dots, k$ .

If considering tetrahedra partitions, as in Example 5, we have two sub-levels of tetrahedra in each level of cuboids. For such deduced tetrahedra partitions  $\mathcal{Q} = \{T\}$ , to verify Condition C) it suffices to consider the ‘tetrahedron’  $P - P_2P_4O$ , with the level decomposition: along the  $x_2$  (opposite) direction  $l = 1, 2, \dots, N$ , and then along the  $x_1$  (opposite) direction  $k = 1, 2, \dots, l$ , and we need only consider either of the two sub-levels of tetrahedra in each level of cuboids, say, along the  $x_3$  (opposite) direction  $i = 1, 2, \dots, 3k$ . We introduce a fourth variable  $j = 1, 2, \dots, k$ , corresponding to the  $j$ -th cuboid in the level of  $k$  cuboids. Then the three tetrahedra:  $i = 3j - 2, 3j - 1, 3j$  belong to the same  $j$ -th cuboid, and these three consecutively connected tetrahedra have the same volume  $h_l^* \times h_k^* \times h_j^*/6$  and have the same diameter  $\approx h_j^*$ , see the right-hand side in Fig. 8. We verify Condition C) as in (3.24) by computing with  $m = 3k$ ,

$$\sum_{i=2}^m \sum_{r=1}^{i-1} \frac{|T_i| h_{T_r}^2}{|T_r|} = 3 \sum_{j=1}^k (h_j^*)^2 + 9 \sum_{j=2}^k \sum_{s=1}^{j-1} h_j^* h_s^* \leq 12 a^2 \leq C |\Omega|^{2/3}.$$

FIGURE 9. A partition with two hanging nodes:  $p_1, p_2$ 

In what follows, we shall address the verification of Hypothesis H), which has the same form for both two- and three-dimensional domains. In practice, we can easily verify Hypothesis H). In fact, the partitions are usually equivalent families [18], in the sense that there is a fixed finite number of reference domains  $\hat{D}_j$  and invertible mappings  $\mathbf{F}_j$ ,  $1 \leq j \leq J$ , such that for any  $D \in \mathcal{P}$  it holds that  $D = \mathbf{F}(\hat{D})$  for some pair  $(\hat{D}, \mathbf{F})$ . Such  $\mathbf{F}_j$  and  $\hat{D}_j$  exist, e.g., when all  $D$  are simplexes (or each  $D$  is composed of a fixed finite number of simplexes),  $\mathbf{F}_j \equiv \mathbf{F}$  is an affine mapping (or a piecewise affine mapping) and  $\hat{D}_j \equiv \hat{D}$  is a simplex (or a polygon/polyhedron composed of a fixed finite number of simplexes).

We verify Hypotheses H) using the standard scaling argument (see [18, 23]): first establishing it on reference domains and then using the invertible mappings to obtain it on physical sub-domains. Note that the constant  $C_2$  appearing in Hypothesis H) possibly depends on  $\sigma_D$  as defined in (1.3), because of the use of the scaling argument. Fortunately, however, for practical polyhedra partitions we need only verify Hypothesis H) for simplexes and other affine equivalent families of partitions (see Remark 6 later on) and in that case, the constant  $C_2$  does not depend on the shape-regular condition (1.3); see Proposition 1 in the below.

Assume that  $\Omega$  is polygonal (or polyhedral) and is partitioned into triangles (or tetrahedra). Let  $\hat{D}$  denote the reference domain (a triangle with vertices  $(0, 0), (1, 0), (0, 1)$  or a tetrahedron with vertices  $(0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1)$ ) such that for any  $D \in \mathcal{P}$  we have  $D = \mathbf{F}_D(\hat{D})$ , where  $\mathbf{F}_D$  is the affine mapping. For any function  $u$  defined on  $D$  we associate a function  $\hat{u}$  defined on  $\hat{D}$  satisfying  $\hat{u} = u \circ \mathbf{F}_D$ . Denote by  $\mathbf{DF}_D \in \mathbb{R}^{d \times d}$  the first-order derivatives matrix of  $\mathbf{F}_D$  on  $\hat{D}$ , with  $(\mathbf{DF}_D)_{kl} = \partial(\mathbf{F}_D)_k / \partial \hat{x}_l$  and by  $\mathbf{J}_\mathbf{F}$  the Jacobian of  $\mathbf{DF}_D$ . Note that  $\mathbf{DF}_D$  and  $\mathbf{J}_\mathbf{F}$  are constant quantities on  $\hat{D}$ . Here we do not assume conforming partitions (i.e., any face of any  $D_1$  in  $\mathcal{P}$  is either a subset of  $\partial\Omega$ , or a face of another  $D_2$  in  $\mathcal{P}$ , cf. [18]). Nonconforming partitions with hanging nodes are allowed (See Fig. 9 below).

**Proposition 1.** *For any partitions whose sub-domains are triangles (or tetrahedra) and for any piecewise polynomials, Hypothesis H) holds, with  $C_2$  being independent of  $D$  (thus, independent of the aspect ratio  $\sigma_D$  defined as in (1.3)).*

*Proof.* Let  $\hat{u} = u \circ \mathbf{F}_D \in \mathcal{P}_s(\hat{D})$ , where  $\mathcal{P}_s(\hat{D})$  is the space of polynomials in  $d$  variables of degree  $\leq s$  (here  $s \geq 1$  is an integer, while for  $s = 0$  Hypothesis H) holds trivially). Noting that  $\int_f u = 0$  leads to (only for affine mappings)

$$\int_f \hat{u} = 0, \quad \text{or} \quad \hat{u}(\hat{p}) = 0 \quad \text{with some } \hat{p} \in \hat{f},$$



and the from Taylor's expansion to get

$$-\hat{u}(\hat{x}) = \hat{u}(\hat{p}) - \hat{u}(\hat{x}) = \nabla \hat{u}(\hat{x}) \cdot (\hat{p} - \hat{x}) + \sum_{2 \leq |\alpha| \leq s} \frac{(\hat{p} - \hat{x})^\alpha}{\alpha!} \left( \frac{\partial}{\partial \hat{x}} \right)^\alpha \hat{u}(\hat{x}),$$

where  $\alpha$  is the usual multi-index of nonnegative integers, and  $|\hat{p} - \hat{x}| \leq h_{\hat{D}} = \sqrt{2}$ , we have

$$(3.27) \quad \int_{\hat{D}} \hat{u}^2 \leq \hat{C} \int_{\hat{D}} |\nabla \hat{u}|^2,$$

where we used the fact that  $|\hat{u}|_{l,\hat{D}} \leq \hat{C} |\hat{u}|_{1,\hat{D}}$ ,  $2 \leq l \leq s$ , for all  $\hat{u} \in \mathcal{P}_s(\hat{D})$ , and the constant  $\hat{C}$  depends only on  $s$  and the  $H^l$ - and  $H^1$  semi-norms on  $\hat{D}$  of the basis functions of  $\mathcal{P}_s(\hat{D})$ , but  $s$  and those semi-norms on  $\hat{D}$  are constants independent of  $D$ . We thus have

$$(3.28) \quad \|u\|_{0,D}^2 = \mathbf{J}_{\mathbf{F}} \int_{\hat{D}} \hat{u}^2 \leq \mathbf{J}_{\mathbf{F}} \hat{C} \|\nabla \hat{u}\|_{0,\hat{D}}^2 \leq C \hat{C} \|\mathbf{DF}_D\|_{\infty,\hat{D}}^2 \|\nabla u\|_{0,D}^2,$$

from which  $\|\mathbf{DF}_D\|_{\infty,\hat{D}} \leq C h_D$  and we conclude that Hypothesis H) holds.  $\square$

*Remark 5.* Note that the objective function  $u$  is in general of piecewise polynomials in practice. For piecewise *linear* polynomials on general partitions of *general shapes*, since  $u(p) = 0$  with some  $p \in f$  from  $\int_f u = 0$ , Hypothesis H) holds with

$$(3.29) \quad \int_D u^2 = \int_D |\nabla u \cdot (p - x)|^2 \leq h_D^2 \int_D |\nabla u|^2.$$

On the other hand, Proposition 1 is still valid for any piecewise  $H^1$  functions in the case where the partitions are conforming. In fact, what we need to do is to show (3.27), and this can be done with the use of the following argument: Setting

$$\bar{u}_{\hat{D}} := \frac{\int_{\hat{D}} \hat{u}}{|\hat{D}|} \text{ and } \bar{u}_{\hat{f}} := \frac{\int_{\hat{f}} \hat{u}}{|\hat{f}|} = 0, \text{ we have}$$

$$\|\hat{u}\|_{0,\hat{D}}^2 = \|\hat{u} - \bar{u}_{\hat{f}}\|_{0,\hat{D}}^2 \leq 2 \|\hat{u} - \bar{u}_{\hat{D}}\|_{0,\hat{D}}^2 + 2 \|\bar{u}_{\hat{D}} - \bar{u}_{\hat{f}}\|_{0,\hat{D}}^2,$$

where, using the trace theorem on  $\hat{D}$ ,

$$(3.30) \quad \int_{\hat{f}} |\hat{v}|^2 \leq \hat{C}_t (\|\hat{v}\|_{0,\hat{D}}^2 + |\hat{v}|_{1,\hat{D}}^2) \quad \forall \hat{v} \in H^1(\hat{D}),$$

where the trace-theorem constant  $\hat{C}_t$  depends on the geometry of  $\hat{D}$ , we have

$$\begin{aligned} \|\bar{u}_{\hat{D}} - \bar{u}_{\hat{f}}\|_{0,\hat{D}}^2 &= |\hat{D}| \left( \frac{\int_{\hat{f}} (\bar{u}_{\hat{D}} - \hat{u})}{|\hat{f}|} \right)^2 \leq \frac{|\hat{D}|}{|\hat{f}|} \int_{\hat{f}} (\bar{u}_{\hat{D}} - \hat{u})^2 \\ &\leq \hat{C}_t \frac{|\hat{D}|}{|\hat{f}|} \left( \|\bar{u}_{\hat{D}} - \hat{u}\|_{0,\hat{D}}^2 + |\hat{u}|_{1,\hat{D}}^2 \right) \\ &\leq \hat{C} (\|\bar{u}_{\hat{D}} - \hat{u}\|_{0,\hat{D}}^2 + |\hat{u}|_{1,\hat{D}}^2), \end{aligned}$$

since  $\frac{|\hat{D}|}{|\hat{f}|} \leq \frac{1}{d}$ , with  $d = 2$  (two dimensions) or  $d = 3$  (three dimensions), for *conforming* partitions with  $|\hat{D}| = 1/2$ ,  $|\hat{f}| = 1$  or  $|\hat{f}| = \sqrt{2}$  ( $d = 2$ ) or  $|\hat{D}| = 1/6$ ,  $|\hat{f}| = 1/2$  or  $|\hat{f}| = \sqrt{3}/2$  ( $d = 3$ ). Hence, we have

$$(3.27)' \quad \|\hat{u}\|_{0,\hat{D}}^2 \leq \hat{C} (\|\bar{u}_{\hat{D}} - \hat{u}\|_{0,\hat{D}}^2 + |\hat{u}|_{1,\hat{D}}^2) \leq \hat{C} |\hat{u}|_{1,\hat{D}}^2,$$

since (cf. [11])  $\|\tilde{u}_{\hat{D}} - \hat{u}\|_{0,\hat{D}} \leq \frac{h_{\hat{D}}}{\pi} |\hat{u}|_{1,\hat{D}} = \frac{\sqrt{2}}{\pi} |\hat{u}|_{1,\hat{D}}$ .

We should note that for *nonconforming* partitions, the above argument does seem applicable, because we do not know how  $\hat{C}_t$  depends on  $\hat{f}$  and because the ratio  $\frac{|\hat{D}|}{|\hat{f}|}$  may be not bounded from above since  $\hat{f}$  may be only a *very small* part of a side or a face of  $\hat{D}$  (for example, in Fig. 9,  $f = \overline{A_1 p_1}$  is a very small part of the side  $\overline{A_1 A_2}$  of the triangle  $D_{A_1 A_2 A_3}$ , and accordingly  $\hat{f} = \mathbf{F}_D^{-1}(f)$  is a very small part of the side  $\overline{\hat{A}_1 \hat{A}_2}$  of the reference triangle  $\hat{D}_{\hat{A}_1 \hat{A}_2 \hat{A}_3} = \mathbf{F}_D^{-1}(D_{A_1 A_2 A_3})$ , where  $\mathbf{F}_D^{-1}$  denotes the inverse of  $\mathbf{F}_D$ ).

*Remark 6.* For other commonly used polyhedra but nonsimplex partitions in practice, for affine mappings (the invertible mapping involved in the equivalent families), the same argument in Proposition 1 can be employed to verify Hypothesis H); for nonaffine mappings such as quadrilaterals partitions, we do not necessarily verify Hypothesis H) for these nonsimplexes, because we can obtain the Poincaré-Friedrichs inequality for such partitions from the one for simplex partitions. Taking quadrilateral partitions as an example, we may divide each quadrilateral into two or four triangles just by connecting the vertices of the quadrilateral, and we then obtain triangles partitions. On such deduced triangles partition we may establish the Poincaré-Friedrichs inequality, and we then obtain the Poincaré-Friedrichs inequality for the original quadrilateral partitions, due to the fact that the objective piecewise  $H^1$  function has zero jumps across interior sides newly added in the interiors of quadrilaterals.

#### 4. THE CASE OF NONCONFORMING ELEMENTS

In this section we derive the Poincaré-Friedrichs inequality for Crouzeix-Raviart (CR) nonconforming linear finite element [21] under the so-called maximum angle condition [27].

Let  $\mathcal{T}_h$  denote the triangulation (conforming partition [18]) of the polygonal or polyhedral  $\Omega \subset \mathbb{R}^d$  ( $d = 2, 3$ ) into triangles or tetrahedra. Denote by  $\mathcal{F}$  the set of all sides or faces in  $\mathcal{T}_h$ . Recall the CR nonconforming linear element [21],

$$(4.1) \quad U_h = \left\{ v \in L^2(\Omega); v|_T \in \mathcal{P}_1(T), \forall T \in \mathcal{T}_h, \int_f [v] = 0, \forall f \in \mathcal{F} \right\}$$

and the lowest-order Raviart-Thomas flux element [29]

$$(4.2) \quad \mathbf{X}_h = \{ \mathbf{v} \in H(\text{div}; \Omega); \mathbf{v}|_T \in (\mathcal{P}_0(T))^d + \mathbf{x} \mathcal{P}_0(T), \forall T \in \mathcal{T}_h \},$$

where

$$H(\text{div}; \Omega) = \{ \mathbf{v} \in (L^2(\Omega))^d; \text{div } \mathbf{v} \in L^2(\Omega) \}.$$

From [29] we know that there exists an operator  $\Pi : (H^1(\Omega))^d \rightarrow \mathbf{X}_h$  such that

$$(4.3) \quad \int_f \Pi \mathbf{v} \cdot \mathbf{n} = \int_f \mathbf{v} \cdot \mathbf{n} \quad \forall f \text{ sides or faces of } T, \forall T \in \mathcal{T}_h,$$

and for all  $\mathbf{v}_h \in \mathbf{X}_h$ ,

$$(4.4) \quad \mathbf{v}_h \cdot \mathbf{n}|_f = \text{constant} \quad \forall f \text{ sides or faces of } T, \forall T \in \mathcal{T}_h,$$

with  $\mathbf{n}$  denoting the outward unit normal vector to  $f$ , and

$$(4.5) \quad \text{div } \Pi \mathbf{v} = P_0 \text{div } \mathbf{v},$$

where  $P_0$  stands for the  $L^2$  orthogonal projection onto the piecewise constant space, satisfying

$$(4.6) \quad \|v - P_0 v\|_{0,T} \leq C h_T \|\nabla v\|_{0,T} \quad \forall v \in H^1(T).$$

**The maximum angle condition in three-dimensions** [27]. Denote by  $\alpha_h^0(T)$  the maximum angle of all triangular faces of the tetrahedron  $T \in \mathcal{T}_h$  and by  $\beta_h^0(T)$  the maximum angle between faces of  $T$ . We require that there exists a constant  $\bar{\gamma}$  such that

$$(4.7) \quad \alpha_h^0(T) \leq \bar{\gamma} < \pi, \quad \beta_h^0(T) \leq \bar{\gamma} < \pi \quad \forall T \in \mathcal{T}_h, \forall h.$$

There is a similar statement of the maximum angle condition in two dimensions [9].

**Lemma 2** ([3]). *Assuming the maximum angle condition. For all  $\mathbf{v} \in (H^1(\Omega))^d$  we have*

$$(4.8) \quad \|\Pi \mathbf{v}\|_{0,T} \leq C \|\mathbf{v}\|_{1,T}$$

where the constant  $C > 0$  depends on  $\bar{\gamma}$ , but it is independent of  $T$  and  $h$ .

**Proposition 2** ([24, 23]). *For any convex polygon or polyhedron  $\Omega$  with the boundary  $\partial\Omega$  the following elliptic problem with any given  $f \in L^2(\Omega)$ ,*

$$-\Delta u = f \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0$$

has a solution  $u \in H^2(\Omega)$ , satisfying

$$\|u\|_2 \leq C \|f\|_0.$$

**Theorem 3.** *Assume that the maximum angle condition holds. There exists a positive constant  $C_{PF}$  only depending on  $\bar{\gamma}$  and  $\Omega$  in (4.7), such that*

$$(4.9) \quad \|v\|_0 \leq C_{PF} \left( \sum_{T \in \mathcal{T}_h} \|\nabla v\|_{0,T}^2 \right)^{\frac{1}{2}} \quad \forall v \in U_h.$$

*Proof.* Let the cuboid  $R$  contain  $\Omega$ , and we may assume that  $\mathcal{T}_h$  is a triangulation of  $R$ . We then define  $v$  on  $R$  by setting its value zero outside  $\Omega$ , and we obtain the Poincaré-Friedrichs inequality (4.9) of  $v \in U_h$  on  $\Omega$  from the one of the extended  $v$  on  $R$ .

We consider the elliptic problem: To find  $u \in H_0^1(R)$  such that

$$-\Delta u = v \quad \text{in } R, \quad u|_{\partial R} = 0.$$

From Proposition 2 we have  $u \in H^2(R)$  and we set  $\mathbf{p} := \nabla u \in (H^1(R))^d$ . We thus have from Lemma 2

$$\begin{aligned}
\|v\|_0^2 &= (v, -\Delta u) = -(v, \operatorname{div} \mathbf{p}) = (v, \operatorname{div} \Pi \mathbf{p} - \operatorname{div} \mathbf{p}) - (v, \operatorname{div} \Pi \mathbf{p}) \\
&= (v, P_0 \operatorname{div} \mathbf{p} - \operatorname{div} \mathbf{p}) + \sum_{T \in \mathcal{T}_h} (\nabla v, \Pi \mathbf{p})_{0,T} - \sum_{f \in \mathcal{F}} \int_f [v] \Pi \mathbf{p} \cdot \mathbf{n} \\
&= (v - P_0 v, -\operatorname{div} \mathbf{p}) + \sum_{T \in \mathcal{T}_h} (\nabla v, \Pi \mathbf{p})_{0,T} \\
(4.10) \quad &\leq C \left( \sum_{T \in \mathcal{T}_h} h_T^2 \|\nabla v\|_{0,T}^2 \right)^{\frac{1}{2}} \|\operatorname{div} \mathbf{p}\|_0 \\
&\quad + C \left( \sum_{T \in \mathcal{T}_h} \|\nabla v\|_{0,T}^2 \right)^{\frac{1}{2}} \|\mathbf{p}\|_1,
\end{aligned}$$

from which  $\|\operatorname{div} \mathbf{p}\|_0 + \|\mathbf{p}\|_1 \leq C \|v\|_0$  (See Proposition 2), and we obtain the desired result.  $\square$

*Remark 7.* It can be seen that the maximum angle condition holds for Examples 2 and 4 in Section 2, and the Poincaré-Friedrichs inequality (4.9) holds for the CR nonconforming linear element (but the constant  $C_{PF}$  therein does not depend on such a constant  $\bar{\gamma}$  in (4.7)). On the other hand, the maximum angle condition is only *sufficient*, because Examples 1 and 3 in Section 2 show that (4.9) holds where the constant  $C_{PF}$  only depends on  $\Omega$  (up to some universe positive constant) and is independent of the maximum angle and the minimum angle, while both conditions are no longer true in these two examples.

*Remark 8.* We put forward an open problem: Does the Poincaré-Friedrichs inequality hold under the maximum angle condition for general piecewise  $H^1$  functions which do not satisfy the jumps condition as in  $U_h$ ? We have proven (4.9) using the argument motivated by [3] and [6] for the CR nonconforming linear element, but this argument seems unable to treat the general case, because the difficulty is that the estimation of the term  $\sum_{f \in \mathcal{F}} \int_f [v] \Pi \mathbf{p} \cdot \mathbf{n}$  relies on the use of the local trace theorem as mentioned in the Introduction where the aspect ratio will be inevitably involved which may grow to infinity as the global mesh size tends to zero in the case of anisotropic meshes. Note that such a term disappears if  $v \in U_h$ .

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#### REFERENCES

1. G. Acosta, *Lagrange and average interpolation over 3D anisotropic elements*. J. Comput. Appl. Math., **135** (2001), 91–109. MR1854446 (2002f:65152)
2. G. Acosta and G. Monzón, *Interpolation error estimates in  $W^{1,p}$  for degenerate  $Q_1$  isoparametric elements*. Numer. Math., **104** (2006), 129–150. MR2242611 (2007m:65098)
3. G. Acosta and R. G. Durán, *The maximum angle condition for mixed and nonconforming elements: Application to the Stokes equations*. SIAM J. Numer. Anal., **37** (2000), 18–36. MR1721268 (2000g:65107)

4. G. Acosta and R. G. Durán, *Error estimates for  $Q_1$  isoparametric elements satisfying a weak angle condition*. SIAM J. Numer. Anal., **38** (2000), 1073–1088. MR1786131 (2001h:65129)
5. T. Apel, *Anisotropic Finite Elements: Local Estimates and Applications*, Advances in Numerical Mathematics, Teubner, Stuttgart, Germany, 1999. MR1716824 (2000k:65002)
6. D. N. Arnold, *An interior penalty finite element method with discontinuous elements*. SIAM J. Numer. Anal., **19** (1982), 742–760. MR664882 (83f:65173)
7. D. N. Arnold and R. S. Falk, *A uniformly accurate finite element method for the Reissner-Mindlin plate*, SIAM J. Numer. Anal., **26** (1989), 1276–1290 MR1025088 (91c:65068)
8. D. N. Arnold, F. Brezzi, B. Cockburn, and L. D. Marini, *Unified analysis of discontinuous Galerkin methods for elliptic problems*. SIAM J. Numer. Anal., **39** (2002), 1749–1779. MR1885715 (2002k:65183)
9. I. Babuška, A. K. Aziz, *On the angle condition in the finite element method*. SIAM J. Numer. Anal., **13** (1976), 214–226. MR0455462 (56:13700)
10. C. Bandle, *Isoperimetric Inequalities and Applications*, Pitman, London, 1980. MR572958 (81e:35095)
11. M. Bebendorf, *A note on the Poincaré inequality for convex domains*. Journal for Analysis and its Applications, **22** (2003), 751–756. MR2036927 (2004k:26025)
12. M. Bebendorf and W. Hackbush, *Existence of  $H$ -matrix approximants to the inverse FE-matrix of elliptic operators with  $L^\infty$ -coefficients*. Numer. Math., **95** (2003), 1–28. MR1993936 (2004e:65128)
13. F. Ben Belgacem, *The mortar finite element method with Lagrange multipliers*. Numer. Math., **84** (1999), 173–197. MR1730018 (2000k:65204)
14. J. Brandts, S. Korotov and M. Křížek, *On the equivalence of regularity criteria for triangular and tetrahedral partitions*. Comput. Math. Appl. **55** (2008), 2227–2233. MR2413688 (2009a:65298)
15. S. Brenner, *Poincaré-Friedrichs inequalities for piecewise  $H^1$  functions*. SIAM J. Numer. Anal., **41** (2003), 306–324. MR1974504 (2004d:65140)
16. S. Brenner, *Korn's inequalities for piecewise  $H^1$  vector fields*. Math. Comput., **73** (2003), 1067–1087. MR2047078 (2005c:65096)
17. F. Brezzi, B. Cockburn, L. D. Marini and E. Süli, *Stabilization mechanisms in discontinuous Galerkin finite element methods*. Comput. Methods Appl. Mech. Engrg., **195** (2006), 3293–3310 MR2220920 (2006m:65256)
18. P. G. Ciarlet, *The Finite Element Method for Elliptic Problems*, North-Holland, Amsterdam, New York, 1978. MR0520174 (58:25001)
19. B. Cockburn and C.-W. Shu, *The local discontinuous Galerkin method for time-dependent convection-diffusion systems*. SIAM J. Numer. Anal., **35** (1998), 2440–2463. MR1655854 (99j:65163)
20. B. Cockburn, G. Karniadakis and C.-W. Shu, *Discontinuous Galerkin Methods: Theory, Computation and Applications*, Lecture Notes in Computational Science and Engineering, volume 11, Springer, 2000. MR1842160 (2002b:65004)
21. M. Crouzeix and P.-A. Raviart, *Conforming and nonconforming finite element methods for solving the stationary Stokes equations*, RAIRO Anal. Numér., **7** (1973), 33–75. MR0343661 (49:8401)
22. R. G. Durán and A. L. Lombardi, *Error estimates on anisotropic  $Q_1$  elements for functions in weighted Sobolev spaces*. Math. Comp., **74** (2005), 1679–1706. MR2164092 (2006e:65216)
23. V. Girault and P.A. Raviart, *Finite Element Methods for Navier-Stokes Equations, Theory and Algorithms*, Springer-Verlag, Berlin, 1986. MR851383 (88b:65129)
24. P. Grisvard, *Elliptic Problems in Nonsmooth Domains*, Pitman, Boston, MA, 1985. MR775683 (86m:35044)
25. O. A. Ladyzhenskaya and N. N. Uraltseva, *Linear and Quasilinear Elliptic Equations*, Academic Press, New York, London, 1968. MR0244627 (39:5941)
26. Y.-J. Liu, C.-W. Shu, E. Tadmor and M. Zhang, *Central discontinuous Galerkin methods on overlapping cells with a nonoscillatory hierarchical reconstruction*. SIAM J. Numer. Anal. **45** (2007), 2442–2467. MR2361897 (2009a:15256)
27. M. Křížek, *On the maximum angle condition for linear tetrahedral elements*, SIAM J. Numer. Anal., **29** (1992), 513–520. MR1154279 (92k:65165)
28. L. E. Payne, *Isoperimetric inequalities and their applications*, SIAM Rev., **9** (1967), 453–488. MR0218975 (36:2058)

29. J. E. Roberts and J.-M. Thomas, *Mixed and Hybrid Methods*, in: Handbook of Numerical Analysis, Vol. II, Finite Element Methods (Part 1), P. G. Ciarlet and J. L. Lions, eds., North-Holland, 1991. MR1115235 (92f:65001)
30. R. Verfürth, *A note on polynomial approximation in Sobolev spaces*, M2AN Math. Model. Numer. Anal., **33** (1999), 715–719. MR1726481 (2000h:41016)
31. B. I. Wohlmuth, *Discretization Methods and Iterative Solvers Based on Domain Decomposition*, Springer-Verlag, Heidelberg, 2001. MR1820470 (2002c:65231)

SCHOOL OF MATHEMATICAL SCIENCES, NANKAI UNIVERSITY, 94 WEIJIN STREET, NANKAI DISTRICT, TIANJIN 300071, PEOPLE'S REPUBLIC OF CHINA

*E-mail address:* `hyduan@nankai.edu.cn`

DEPARTMENT OF MATHEMATICS, NATIONAL UNIVERSITY OF SINGAPORE, 2 SCIENCE DRIVE 2, SINGAPORE 117543

*E-mail address:* `scitance@nus.edu.sg`